Exercise Sheet No. 1
– with solutions –

Exercise 1:
Consider the following sets of real numbers: $A = [-13, 2]$, $B = [2, 5]$, $C = \{0, 2, 8\}$ und $D = \{0, 4\}$.
(a) Determine the sets $A \cup B$, $A \cap C$ and $(B \backslash C) \cap (A \cup D)$.

(b) Determine all subsets of the set $\{\alpha, 2, \Box\}$.

Solution 1:

(a) We have

$$\begin{aligned} A \cup B &= [-13, 2] \cup [2, 5] = [-13, 5], \\ A \cap C &= [-13, 2] \cap \{0, 2, 8\} = \{0, 2\}, \\ (B\backslash C) \cap (A \cup D) &= (2, 5] \cap ([-13, 2] \cup \{4\}) = \{4\}. \end{aligned}$$

(b) There are $2^3 = 8$ subsets, namely $\emptyset, \{\alpha\}, \{2\}, \{\Box\}, \{\alpha, 2\}, \{\alpha, \Box\}, \{2, \Box\}, \{\alpha, 2, \Box\}$.

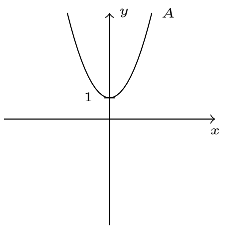
Exercise 2: Let the sets A, B and C be given by:

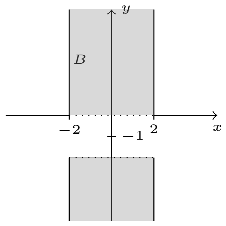
$$\begin{aligned} A &= \{(x, y) \in \mathbb{R}^2 : \ y = x^2 + 1\}, \\ B &= \{(x, y) \in \mathbb{R}^2 : \ |x| \le 2, \ |y + 1| > 1\}, \\ C &= \{(x, y) \in \mathbb{R}^2 : \ |x| > 2\}. \end{aligned}$$

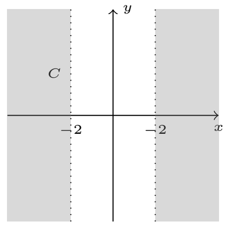
(a) Sketch the sets A, B and C.

(b) Give a formal description of the sets $A \cap B$, $A \cap C$, $B \cap C$ and $\mathbb{R}^2 \setminus (B \cup C)$ and sketch them.

Solution 2:

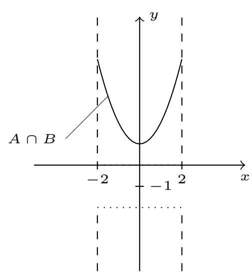






(a)

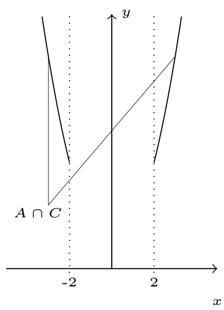
(b) <u>$A \cap B$</u> :



$(x, y) \in A \Rightarrow y = x^2 + 1$. $(x, y) \in B \Rightarrow -2 \le x \le 2$ and: $y + 1 < -1$ or $y + 1 > 1$, so $y < -2$ or $y > 0$. If $(x, y) \in A \cap B$, since $0 \le |x| \le 2$, $1 \le x^2 + 1 \le 5$, thus, $1 \le y \le 5$. Together this gives the condition $-2 \le x \le 2, \ 1 \le y \le 5, \ y = x^2 + 1$ if $(x, y) \in A \cap B$.
Thus, $A \cap B \subset \{(x, y) \in \mathbb{R}^2 \mid -2 \le x \le 2, \ 1 \le y \le 5, \ y = x^2 + 1\}$.
Since also $\{(x, y) \in \mathbb{R}^2 \mid -2 \le x \le 2, \ 1 \le y \le 5, \ y = x^2 + 1\} \subset A \cap B$, we have

$$\{(x, y) \in \mathbb{R}^2 \mid -2 \le x \le 2, \ 1 \le y \le 5, \ y = x^2 + 1\} = A \cap B.$$

<u>$A \cap C$</u> :



$(x, y) \in A \Rightarrow y = x^2 + 1$. $(x, y) \in C \Rightarrow -2 > x$ or $2 < x$. Thus,
$A \cap C \subset \{(x, y) \in \mathbb{R}^2 \mid x \in (-\infty, -2) \cup (2, \infty), \ y = x^2 + 1\}$.
Since also $\{(x, y) \in \mathbb{R}^2 \mid x \in (-\infty, -2) \cup (2, \infty), \ y = x^2 + 1\} \subset A \cap C$, we have

$$\{(x, y) \in \mathbb{R}^2 \mid x \in (-\infty, -2) \cup (2, \infty), \ y = x^2 + 1\} = A \cap C.$$

