Exercise Sheet No. 2
– with solutions –

Exercise 1:
Solve the following inequalities for x ($x \in \mathbb{R}$):

(a) $(x-5)^3(x+1) \geq 0,$ (b) $\dfrac{(x+1)(3-x)}{(x+5)^2} \leq 0,$ (c) $|x| = x^3 + 2x^2 - 3x.$

Solution 1:
a) $(x-5)^3(x+1) \geq 0 \Leftrightarrow ((x-5)^3 \geq 0,\ x+1 \geq 0)$ or $((x-5)^3 \leq 0,\ x+1 \leq 0) \Leftrightarrow (x \geq 5,\ x \geq -1)$ or $(x \leq 5,\ x \leq -1) \Leftrightarrow x \geq 5$ oder $x \leq -1$

b) The denominator gives the condition $x \neq -5$. Numerator > 0, thus $(x+1)(3-x) \leq 0$ suffices. Thus, $(x+1 \leq 0,\ 3-x \geq 0)$ or $(x+1 \geq 0,\ 3-x \leq 0) \Leftrightarrow (x \leq -1,\ x \leq 3)$ or $(x \geq -1,\ x \geq 3) \Leftrightarrow x \leq -1$ or $x \geq 3$.
Set of solutions: $\mathbb{L} = (-\infty, -1] \cup [3, \infty) \backslash \{-5\}$.

c) $x \geq 0$: $|x| = x^3 + 2x^2 - 3x \Leftrightarrow 0 = x^3 + 2x^2 - 4x = x(x^2 + 2x - 4) = x((x+1)^2 - 5) \Leftrightarrow x = 0$ or $(x+1)^2 = 5 \Leftrightarrow x = 0$ or $|x+1| = \sqrt{5} \Leftrightarrow x = 0$ or $x = -1 + \sqrt{5}$ or $x = -1 - \sqrt{5}$.
Since $x \geq 0$, we have $x = 0$ or $x = -1 + \sqrt{5}$.
$x < 0$: $|x| = x^3 + 2x^2 - 3x \Leftrightarrow 0 = x^3 + 2x^2 - 2x = x(x^2 + 2x - 2) = x((x+1)^2 - 3) \Leftrightarrow x = 0$ or $(x+1)^2 = 3 \Leftrightarrow x = 0$ or $|x+1| = \sqrt{3} \Leftrightarrow x = 0$ or $x = -1 + \sqrt{3}$ or $x = -1 - \sqrt{3}$.
Since $x < 0$, we have $x = -1 - \sqrt{3}$. Thus, the set of solutions is $\mathbb{L} = \{0, -1 + \sqrt{5}, -1 - \sqrt{3}\}$.

Exercise 2:
Prove the following identities:

(a) $\begin{pmatrix} n \\ r \end{pmatrix} = \dfrac{n}{r} \cdot \begin{pmatrix} n-1 \\ r-1 \end{pmatrix}$, $n \geq r \geq 1$;

(b) $\begin{pmatrix} n \\ m \end{pmatrix} \cdot \begin{pmatrix} m \\ r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} \cdot \begin{pmatrix} n-r \\ m-r \end{pmatrix}$, $n \geq m \geq r \geq 0$.

Solution 2:

(a) Applying the definition of the binomial coefficient we obtain:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n}{r} \cdot \frac{(n-1)!}{(r-1)!\,([n-1]-[r-1])!} = \frac{n}{r} \cdot \binom{n-1}{r-1}.$$

(b) Applying the defintion and simplifying we obtain:

$$\binom{n}{m} \cdot \binom{m}{r} = \frac{n!}{m!\,(n-m)!} \cdot \frac{m!}{r!\,(m-r)!} = \frac{n!}{r!\,(n-m)!\,(m-r)!}$$
$$= \frac{n!}{r!\,(n-r)!} \cdot \frac{(n-r)!}{([n-r]-[m-r])!\,(m-r)!} = \binom{n}{r} \cdot \binom{n-r}{m-r}.$$

Exercise 3: Prove by induction on $n \in \mathbb{N}$ that

(a) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n},$ (b) $\displaystyle\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}.$

Solution 3:

(a) *Base* $n = 1$:

$$\sum_{k=1}^{1} \frac{1}{k^2} = 1 \leq 1 = 2 - \frac{1}{1}.$$

Thus, the claim holds for $n = 1$.

Step $n \curvearrowright n+1$: Assume that $\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$ holds for some $n \in \mathbb{N}$ *(induction hypothesis)*. We want to show the claim for $n+1$, i.e.

$$\sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}.$$

We split the sum so we can apply our induction hypothesis:

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$

Then we find a common denominator for the expression on the righthand side and expand it:

$$\sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 + \frac{n-(n+1)^2}{n(n+1)^2} = 2 - \frac{n^2+n+1}{n(n+1)^2}$$

If we look at the fraction on the righthand side and compare it to the claim for $n+1$, we see that we only need to get rid of the 1 in the numerator. We have:

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{k^2} &\leq 2 - \frac{n^2+n+1}{n(n+1)^2} = 2 - \frac{n(n+1)+1}{n(n+1)^2} = 2 - \frac{n(n+1)}{n(n+1)^2} - \frac{1}{n(n+1)^2} \\ &\leq 2 - \frac{n(n+1)}{n(n+1)^2} = 2 - \frac{1}{n+1}.\end{aligned}$$

We can do this, since $-\frac{1}{n(n+1)^2} \leq 0$.

(b) *Base* $n = 1$:

$$\sum_{k=1}^{1} \frac{1}{(3k-2)(3k+1)} = \frac{1}{1 \cdot 4} = \frac{1}{3+1}$$

Thus, the claim holds for $n = 1$.

