Exercise Sheet No. 10
– with solutions –

Exercise 46:
Determine the derivatives of

(i) $f_1(x) = x^x, \quad x \in \mathbb{R}_{>0},$ (ii) $f_2(x) = (\sqrt{x}+1)\left(\frac{1}{\sqrt{x}} - 1\right), \quad x \in \mathbb{R}_{>0},$

(iii) $f_3(x) = \frac{\sin x}{\sin x + \cos x}, \quad x \in \left[0, \frac{\pi}{2}\right]$ (iv) $f_4(x) = e^{(\sin x)^2} + e^{\sin(x^2)} + \left(e^{\sin x}\right)^2, \quad x \in \mathbb{R}.$

Solution 46: Using the known rules for finding derivatives we obtain

(i) $f_1(x) = e^{x \ln x} \Rightarrow f_1'(x) = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1).$

(ii) $f_2'(x) = \left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)' = -\frac{1}{2x^{3/2}} - \frac{1}{2x^{1/2}} = -\frac{1+x}{2x^{3/2}}.$

(iii) $f_3'(x) = \frac{\cos x(\sin x + \cos x) - \sin x(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{\cos^2 x + \sin^2 x}{(\sin x + \cos x)^2} = \frac{1}{(\sin x + \cos x)^2}.$

(iv) $f_4'(x) = e^{(\sin x)^2} 2 \sin x \cos x + e^{\sin(x^2)} \cos(x^2)\, 2x + 2e^{2 \sin x} \cos x.$

Exercise 47:
Consider the function

$$f(x) = \begin{cases} e^x, & \text{if } x \le 0, \\ \cos(x) + x, & \text{if } x > 0. \end{cases}$$

(a) Show that f is continuously differentiable at $x \neq 0$ arbitrarily many times.

(b) Show that f is (once) continuously differentiable at $x = 0$.

(c) Is f twice continuously differentiable at $x = 0$?

Solution 47:

(a) The functions e^x and $x + \cos x$ are continuously differentiable arbitrarily many times on $\mathbb{R}$, and thus, the same is true for f.

(b) For $x \neq 0$ we have

$$f'(x) = \begin{cases} e^x, & x < 0, \\ 1 - \sin x, & 0 < x \end{cases}.$$

Thus, we have

$$\lim_{x \to 0-} f'(x) = 1 = \lim_{x \to 0+} f'(x).$$

Thus, the derivative f' is continuously extendable at $x = 0$. Thus, by Thm 7.4 f is continuosly differentibale at $x = 0$.

(c) We have for $x \neq 0$

$$f''(x) = \begin{cases} e^x, & x < 0, \\ -\cos x, & 0 < x. \end{cases}$$

Here we have

$$\lim_{x \to 0-} f''(x) = 1 \neq -1 = \lim_{x \to 0+} f''(x).$$

Thus, f'' is not continuously extendable at $x_0 = 0$ and thus, not twice continuously differentiable.

Exercise 48:

(a) Show that the function given by $f(x) = \sinh(x)$ is strongly monotonically increasing on $\mathbb{R}$ and is thus injective.

(b) Show that f is bijective, and thus has an inverse.

(c) Use the representation $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ to determine the inverse $f^{-1}(y) = \operatorname{Arsinh}(y)$ and its derivative $(f^{-1})'(y)$.

Solution 48:

(a) We have $(\sinh x)' = \cosh(x) > 0$ for all $x \in \mathbb{R}$, so f is strongly monotonically increasing, and thus injective on $\mathbb{R}$.

(b) We already now that the function is increasing and continuous. Surjectivity follows from $\lim_{x\to+\infty} \sinh x = +\infty$ and $\lim_{x\to-\infty} \sinh x = -\infty$.

(c) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$. We solve for x:

$$\begin{aligned} 2y = e^x - e^{-x} \iff e^{2x} - 2ye^x - 1 = 0 \iff \\ (e^x - y)^2 = 1 + y^2 \iff e^x = y \pm \sqrt{1+y^2} \end{aligned}$$

Since $e^x > 0$ and $y - \sqrt{1+y^2} < y - \sqrt{y^2} = 0$ we must have $e^x = y + \sqrt{1+y^2}$. Thus, we have

$$x = \ln(y + \sqrt{1+y^2}), \quad \text{i.e.} \quad f^{-1}(y) = \ln(y + \sqrt{y^2+1}) = \operatorname{Arsinh}(y).$$

Die Ableitung ergibt

$$\begin{aligned} \left(f^{-1}(y)\right)' &= \frac{1}{y+\sqrt{y^2+1}}(y+\sqrt{y^2+1})' = \frac{1+\frac{1}{2}(y^2+1)^{-\frac{1}{2}}2y}{y+\sqrt{y^2+1}} \\ &= \frac{\sqrt{y^2+1}+y}{(y+\sqrt{y^2+1})\sqrt{y^2+1}} = \frac{1}{\sqrt{y^2+1}}, \end{aligned}$$

where from the first to the second row we expanded by $(y^2+1)^{\frac{1}{2}}$.