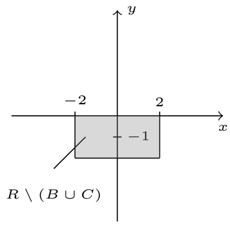
<u>$B \cap C$</u> :

$(x, y) \in B \Rightarrow -2 \le x \le 2$. $(x, y) \in C \Rightarrow -2 > x$ or $2 < x$. That is a contradiction, so we have $B \cap C = \emptyset$.

$\underline{\mathbb{R}^2 \setminus (B \cup C)}$:

$(x, y) \in \mathbb{R}^2 \setminus (B \cup C) \Rightarrow (x, y) \in \mathbb{R}^2$ and $(x, y) \notin (B \cup C)$.
Thus, $(x, y) \in \mathbb{R}^2$, $(x, y) \notin B$ und $(x, y) \notin C$.
If $(x, y) \notin B$, $|x| > 2$ or $-2 \leq y \leq 0$.
If $(x, y) \notin C$, $|x| \leq 2$.
Combined we have $\mathbb{R}^2 \setminus (B \cup C) \subset \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2,\ -2 \leq y \leq 0\}$.
It is easy to see that $\{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2,\ -2 \leq y \leq 0\} \subset \mathbb{R}^2 \setminus (B \cup C)$.
Thus,

$$\{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2,\ -2 \leq y \leq 0\} = \mathbb{R}^2 \setminus (B \cup C).$$



Exercise 3:

(a) Solve the following inequalities for x:

$$(1) \quad (2 - x)(4 - x) \leq 0, \qquad (2) \quad 1 - |x - 2| < \frac{1}{2} |x - 3|.$$

(b) Which $x \in \mathbb{R}$ are solutions to both inequalities? Which $x \in \mathbb{R}$ are solutions to at least one inequality? Which $x \in \mathbb{R}$ are solutions to (1) but not (2)?

Write all your solutions as (a union of) intervals.

Solution 3:

(a) (1) A product of two factors is ≤ 0 if and only if one factor is ≤ 0 and the other one is ≥ 0:

$$\left.\begin{array}{l} 1.\ 2 - x \leq 0,\ 4 - x \geq 0 \Leftrightarrow x \geq 2,\ x \leq 4 \Leftrightarrow 2 \leq x \leq 4 \\ 2.\ 2 - x \geq 0,\ 4 - x \leq 0 \Leftrightarrow x \leq 2,\ x \geq 4 \quad \text{impossible} \end{array}\right\} L_1 = [2, 4]$$

(2) We need to distinguish the following 4 cases:

$$\begin{aligned} (x - 2 \geq 0 \Longleftrightarrow x \geq 2) \quad &\text{and} \quad (x - 3 \geq 0 \Longleftrightarrow x \geq 3), \\ x \geq 2 \quad &\text{and} \quad (x - 3 < 0 \Longleftrightarrow x < 3), \\ (x - 2 < 0 \Longleftrightarrow x < 2) \quad &\text{and} \quad x \geq 3, \\ x < 2 \quad &\text{and} \quad x < 3 \end{aligned}$$

Since the case $x < 2$ and $x \geq 3$ is contradictory, we do not need to consider it. Also, in the case $x \geq 3$ we already have $x \geq 2$, and similarly $x < 2$ implies $x < 3$. Thus, for the three cases only the following conditions remain:

$$x < 2, \qquad 2 \leq x < 3, \qquad 3 \leq x.$$

In the case $x < 2$ we have $|x - 2| = -(x - 2)$ and $|x - 3| = -x + 3$. Thus, here the inequality can be written as $1 - (2 - x) < \frac{1}{2}(3 - x)$. Solving for x this is $x < \frac{5}{3}$, so in this case we obtain

$$x \in (-\infty, 2) \cap (-\infty, \tfrac{5}{3}) = (-\infty, \tfrac{5}{3}).$$

In the case $2 \leq x < 3$ we have $|x - 2| = x - 2$ and $|x - 3| = -x + 3$. Rewriting the inequality accordingly we obtain $x > 3$, a contradiction to the assumption $x < 3$. Thus, in the interval $[2, 3)$ there are no solutions.

