Exercise Sheet No. 11
– with solutions –

Exercise 51: Use the mean value theorem to prove the following statements:

(a) Show the inequality

$$\ln(1+x) \le \frac{x}{\sqrt{1+x}} \qquad \text{for } x > 0\,.$$

Hint: Consider the function $f(t) = \ln(1+t) - \frac{t}{\sqrt{1+t}}$ in the interval $[0, x]$.

(b) Show the estimates

$$1 - \frac{1}{x} < \ln x < x - 1, \quad x \in (1, \infty).$$

Can you give an upper and lower bound on the real number $a = 2\ln 3 - 3\ln 2$ using these inequalities?
Hint: Consider the function $f(t) = \ln t$ for $t \in (1, x)$.

(c) Show Lipschitz continuity of the function

$$f(x) = \sqrt{1+x}\,, \qquad 0 \le x < 3\,,$$

and compute a Lipschitz constant.

Solution 51:

(a) According to the mean value theorem we have

$$\frac{f(x) - f(0)}{x - 0} = f'(t_0)$$

for some (unknown) $t_0 \in [0, x]$, and thus, $f(x) = x\, f'(t_0)$ (*). We obtain

$$f'(t) = \frac{1}{1+t} - \frac{1}{\sqrt{1+t}} + t\frac{1}{2\,(1+t)^{3/2}} = \frac{\sqrt{1+t} - (1 + t/2)}{(1+t)\sqrt{1+t}}.$$

The numerator of this fraction is positive, the denominator negative, since $1 + t/2 \ge \sqrt{1+t}$ (Bernoulli's inequality). Thus, the righthand side of (*) is negative, $f(x) < 0$, which proves the claim.

(b) Let $x \in (1, \infty)$. We set $f(t) = \ln t$, for $t \in (1, x)$. Then we have $f'(t) = \frac{1}{t}$ for some $t \in (1, x)$ and with the mean value theorem we obtain

$$\ln x = \ln x - \ln 1 = f(x) - f(1) = f'(\xi)(x-1) = \frac{1}{\xi}(x-1)$$

for some $\xi \in (1, x)$. Since $1 < \xi < x$, we have $\frac{1}{x} < \frac{1}{\xi} < 1$ and from the last equation we obtain

$$\frac{x-1}{x} < \ln x < x - 1.$$

Since $a = 2\ln 3 - 3\ln 2 = \ln(3^2) - \ln(2^3) = \ln(\frac{9}{8})$, we have $\frac{1}{9} < \ln\frac{9}{8} < \frac{1}{8}$.

(c) A function $f\colon D \to \mathbb{R}$ is Lipschitz-continuous on D if there is a constant $L > 0$, s.t.

$$|f(x) - f(y)| \le L|x - y| \text{ für alle } x, y \in D.$$

Here we have $f(x) = \sqrt{1+x}$. From the mean value theorem we obtain $|f(x) - f(y)| = |f'(\xi)||x - y|$ for some $\xi \in (x, y)$. The derivative $f'(\xi) = \frac{1}{2\sqrt{1+\xi}}$ can be estimated on $(0, 3)$ by $f'(\xi) \le \frac{1}{2\sqrt{1+0}}$ and thus,

$$\left|\sqrt{1+x} - 1\right| = \frac{1}{2\sqrt{1+\xi}} \cdot |x - 0| \le \frac{1 \cdot |x|}{2\sqrt{1+0}} = \frac{x}{2},$$

i.e. $L = \frac{1}{2}$ is a Lipschitz constant.

Exercise 52:

(a) Calculate the following limits

$$\text{(i)} \quad \lim_{x\to 0}\left[\frac{1}{e^x-1}-\frac{1}{x}\right], \qquad \text{(ii)} \quad \lim_{x\to\infty}\frac{x^2e^x}{(e^x-1)^2}, \qquad \text{(iii)} \quad \lim_{x\to 0+}(e^x+3x)^{\frac{1}{x}}.$$

(b) Find a value $c \in \mathbb{R}$, such that the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x-1} - \frac{1}{\ln(x)}, & \text{if } x \neq 1, \\ c, & \text{if } x = 1. \end{cases}$$

is continuous.

Solution 52:

(a) We use l'Hôspital's rules for all parts. The existence of the limits is obtained by reading from right to left.

