Exercise Sheet No. 3
– with solutions –

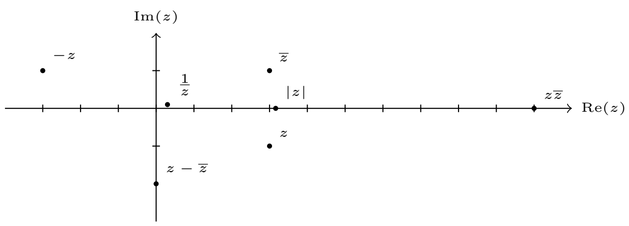
Exercise 11: Given the complex numbers

$$\text{(i) } z = 3 - \mathrm{i}, \qquad \text{(ii) } z = 3 + 4\mathrm{i},$$

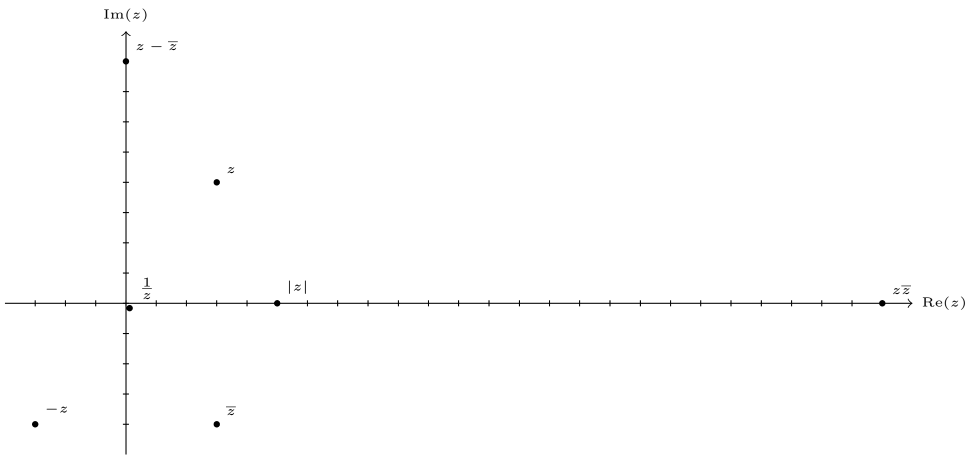
resolve each of the numbers $\overline{z}$, $-z$, $z\overline{z}$, $\frac{1}{z}$, $z - \overline{z}$, $|z|$ into their real and imaginary parts and plot them in the complex plane.

Solution 11:
The solutions are

z	$\overline{z}$	$-z$	$z\overline{z} = \|z\|^2$	$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{\|z\|^2}$	$z - \overline{z} = 2\mathrm{i}\mathrm{Im}(z)$	$\|z\| = \sqrt{z\overline{z}}$
$3 - \mathrm{i}$	$3 + \mathrm{i}$	$-3 + \mathrm{i}$	$3^2 + 1^2 = 10$	$\frac{3+\mathrm{i}}{10} = \frac{3}{10} + \frac{\mathrm{i}}{10}$	$-2\mathrm{i}$	$\sqrt{10}$
$3 + 4\mathrm{i}$	$3 - 4\mathrm{i}$	$-3 - 4\mathrm{i}$	$3^2 + 4^2 = 25$	$\frac{3-4\mathrm{i}}{25} = \frac{3}{25} - \mathrm{i}\frac{4}{25}$	$8\mathrm{i}$	$\sqrt{25} = 5$



(i)



(ii)

Exercise 12:

(a) Compute a polar coordinate representation $r(\cos(\varphi) + \mathrm{i}\sin(\varphi))$ for each of the following numbers:

$$\text{(i) } \mathrm{i}, \qquad \text{(ii) } -1 + \mathrm{i}, \qquad \text{(iii) } \frac{1 - \mathrm{i}}{\mathrm{i} + 2}.$$

(b) Let $z, w \in \mathbb{C}$. Compute $\mathrm{Re}(z\overline{w})$ by

(i) using Cartesian coordinates: $z = x + \mathrm{i}y$, $w = a + ib$ with $x, y, a, b \in \mathbb{R}$.

(ii) using polar coordinates: $z = r(\cos(\varphi) + \mathrm{i}\sin(\varphi))$, $w = q(\cos(\psi) + \mathrm{i}\sin(\psi))$ with $r, q \geq 0$, $\varphi, \psi \in (-\pi, \pi]$.

Solution 12:

(a) (i) Since $|\mathrm{i}| = 1$ and $\mathrm{Arg}(\mathrm{i}) = \frac{\pi}{2}$ we have $\mathrm{i} = \cos(\frac{\pi}{2}) + \mathrm{i}\sin(\frac{\pi}{2})$.

(ii) We have $|-1+\mathrm{i}| = \sqrt{2}$ and $\mathrm{Arg}(-1+\mathrm{i}) = \frac{3\pi}{4}$, thus, $-1+\mathrm{i} = \sqrt{2}(\cos(\frac{3\pi}{4}) + \mathrm{i}\sin(\frac{3\pi}{4}))$.

