Exercise Sheet No. 12
– with solutions –

Exercise 56:
Use integration by parts in order to calculate the following indefinite integrals

$$\text{(a)} \quad \int \sin^2(x)\, dx, \qquad \text{(b)} \quad \int x^2 \cdot \ln(x)\, dx.$$

Use integration by parts in order to calculate the following definite integrals

$$\text{(c)} \quad \int_0^1 x \cdot \arctan(x)\, dx, \qquad \text{(d)} \quad \int_0^{\frac{\pi}{2}} \cos^4(x)\, dx.$$

Solution 56:

(a) We set $u = \sin(x)$ and $v' = \sin(x)$, i.e. $u' = \cos(x)$ and $v = -\cos(x)$. Then we have

$$\begin{aligned}\int \sin^2(x)\,\mathrm{d}x &= -\sin(x)\cos(x) + \int \cos^2(x)\,\mathrm{d}x = -\sin(x)\cos(x) + \int (1-\sin^2(x))\,\mathrm{d}x \\ &= -\sin(x)\cos(x) + x - \int \sin^2(x)\,\mathrm{d}x.\end{aligned}$$

Using again integration by parts on $\int \cos^2(x)\,\mathrm{d}x$ does *not* work. Instead, we have to use the identity $1 = \sin^2(x) + \cos^2(x)$. Adding $\sin^2(x)$ on both sides of the equation we obtain the solution

$$\int \sin^2(x)\mathrm{d}x = \frac{x}{2} - \frac{1}{2}\sin(x)\cos(x) + C.$$

(b) We set $u = \ln(x)$ and $v' = x^2$, i.e. $u' = \frac{1}{x}$ and $v = \frac{x^3}{3}$. Then we have

$$\int x^2 \ln(x)\,\mathrm{d}x = \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}\,\mathrm{d}x = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C.$$

Here taking the derivative of the logarithm gives the solution.

(c) We set

$$u(t) = \arctan(t), \qquad u'(t) = \frac{1}{1+t^2}$$

and

$$v'(t) = t, \qquad v(t) = \frac{t^2+1}{2}.$$

Then we have

$$\begin{aligned}\int_0^1 t \cdot \arctan(t)\,\mathrm{d}t &= \left[\arctan(t) \cdot \frac{t^2+1}{2}\right]_0^1 - \int_0^1 \frac{1}{2}\,\mathrm{d}t \\ &= \arctan(1) - 1/2 = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

Alternative: If we set $v(t) = \frac{1}{2}t^2$, we obtain (with $\frac{t^2}{1+t^2} = \frac{1+t^2-1}{1+t^2} = 1 - \frac{1}{1+t^2}$):

$$\begin{aligned}\int_0^1 t \cdot \arctan(t)\,\mathrm{d}t &= \left[\frac{1}{2}t^2 \cdot \arctan(t)\right]_0^1 - \frac{1}{2}\int_0^1 \frac{t^2}{1+t^2}\,\mathrm{d}t \\ &= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left[1 - \frac{1}{1+t^2}\right]\mathrm{d}t = \frac{\pi}{8} - \frac{1}{2}t\Big|_0^1 + \frac{1}{2}\arctan(t)\Big|_0^1 \\ &= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

(d) Integration by parts and $\sin^2(t) + \cos^2(t) = 1$ yields

$$\int_0^{\frac{\pi}{2}} \cos^3(t) \cdot \cos(t)\,\mathrm{d}t = \underbrace{\cos^3(t)\sin(t)\Big|_0^{\frac{\pi}{2}}}_{0} + 3\int_0^{\frac{\pi}{2}} \cos^2(t)\sin^2(t)\ \mathrm{d}t = 3\int_0^{\frac{\pi}{2}} \cos^2(t)\ \mathrm{d}t - 3\int_0^{\frac{\pi}{2}} \cos^4(t)\ \mathrm{d}t,$$

and rearraning gives $4\int_0^{\frac{\pi}{2}} \cos^4(t)\,\mathrm{d}t = 3\int_0^{\frac{\pi}{2}} \cos^2(t)\,\mathrm{d}t$. We further have

$$\int_0^{\frac{\pi}{2}} \cos^2(t)\,\mathrm{d}t = \underbrace{\cos(t)\sin(t)|_0^{\frac{\pi}{2}}}_{0} + \int_0^{\frac{\pi}{2}} \sin^2(t)\,\mathrm{d}t$$

In part (a) we determined an anti-derivative of $\sin^2$, which we use to calculate the last integral: We have $\int_0^{\frac{\pi}{2}} \sin^2(t)\,\mathrm{d}t = \frac{\pi}{4}$. Combining everything we obtain

$$\int_0^{\frac{\pi}{2}} \cos^4(t)\,\mathrm{d}t = \frac{3}{4}\int_0^{\frac{\pi}{2}} \cos^2(t)\,\mathrm{d}t = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2(t)\,\mathrm{d}t = \frac{3\pi}{16}.$$

Exercise 57:
Use integration by substitution to calculate the following anti-derivatives

(a) $\displaystyle\int \frac{1}{x \ln x}\, dx$ on $(1, \infty)$, (b) $\displaystyle\int \frac{x}{\sqrt{x^2 - 1}}\, dx$ on $(1, \infty)$.

Calculate the following integral using first substitution and then integration by parts

(c) $\displaystyle\int_1^4 \arctan\sqrt{\sqrt{\mathrm{x}} - 1}\,\mathrm{dx}.$

Solution 57:

(a) Consider the integral $\int \frac{1}{x \ln x}\mathrm{d}x = \int \frac{1}{x} \cdot \frac{1}{\ln x}\mathrm{d}x$. Note that the fraction $\frac{1}{\ln x}$ gets multiplied by the derivative of the numerator (namely $\frac{1}{x}$). We substitute $u = \ln x$. Then we have $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and

$$\int \frac{1}{x \ln x}\mathrm{d}x = \int \frac{1}{\ln(x)} \cdot \frac{1}{x}\mathrm{d}x = \int \frac{1}{u}\mathrm{d}u = \ln u + C = \ln(\ln(x)) + C.$$

!! Resubstitution in the last step !!