(i) We first find a common denominator, then we apply l'Hôspital's first rule:

$$\lim_{x\to 0}\left[\frac{1}{e^x-1}-\frac{1}{x}\right] = \lim_{x\to 0}\frac{x-e^x+1}{(e^x-1)x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{1-e^x}{e^x-1+xe^x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{-e^x}{e^x+e^x+xe^x} = \frac{-1}{2+0} = -\frac{1}{2}$$

(ii)

$$\lim_{x\to\infty}\frac{x^2e^x}{(e^x-1)^2} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{2xe^x+x^2e^x}{2(e^x-1)e^x} = \lim_{x\to\infty}\frac{2x+x^2}{2(e^x-1)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{2+2x}{2e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{2}{2e^x} = 0.$$

(iii) Here we additionally need continuity of exp.

$$\begin{aligned} \lim_{x\to 0+}(e^x+3x)^{\frac{1}{x}} &= \lim_{x\to 0+}\exp\left(\tfrac{1}{x}\ln(e^x+3x)\right) = \exp\left(\lim_{x\to 0+}\frac{\ln(e^x+3x)}{x}\right) \\ &\overset{\frac{0}{0}}{=} \exp\left(\lim_{x\to 0+}\frac{\frac{1}{e^x+3x}(e^x+3)}{1}\right) = \exp\left(\frac{1+3}{1}\right) = e^4 \end{aligned}$$

(b) With l'Hôspital's first rule we obtain

$$\lim_{x\to 1}\frac{1}{x-1}-\frac{1}{\ln x} = \lim_{x\to 1}\frac{\ln x-(x-1)}{(x-1)\ln x} \overset{\frac{0}{0}}{=} \lim_{x\to 1}\frac{\frac{1}{x}-1}{\ln x+1-\frac{1}{x}} \overset{\frac{0}{0}}{=} \lim_{x\to 1}\frac{-\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = -\frac{1}{2}.$$

Thus, the function is continuos for $c = -1/2$.

Exercise 53: Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin^2(x)$.
Find the Taylor polynomial of degree 5 for f at $x_0 = 0$. Use this Taylor polynomial to approximate $\sin^2(\frac{1}{10})$ and show that the error of approximation is smaller than 10^{-6}.

Solution 53:
We have

$$\begin{aligned} f(x) = \sin^2 x = \frac{1-\cos(2x)}{2}, \quad f'(x) = \sin 2x, \quad f''(x) = 2\cos 2x, \\ f'''(x) = -4\sin 2x, \quad f^{(4)}(x) = -8\cos 2x, \quad f^{(5)}(x) = 16\sin 2x, \quad f^{(6)}(x) = 32\cos 2x. \end{aligned}$$

Thus, the taylor polynomial of degree 5 is

$$p_5(x) = \sum_{k=0}^{5}\frac{f^{(k)}(0)}{k!}x^k = \frac{2}{2!}x^2 - \frac{8}{4!}x^4 = x^2 - \frac{1}{3}x^4$$

(here we have $p_5 = p_4$, thus, degree 4) and the remainder term R_5

$$R_5(x,0) = \frac{x^6}{6!}32\cos 2z,$$

with some $z \in [0, x]$. Thus, we have the estimate

$$|R_5(\tfrac{1}{10}, 0)| \leq 10^{-6} \cdot \frac{32}{720} < 10^{-6}.$$

The taylor formula is

$$\sin^2 x = p_5(x) + R_5(x)$$

and we can approximate $\sin^2(\frac{1}{10})$ by

$$\sin^2(\frac{1}{10}) = (\frac{1}{10})^2 - \frac{1}{3}(\frac{1}{10})^4 + R_5(\tfrac{1}{10}, 0) \approx (\frac{1}{10})^2 - \frac{1}{3}(\frac{1}{10})^4 = \frac{1}{100} - \frac{1}{30000} = \frac{299}{30\,000}.$$

Exercise 54: Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \cosh \frac{x}{2}.$$

Determine all derivatives $f^{(n)}$, $n = 0, 1, 2, \ldots$ of the function f and give the Taylor series for f with centre of expansion $x_0 = 0$. Where does the series converge?