In the case $3 \geq x$ we have $|x - 2| = x - 2$ and $|x - 3| = -x + 3$. As before we insert that into the inequality and solve for x, which yields $x > 3$, which means

$$x \in [3, \infty) \cap (3, \infty) = (3, \infty).$$

Combining the sets we found in the first and third case, we obtain

$$L_2 = (-\infty, \tfrac{5}{3}) \text{ or } x \in (3, \infty) \quad \Longleftrightarrow \quad x \in (-\infty, \tfrac{5}{3}) \cup (3, \infty).$$

(b) We have that $x \in \mathbb{R}$ is a solution to both (1) and (2), if it lies in the intersection of the solution sets, which is given by

$$L_1 \cap L_2 = [2, 4] \cap \big((-\infty, \tfrac{5}{3}) \cup (3, \infty)\big) = (3, 4].$$

Being a solution to at least one of the inequalities means that $x \in \mathbb{R}$ is in the intersection of the two solution sets, which is

$$L_1 \cup L_2 = [2,4] \cup \big((-\infty, \tfrac{5}{3}) \cup (3,\infty)\big) = (-\infty, \tfrac{5}{3}) \cup [2,\infty).$$

An $x \in \mathbb{R}$ is a solution to (1) but not (2) if and only if it lies in the following difference:

$$L_1 \setminus L_2 = [2,4] \setminus \big((-\infty, \tfrac{5}{3}) \cup (3,\infty)\big) = [2,3].$$

Exercise 4: Evaluate the following sums:

$$\text{(a)} \quad \sum_{m=5}^{9} (m^2 - m), \qquad \text{(b)} \quad \sum_{\nu=1}^{4} \sum_{k=1}^{\nu} \nu\,(\nu - k), \qquad \text{(c)} \quad \sum_{n=4}^{27} 4 \left(\frac{1}{2}\right)^n.$$

Solution 4:

(a) We split the sum and evaluate the two resulting sums:

$$\begin{aligned}\sum_{m=5}^{9} (m^2 - m) &= \sum_{m=5}^{9} m^2 - \sum_{m=5}^{9} m \\ &= (25 + 36 + 49 + 64 + 81) - (5 + 6 + 7 + 8 + 9) \\ &= 255 - 35 = 220.\end{aligned}$$

(b) The factor ν is independent of the index variable k, so we can take it out of the inner sum. Then we evaluate the inner sum and obtain:

$$\sum_{\nu=1}^{4} \sum_{k=1}^{\nu} \nu(\nu - k) = \sum_{\nu=1}^{4} \nu \sum_{k=0}^{\nu-1} k = \sum_{\nu=1}^{4} \nu \frac{1}{2}(\nu - 1)\nu = \frac{1}{2}(4 + 18 + 48) = 35.$$

(c) We can take the 4 out of the sum and shift the index ($n' = n - 4$). Then we split the product $(\frac{1}{2})^{n'+4}$ and take out the factor $(\frac{1}{2})^4$. Then we use the sum of geometric progression and simplify the expression:

$$\begin{aligned}\sum_{n=4}^{27} 4 \left(\frac{1}{2}\right)^n &= 4 \sum_{n'=0}^{23} \left(\frac{1}{2}\right)^{n'+4} = 4 \sum_{n=0}^{23} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^4 \\ &= \frac{4}{16} \frac{1 - (\frac{1}{2})^{24}}{1 - \frac{1}{2}} = \frac{2}{4}\left(1 - \left(\frac{1}{2}\right)^{24}\right) \\ &= \frac{2^{24} - 1}{2^{25}}\end{aligned}$$

Exercise 5:

(a) Determine i, j, i', j', such that

$$1 + 9 + 25 + \cdots + (2n+1)^2 = \sum_{k=i}^{j} (2k+1)^2 = \sum_{k=i'}^{j'} (2k-1)^2.$$

(b) Evaluate the sum.

Solution 5:

(a) $1 + 9 + 25 + \ldots + (2n+1)^2 = \sum_{k=0}^{n} (2k+1)^2 = \sum_{k=1}^{n+1} (2k-1)^2.$

(b) $\displaystyle\sum_{k=0}^{n} (2k+1)^2 = 4 \sum_{k=0}^{n} k^2 + 4 \sum_{k=0}^{n} k + \sum_{k=0}^{n} 1 = \frac{2n(n+1)(2n+1)}{3} + 2n(n+1) + (n+1) = \frac{(n+1)(2n+1)(2n+3)}{3}$