(b) We substitute $u = x^2 - 1$. (attention: The substitution $x = \sin u$ doe snot work here, since x is not in $(-1, 1)$, but in $(1, \infty)$.) Thus, we have

$$\int \frac{x}{\sqrt{x^2 - 1}}\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{u}}\mathrm{d}u = \frac{1}{2}\int u^{-1/2} = u^{1/2} + C = \sqrt{x^2 - 1} + C.$$

(c) The substitution $u(x) = \sqrt{\sqrt{x} - 1} \Leftrightarrow (u^2 + 1)^2 = x$ gives $\mathrm{d}x = 4u(u^2 + 1)\ \mathrm{d}u$ and we obtain

$$\int_1^4 \arctan\sqrt{\sqrt{x} - 1}\ \mathrm{d}x = \int_0^1 4u(u^2 + 1)\arctan u\ \mathrm{d}u.$$

Integration by parts with $f'(u) = 4u(u^2+1)$, $f(u) = (u^2+1)^2$ und $g(u) = \arctan u$, $g'(u) = \frac{1}{1+u^2}$ yields

$$[(u^2 + 1)^2 \arctan u]_0^1 - \int_0^1 (u^2 + 1)\ \mathrm{d}u = 4\arctan 1 - [\frac{1}{3}u^3 + u]_0^1 = 4 \cdot \frac{\pi}{4} - \frac{1}{3} - 1 = \pi - \frac{4}{3}.$$

Exercise 58:
Find the derivative F' of the following function:

$$F : [0,1] \to \mathbb{R}, \quad F(x) = \int\limits_{\ln(x)}^{x^2} \sin(\cos(t))\, dt.$$

Solution 58:
We can write F as:

$$F(x) = \int\limits_{\ln x}^{0} \sin(\cos t)\, \mathrm{d}t + \int\limits_{0}^{x^2} \sin(\cos t)\, \mathrm{d}t = -\int\limits_{0}^{\ln x} \sin(\cos t)\, \mathrm{d}t + \int\limits_{0}^{x^2} \sin(\cos t)\, \mathrm{d}t.$$

We define $G\colon [0,1] \to \mathbb{R}$ by:

$$G(x) := \int\limits_{0}^{x} \sin(\cos t)\, \mathrm{d}t.$$

Thus, $F(x) = -G(\ln x) + G(x^2)$.
By the first fundamental theorem of differential and integral calculus (Thm. 8.11 lecture notes) we have

$$G'(x) = \sin(\cos x).$$

Now, by the chain rule we have

$$F'(x) = -G'(\ln x)\frac{1}{x} + G'(x^2)2x = -\frac{\sin(\cos(\ln x))}{x} + 2x\sin(\cos(x^2)).$$

Exercise 59:
Solve the following initial value problem

$$y'(x) - y(x) - 2xe^x = 0, \quad y(0) = 1.$$

Solution 59: We have a first order linear differential equation. Using the solution method on page 109, we first determine the integrating factor μ:

$$\mu(x) = \exp(\int -1 dx) = e^{-x}.$$

Multiplying our equation with μ, we obtain

$$\begin{aligned} (\mu(x)y(x))' &= (2xe^x)\mu(x) \\ \Leftrightarrow e^{-x}y(x) &= \int 2xe^x e^{-x} = x^2 + C. \end{aligned}$$

Thus, the solution is $y = (x^2 + C)e^x$. For the initial value $y(0) = 1$ we obtain

$$y(0) = (0 + C)e^0 = C \stackrel{!}{=} 1,$$

i.e.

$$y(x) = (x^2 + 1)e^x, \qquad x \in \mathbb{R}.$$

Exercise 60:
Solve the initial value problem

$$y^3(x) - x^2 + xy^2(x)y'(x) = 0, \qquad y(1) = 1.$$

Hint: Link this problem to Bernoulli-type ODEs.

Solution 60: We multiply the equation by $\frac{y^{-2}}{x}$ and obtain a Bernoulli differential equation

$$y' + \frac{1}{x} = y^{-2}x.$$

with $\alpha = -2$. Substituting $u(x) = (y(x))^{1-\alpha} = (y(x))^3$ and thus, $y(x) = u(x)^{1/3}$, we obtain the linear differential equation

$$u' + (1-\alpha)\frac{1}{x} = (1-\alpha)x,$$

i.e.

$$u' + \frac{3}{x}u = 3x.$$

We solve this by first calculating the integrating factor:

$$\mu(x) = \exp(\int \frac{3}{x}dx) = \exp(3\ln x) = x^3.$$

Multiplying the differential equation with μ, we obtain $(\mu u)' = 3x\mu$, i.e., by integrating both sides,

$$\mu u = \int 3x^4 = \frac{3}{5}x^5 + C,$$

i.e. $u(x) = \frac{3}{5}x^2 + Cx^{-3}$ for some $C \in \mathbb{R}$. Resubstituting $y(x) = u(x)^{-3}$, we have

$$y(x) = (\frac{3}{5}x^2 + Cx^{-3})^{1/3}.$$

With our initial value condition we obtain

$$y(1) = (\frac{3}{5} + C)^{-3} \overset{!}{=} 1,$$

which gives the solution $C = \frac{2}{5}$. Thus, we have the solution

$$y(x) = \left(\frac{3}{5}x^2 + \frac{2}{5x^3}\right)^{1/3}.$$