Solution 54:
We calculate the derivatives for $k = 1, 2, 3$ and make the following claim:

$$f^{(k)}(x) = \begin{cases} \frac{1}{2^k} \sinh\left(\frac{x}{2}\right), & k \text{ odd}, \\ \frac{1}{2^k} \cosh\left(\frac{x}{2}\right), & k \text{ even}. \end{cases}$$

We prove the claim by induction:

- $k = 1$: $f'(x) = \frac{1}{2}\sinh(\frac{x}{2})$ ✓
- $k = 2$: $f''(x) = \frac{1}{4}\cosh(\frac{x}{2})$ ✓
- $k \to k + 1$:
  Case 1: k odd: $f^{(k+1)}(x) = \left(\frac{1}{2^k}\sinh(\frac{x}{2})\right)' = \frac{1}{2^{k+1}}\cosh(\frac{x}{2})$,
  Case 2: k even: $f^{(k+1)}(x) = \left(\frac{1}{2^k}\cosh(\frac{x}{2})\right)' = \frac{1}{2^{k+1}}\sinh(\frac{x}{2})$. □

We want to expand the Taylor series about the point $x_0 = 0$. Since $\sinh(0) = 0$ and $\cosh(0) = 1$, we have $f^{(2k+1)}(x) = 0$ and $f^{(2k)}(x) = \frac{1}{2^{2k}}$, so for the Taylor series about the expansion point $x_0 = 0$ we obtain

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!}(x - 0)^{2k} = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(2k)!}.$$

For the convergence we use the ratio test. With $a_k = \frac{x^{2k}}{2^{2k}(2k)!}$ we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{2k}(2k)!|x|^{2k+2}}{2^{2k+2}(2k+2)!|x|^{2k}} = \frac{|x|^2}{4(2k+2)(2k+1)} \to 0 \quad (k \to \infty).$$

Thus, the series converges on $\mathbb{R}$.

Exercise 55: Let the function $f : (-1, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2+5x+4}$.
Find the Taylor series for f at $x_0 = 0$ and compute $f^{(5)}(0)$.
Hint: $f(x) = \frac{1/3}{x+1} - \frac{1/3}{x+4}$.

Solution 55:
We start with a partial fraction decomposition:

$$\frac{1}{x^2 + 5x + 4} = \frac{1}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4} \Rightarrow A = \frac{1}{3},\ B = -\frac{1}{3}.$$

Now we find a power series expansion of $\frac{1}{x+1}$ about 0:

$$\frac{1}{x+1} = \sum_{k=0}^{\infty} a_k x^k \Rightarrow 1 = \sum_{k=0}^{\infty} a_k (x^{k+1} + x^k) \Rightarrow 1 = a_0 + \sum_{k=1}^{\infty} (a_{k-1} + a_k) x^k$$

Comparison of coefficients yields $a_0 = 1,\ a_k = -a_{k-1} \Rightarrow a_k = (-1)^k \Rightarrow \frac{1}{x+1} = \sum_{k=0}^{\infty} (-1)^k x^k$.

Now we expand $\frac{1}{x+4}$ as a power series about 0:

$$\frac{1}{x+4} = \sum_{k=0}^{\infty} b_k x^k \Rightarrow 1 = \sum_{k=0}^{\infty} b_k (x^{k+1} + 4x^k) \Rightarrow 1 = 4b_0 + \sum_{k=1}^{\infty} (b_{k-1} + 4b_k) x^k$$

Comparison of coefficients yields $b_0 = \frac{1}{4},\ b_k = -\frac{b_{k-1}}{4} \Rightarrow b_k = (-1)^k \frac{1}{4^{k+1}} \Rightarrow \frac{1}{x+4} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{4^{k+1}} x^k$.

In conclusion, we have

$$\frac{1}{x^2+5x+4} = \frac{1}{3}\frac{1}{x+1} - \frac{1}{3}\frac{1}{x+4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{3} \left(1 - \frac{1}{4^{k+1}}\right) x^k.$$

This is the Taylor series of f about 0. We obtain

$$\frac{f^{(5)}(0)}{5!} = \frac{(-1)^5}{3} \left(1 - \frac{1}{4^6}\right) = -\frac{1}{3} \left(1 - \frac{1}{4^6}\right) \Rightarrow f^{(5)}(0) = -5! \left(\frac{1}{3} \left(1 - \frac{1}{4^6}\right)\right) = -\frac{20475}{512}.$$