Exercise 1:
Solve the following inequalities for \( x \in \mathbb{R} \):

(a) \((x - 5)^3(x + 1) \geq 0\), \hspace{1cm} (b) \(\frac{(x + 1)(3 - x)}{(x + 5)^2} \leq 0\), \hspace{1cm} (c) \(|x| = x^3 + 2x^2 - 3x\).

Solution 1:
(a) \((x - 5)^3(x + 1) \geq 0 \Leftrightarrow ((x - 5)^3 \geq 0, x + 1 \geq 0) \text{ or } ((x - 5)^3 \leq 0, x + 1 \leq 0) \Leftrightarrow (x \geq 5, x \geq -1) \text{ or } (x \leq 5, x \leq -1) \Leftrightarrow x \geq 5 \text{ oder } x \leq -1\).

(b) The denominator gives the condition \( x \neq -5 \). Numerator > 0, thus \((x + 1)(3 - x) \leq 0\) suffices. Thus, \((x + 1 \leq 0, 3 - x \geq 0) \text{ or } (x + 1 \geq 0, 3 - x \leq 0) \Leftrightarrow (x \leq -1, x \leq 3) \text{ or } (x \geq -1, x \geq 3) \Leftrightarrow x \leq -1 \text{ or } x \geq 3\).

Set of solutions: \( \mathbb{L} = (-\infty, -1] \cup [3, \infty) \setminus \{-5\} \).

(c) \(x \geq 0 : \left|x\right| = x^3 + 2x^2 - 3x \Leftrightarrow 0 = x^3 + 2x^2 - 4x = x(x^2 + 2x - 4) = (x + 1)^2 - 5 \Leftrightarrow x = 0 \text{ or } (x + 1)^2 = 5 \Leftrightarrow x = 0 \text{ or } |x + 1| = \sqrt{5} \Leftrightarrow x = 0 \text{ or } x = -1 + \sqrt{5} \text{ or } x = -1 - \sqrt{5} \).

Since \( x \geq 0 \), we have \( x = 0 \) or \( x = -1 + \sqrt{5} \).

\( x < 0 : \left|x\right| = x^3 + 2x^2 - 3x \Leftrightarrow 0 = x^3 + 2x^2 - 2x = x(x^2 + 2x - 2) = x((x + 1)^2 - 3) \Leftrightarrow x = 0 \text{ or } (x + 1)^2 = 3 \Leftrightarrow x = 0 \text{ or } x = -1 + \sqrt{3} \text{ or } x = -1 - \sqrt{3} \).

Since \( x < 0 \), we have \( x = -1 - \sqrt{3} \). Thus, the set of solutions is \( \mathbb{L} = \{0, -1 + \sqrt{5}, -1 - \sqrt{3}\} \).

Exercise 2:
Prove the following identities:

(a) \(\binom{n}{r} = \frac{n}{r} \cdot \binom{n - 1}{r - 1}, \quad n \geq r \geq 1 \);

(b) \(\binom{n}{m} \cdot \binom{m}{r} = \binom{n}{r} \cdot \binom{n - r}{m - r}, \quad n \geq m \geq r \geq 0 \).

Solution 2:
(a) Applying the definition of the binomial coefficient we obtain:

\[ \binom{n}{r} = \frac{n!}{r!(n - r)!} = \frac{n}{r} \cdot \frac{(n - 1)!}{(r - 1)!(n - 1 - (r - 1))!} = \frac{n}{r} \cdot \binom{n - 1}{r - 1}. \]

(b) Applying the definition and simplifying we obtain:

\[ \binom{n}{m} \cdot \binom{m}{r} = \frac{n!}{m!(n - m)!} \cdot \frac{m!}{r!(m - r)!} = \frac{n}{r} \cdot \frac{(n - m)!}{(n - m - (r - 1))!} \cdot \frac{(m - r)!}{(m - r - (m - r))!} = \frac{n}{r} \cdot \binom{n - r}{m - r}. \]

Exercise 3: Prove by induction on \( n \in \mathbb{N} \) that

(a) \(\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}\), \hspace{1cm} (b) \(\sum_{k=1}^{n} \frac{1}{(3k - 2)(3k + 1)} = \frac{n}{3n + 1}\).