Exercise 49:
Prove the following inequalities:

(a) $\ln x > \frac{2(x-1)}{x+1}$ for all $x \in \mathbb{R},\ x > 1$,

(b) $5x + \frac{1}{x^5} \geq 6$ for all $x \in \mathbb{R},\ x > 0$.

Solution 49:

(a) We consider the difference of the two expressions:

$$f\colon [1,\infty) \to \mathbb{R},\ x \mapsto \ln x - \frac{2(x-1)}{x+1}$$

and want to show that $f(x) > 0$ for all $x > 1$. We can add the point $x = 1$ to the domain of f, since all the expressions are defined at $x = 1$. f is continuous on $[1,\infty)$ an differentiable on $(1,\infty)$. Now we determine the derivative to find extrema of f:

$$f'(x) = \frac{1}{x} - \frac{2}{x+1} + \frac{2(x-1)}{(x+1)^2} = \frac{(x-1)^2}{x(x+1)^2}$$

We see that $f'(x) > 0$ for $x \neq 1$, thus, f is strictly monotonically increasing. Combining this with the fact that $f(1) = 0$, we obtain that $f(x) > 0$ for $x > 1$.which completes the proof.

(b) We consider the lefthand side as a function

$$f\colon (0,\infty) \to \mathbb{R},\, x \mapsto 5x + \frac{1}{x^5}$$

and want to show that $f(x) \geq 6$ for all $x > 0$. We look at the derivative of f to determine extrema:

$$f'(x) = 5 - \frac{5}{x^6} = \frac{5}{x^6}(x^6 - 1) \qquad \text{and} \qquad f'(x) = 0 \Longleftrightarrow x^6 = 1 \Longleftrightarrow x = 1 \quad (x > 0).$$

We further have $f'(x) > 0$ if $x^6 > 1$, i.e. $x > 1$ and $f'(x) < 0$ if $x^6 < 1$, i.e. $x < 1$. Thus, f is strictly monotonically decreasing on $(0, 1]$ and increasing on $[1, \infty)$. Thus, f has a global minimum at $x = 1$ and $f(1) = 6$, so $f(x) \geq 6$ for all $x > 0$, which proves the inequality.

Exercise 50:
Consider the function $f : [-5, 5] \to \mathbb{R}$ and $g : [\frac{1}{2}, \frac{5}{2}] \to \mathbb{R}$ given by

$$f(x) = \frac{x+2}{x^2+4x+5} \qquad \text{and} \qquad g(x) = 2x^3 - 9x^2 + 12x.$$

Determine the images of f and g.

Solution 50:
The functions f and g are continuously differentiable in the interior of their domains. To determine the range, we need to find all extrema and the values at the boundary points.

(i) For f we have $f(-5) = -3/10$ and $f(5) = 7/50$ for the boundary points. Since the denominator has no zeros, f is differentiable:

$$f'(x) = -\frac{x^2+4x+3}{(x^2+4x+5)^2} \qquad \text{und} \qquad f'(x) = 0 \Longleftrightarrow x^2 + 4x + 3 = 0 \Longleftrightarrow x = -1,\, x = -3$$

At the extremal points we thus have

$$f(-3) = -1/2 \qquad \text{and} \qquad f(-1) = 1/2.$$

Thus, the image is $f([-5, 5]) = [-1/2, 1/2]$.

(ii) For the boundary points of the domain of g we have: $g(1/2) = 4$ and $g(5/2) = 5$. g is a degree 3 polynomial, and thus differentiable:

$$g'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-2)(x-1) \qquad \text{and} \qquad g'(x) = 0 \Longleftrightarrow x = 1,\, x = 2.$$

The extremal values are $g(1) = 5$ and $g(2) = 4$. Thus, we have $g([\frac{1}{2}, \frac{5}{2}]) = [4, 5]$.