Step $n \curvearrowright n+1$: Assume that $\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}$ holds for some $n \in \mathbb{N}$ *(induction hypothesis)*. We show that it then holds for $n+1$, i.e. that

$$\sum_{k=1}^{n+1} \frac{1}{(3k-2)(3k+1)} = \frac{n+1}{3(n+1)+1} = \frac{n+1}{3n+4}.$$

We start by splitting the sum by separating the $(n+1)$st summand, in order to apply our induction hypothesis. Then finding a common denominator and simplifying the expression gives the desired identity:

$$\begin{aligned}\sum_{k=1}^{n+1} \frac{1}{(3k-2)(3k+1)} &= \sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(n+1)-2)(3(n+1)+1)} \\ &= \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} \\ &= \frac{n(3n+4)+1}{(3n+1)(3n+4)} = \frac{3n^2+4n+1}{(3n+1)(3n+4)} \\ &= \frac{(3n+1)(n+1)}{(3n+1)(3n+4)} = \frac{n+1}{3n+4}\end{aligned}$$

Exercise 4: Consider the following identity

$$\sum_{k=1}^{n} \frac{(2k)! - 2\big((2k-2)!\big)}{2^k} = \frac{(2n)!}{2^n} - 1.$$

(a) Prove this identity for $n \in \mathbb{N}$ by induction.

(b) Prove this identity for $n \in \mathbb{N}$ by shifting the index.

Solution 4:

(a) *Base:* for $n = 1$ we have

$$\sum_{k=1}^{1} \frac{(2k)! - 2\,(2k-2)!}{2^k} = \frac{2-2}{2} = 0 = \frac{2!}{2} - 1.$$

Step: Assume the claim holds for some $n \in \mathbb{N}$. Then for $n+1$ we have

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{(2k)! - 2(2k-2)!}{2^k} &= \sum_{k=1}^{n} \frac{(2k)! - 2(2k-2)!}{2^k} + \frac{(2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}} \\
&= \frac{(2n)!}{2^n} - 1 + \frac{(2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}} \\
&= \frac{2(2n)! + (2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}} - 1 \\
&= \frac{2(2n)! + (2(n+1))! - 2(2n)!}{2^{n+1}} - 1 \\
&= \frac{(2(n+1))!}{2^{n+1}} - 1,
\end{aligned}$$

where we first split the sum, then applied the induction hypothesis in the 2nd line and then brought the expression to the desired form.

(b)

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(2k)! - 2\big[(2k-2)!\big]}{2^k} &= \sum_{k=1}^{n} \frac{(2k)!}{2^k} - \sum_{k=1}^{n} \frac{(2k-2)!}{2^{k-1}} \\
&= \sum_{k=1}^{n} \frac{(2k)!}{2^k} - \sum_{k=0}^{n-1} \frac{(2(k+1)-2)!}{2^{(k+1)-1}} \\
&= \sum_{k=1}^{n} \frac{(2k)!}{2^k} - \sum_{k=0}^{n-1} \frac{(2k)!}{2^k} \\
&= \sum_{k=1}^{n-1} \frac{(2k)!}{2^k} + \frac{(2n)!}{2^n} - \frac{(2\cdot 0)!}{2^0} - \sum_{k=1}^{n-1} \frac{(2k)!}{2^k} \\
&= \frac{(2n)!}{2^n} - 1
\end{aligned}$$

Exercise 5: Prove by induction on $n \in \mathbb{N}$ that

(a) $2^n > 2n + 1, \text{ for } n \geq 3$ (b) $2^n \geq n^2, \text{ for } n \geq 4$ (c) $\sum_{l=0}^{n} \binom{n}{l} = 2^n.$

Solution 5:

a) *Base* for $n = 3$: $2^3 = 8 > 7 = 2 \cdot 3 + 1$

Step $n \curvearrowright n+1$: Assume that $2^n > 2n + 1$ holds for some $n \in \mathbb{N}_{\geq 3}$ *(induction hypothesis)*. We want to show that it holds for $n+1$:

$$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n \overset{\text{I.H.}}{>} (2n+1) + (2n+1) > (2n+1) + 2 = 2n + 3 = 2(n+1) + 1.$$

b) *Base* for $n = 4$: $2^4 = 16 \geq 16 = 4^2$.

Step $n \curvearrowright n+1$: Assume $2^n \geq n^2$ holds for some $n \in \mathbb{N}_{\geq 4}$ *(induction hypothesis)*. Then we have

$$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n \overset{\text{I.V.}}{\geq} n^2 + 2^n \overset{\text{(a)}}{\geq} n^2 + 2n + 1 = (n+1)^2.$$

c) *Base:* $\sum_{l=0}^{1} \binom{1}{l} = \frac{1!}{0!\,1!} + \frac{1!}{1!\,0!} = 1 + 1 = 2^1.$

Step $n \curvearrowright n+1$: Assume $\sum_{l=0}^{n} \binom{n}{l} = 2^n$ holds for some $n \in \mathbb{N}$ *(Induktionsvoraussetzung)*.

Rearranging yields:

$$\begin{aligned}\sum_{l=0}^{n+1} \binom{n+1}{l} &= \binom{n+1}{0} + \sum_{l=1}^{n} \binom{n+1}{l} + \binom{n+1}{n+1} = \binom{n}{0} + \sum_{l=1}^{n} \left[\binom{n}{l-1} + \binom{n}{l} \right] + \binom{n}{n} \\ &= \binom{n}{0} + \sum_{l=0}^{n-1} \binom{n}{l} + \sum_{l=1}^{n} \binom{n}{l} + \binom{n}{n} = 2 \sum_{l=0}^{n} \binom{n}{l}.\end{aligned}$$

Thus, by the induction hypothesis we have $\sum_{l=0}^{n+1} \binom{n+1}{l} = 2^{n+1}$.