Solution 3:
(a) Base $n = 1$:

$$\sum_{k=1}^{1} \frac{1}{k^2} = 1 \leq 1 = 2 - \frac{1}{1}.$$ 

Thus, the claim holds for $n = 1$.

Step $n \mapsto n+1$: Assume that $\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$ holds for some $n \in \mathbb{N}$ \textit{(induction hypothesis)}. We want to show the claim for $n+1$, i.e.

$$\sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}.$$ 

We split the sum so we can apply our induction hypothesis:

$$\sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}.$$ 

Then we find a common denominator for the expression on the righthand side and expand it:

$$\sum_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{n^2 + n + 1}{n(n+1)^2} = 2 - \frac{n(n+1)}{n(n+1)^2} = \frac{2}{n(n+1)^2} - \frac{1}{n+1}.$$ 

We can do this, since $-\frac{1}{n(n+1)^2} \leq 0$.

(b) Base $n = 1$:

$$\sum_{k=1}^{1} \frac{1}{(3k)^2(3k+1)} = \frac{1}{1 \cdot 4} = \frac{1}{3 + 1}.$$ 

Thus, the claim holds for $n = 1$.

Step $n \mapsto n+1$: Assume that $\sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} = \frac{n}{3n+1}$ holds for some $n \in \mathbb{N}$ \textit{(induction hypothesis)}. We show that it then holds for $n+1$, i.e. that

$$\sum_{k=1}^{n+1} \frac{1}{(3k-2)(3k+1)} = \frac{n+1}{3(n+1)+1} = \frac{n+1}{3n+4}.$$ 

We start by splitting the sum by separating the $(n+1)$st summand, in order to apply our induction hypothesis. Then finding a common denominator and simplifying the expression gives the desired identity:

$$\sum_{k=1}^{n+1} \frac{1}{(3k-2)(3k+1)} \equiv \sum_{k=1}^{n} \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3(n+1)-2)(3(n+1)+1)}$$

$$= \sum_{k=1}^{n} \frac{1}{(3k+1)(3n+4)} + \frac{1}{3(n+1)+1}$$

$$= \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} + \frac{1}{(3n+1)(3n+4)}$$

$$= \frac{n}{(3n+1)(3n+4)} + \frac{1}{(3n+1)(3n+4)}$$

$$= \frac{n+1}{3n+4}.$$ 

Exercise 4: Consider the following identity

$$\sum_{k=1}^{n} \frac{(2k)! - 2(2k-2)!}{2^k} = \frac{(2n)!}{2^n} - 1.$$
(a) Prove this identity for \( n \in \mathbb{N} \) by induction.

(b) Prove this identity for \( n \in \mathbb{N} \) by shifting the index.

**Solution 4:**

(a) \textit{Base:} for \( n = 1 \) we have

\[
\sum_{k=1}^{1} \frac{(2k)! - 2(2k-2)!}{2^k} = \frac{2 - 2}{2} = 0 = \frac{2!}{2} - 1.
\]

\textit{Step:} Assume the claim holds for some \( n \in \mathbb{N} \). Then for \( n + 1 \) we have

\[
\sum_{k=1}^{n+1} \frac{(2k)! - 2(2k-2)!}{2^k} = \sum_{k=1}^{n} \frac{(2k)! - 2(2k-2)!}{2^k} + \frac{(2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}}
\]

\[
= \frac{(2n)!}{2^n} - 1 + \frac{(2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}}
\]

\[
= \frac{2(2n)! + (2(n+1))! - 2(2(n+1)-2)!}{2^{n+1}} - 1
\]

\[
= \frac{(2(n+1))!}{2^{n+1}} - 1,
\]

where we first split the sum, then applied the induction hypothesis in the 2nd line and then brought the expression to the desired form.

(b)

\[
\sum_{k=1}^{n} \frac{(2k)! - 2(2k-2)!}{2^k} = \sum_{k=1}^{n} \frac{(2k)!}{2^k} - \sum_{k=1}^{n} \frac{(2k-2)!}{2^{k-1}}
\]

\[
= \sum_{k=1}^{n} \frac{(2k)!}{2^k} - \sum_{k=0}^{n-1} \frac{(2(k+1)-2)!}{2^{(k+1)-1}}
\]

\[
= \sum_{k=1}^{n} \frac{(2k)!}{2^k} - \sum_{k=0}^{n-1} \frac{(2k)!}{2^k}
\]

\[
= \sum_{k=1}^{n-1} \frac{(2k)!}{2^k} + \frac{(2n)!}{2^n} - \frac{(2\cdot 0)!}{2^0} - \sum_{k=1}^{n-1} \frac{(2k)!}{2^k}
\]

\[
= \frac{(2n)!}{2^n} - 1
\]

**Exercise 5:** Prove by induction on \( n \in \mathbb{N} \) that

(a) \( 2^n > 2n + 1 \), for \( n \geq 3 \) \hspace{1cm} (b) \( 2^n \geq n^2 \), for \( n \geq 4 \) \hspace{1cm} (c) \( \sum_{l=0}^{n} \left( \begin{array}{c} n \\end{array} \right) \right) = 2^n \).

**Solution 5:**

\textit{a) Base} for \( n = 3: 2^3 = 8 > 7 = 2 \cdot 3 + 1 \)

\textit{Step} \( n \to n+1: \) Assume that \( 2^n > 2n + 1 \) holds for some \( n \in \mathbb{N}_{\geq 3} \) (\textit{induction hypothesis}). We want to show that it holds for \( n + 1: \)

\[
2^{n+1} = 2 \cdot 2^n = 2^n + 2^n > (2n + 1) + (2n + 1) > (2n + 1) + 2 = 2n + 3 = 2(n+1) + 1.
\]

\textit{b) Base} for \( n = 4: 2^4 = 16 \geq 16 = 4^2 \).

\textit{Step} \( n \to n+1: \) Assume \( 2^n \geq n^2 \) holds for some \( n \in \mathbb{N}_{\geq 4} \) (\textit{induction hypothesis}). Then we have

\[
2^{n+1} = 2 \cdot 2^n = 2^n + 2^n \geq n^2 + 2^n \geq n^2 + 2n + 1 = (n+1)^2.
\]
c) Base: \( \sum_{l=0}^{1} \binom{1}{l} = \frac{1!}{0!1!} + \frac{1!}{1!0!} = 1 + 1 = 2^1. \)

Step \( n \rightarrow n + 1 \): Assume \( \sum_{l=0}^{n} \binom{n}{l} = 2^n \) holds for some \( n \in \mathbb{N} \) (Induktionsvoraussetzung).

Rearranging yields:

\[
\sum_{l=0}^{n+1} \binom{n+1}{l} = \binom{n+1}{0} + \sum_{l=1}^{n} \left( \binom{n+1}{l} + \binom{n+1}{l} \right) = \binom{n}{0} + \sum_{l=1}^{n} \left[ \binom{n}{l-1} + \binom{n}{l} \right] + \binom{n}{n}
\]

\[
= \binom{n}{0} + \sum_{l=0}^{n-1} \binom{n}{l} + \sum_{l=1}^{n} \binom{n}{l} + \binom{n}{n} = 2 \sum_{l=0}^{n} \binom{n}{l}. 
\]

Thus, by the induction hypothesis we have \( \sum_{l=0}^{n+1} \binom{n+1}{l} = 2^{n+1}. \)