(iii) We simplify

$$\frac{1-\mathrm{i}}{\mathrm{i}+2} = \frac{(1-\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})} = \frac{2-\mathrm{i}-2\mathrm{i}-1}{5} = \frac{1-3\mathrm{i}}{5}.$$

The absolute value is $|\frac{1-3\mathrm{i}}{5}| = \frac{\sqrt{10}}{5}$ and according to the lecture notes, the principal argument is $\mathrm{Arg}(\frac{1-3\mathrm{i}}{5}) = \arctan(\frac{-3/5}{1/5}) = \arctan(-3)$. That corresponds to approximately $-71°$, i.e.

$$\frac{1-\mathrm{i}}{\mathrm{i}+2} = \frac{\sqrt{10}}{5}\left(\cos(\arctan(-3) + \mathrm{i}\sin(\arctan(-3))\right).$$

(b) (i) Let $z = x + \mathrm{i}y$, $w = a + ib$ with $x, y, a, b \in \mathbb{R}$. Then we have $\overline{w} = a - ib$. We calculate the product $z\overline{w} = (x+\mathrm{i}y)(a-\mathrm{i}b) = xa + yb + \mathrm{i}(-xb + ay)$. There we can see that the real part is $\mathrm{Re}(z\overline{w}) = xa + yb$.

(ii) Let $z = r(\cos(\varphi) + \mathrm{i}\sin(\varphi))$, $w = q(\cos(\psi) + \mathrm{i}\sin(\psi))$. Then the complex conjugate of w is

$$\overline{w} = \overline{q\cos(\psi) + \mathrm{i}q\sin(\psi))} = q(\cos(\psi) - \mathrm{i}\sin(\psi)) = q(\cos(-\psi) + \mathrm{i}\sin(-\psi)).$$

Applying the definition of the multiplication we obtain

$$\mathrm{Re}(z\overline{w}) = \mathrm{Re}\big(rq(\cos(\varphi-\psi) + \mathrm{i}\sin(\varphi-\psi))\big) = rq\cos(\varphi-\psi).$$

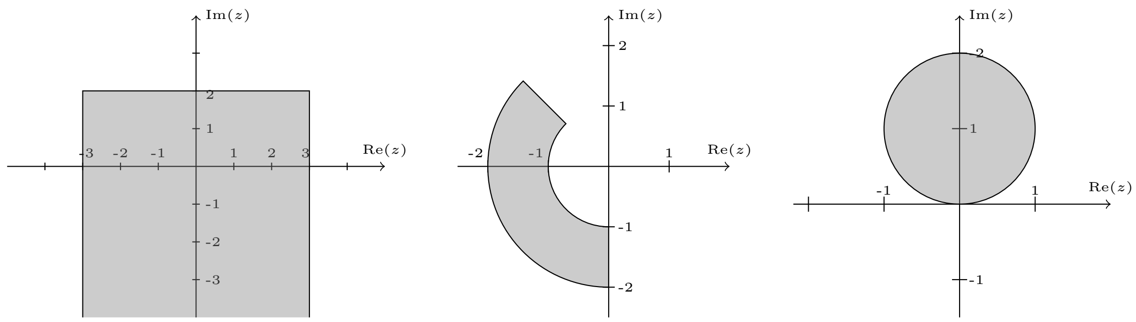
Exercise 13: Sketch the following subsets of the complex plane:

(a) $A = \{z \in \mathbb{C} \ : \ |\mathrm{Re}\, z| \leq 3,\ \mathrm{Im}\, z \leq 2\}$,

(b) $B = \big\{z \in \mathbb{C} \ : \ 1 \leq |z| \leq 2, \mathrm{Arg}\, z \in (-\pi, -\frac{\pi}{2}] \cup [\frac{3\pi}{4}, \pi]\big\}$,

(c) $C = \big\{z \in \mathbb{C} \ : \ |z|^2 \leq 2\mathrm{Im}\, z\big\}$.

Solution 13:



A B C

(a) Using Cartesian coordinates $z = x + \mathrm{i}y$, A can be represented as $\{(x, y) \in \mathbb{R}^2 : -3 \leq x \leq 3,\ y \leq 2\}$.

(b) Using polar coordinates we can represent z as $z = r(\cos\varphi + \mathrm{i}\sin\varphi)$. Now we determina the "boundary" of the set B. If we fix $r = 1$ and take all angles $\varphi \in [3\pi/4, 3\pi/2]$, this gives us the inner circle segment. For $r = 2$ and $\varphi \in [3\pi/4, 3\pi/2]$ we land on the outer circle segment. Similarly, for a fixed angle $\varphi = 3\pi/4$ or $\varphi = -\pi/2$ and radius $r \in \{1, 2\}$ we obtain the two segments joining the points -1 and -2 and the points $(-1+\mathrm{i})/\sqrt{2}$ and $(-\sqrt{2} + \sqrt{2}\mathrm{i})$.

(c) With $z = x + \mathrm{i}y$ we obtain by completing the square

$$|z|^2 \leq 2\,\mathrm{Im}\, z \Longleftrightarrow x^2 + y^2 \leq 2y \Longleftrightarrow x^2 + (y-1)^2 \leq 1 \Longleftrightarrow |z - \mathrm{i}|^2 \leq 1.$$

Thus, the set C describes all points on a disc with centre i with radius 1.

Exercise 14: Find all solutions $z \in \mathbb{C}$ of the equation by completing the square:

$$z^2 + (2 - \mathrm{i}2\sqrt{2})z - 7 - \mathrm{i}(8 + 2\sqrt{2}) = 0.$$

Solution 14: We want to write the equation as $z^2 + 2uz + v$. That gives us

$$u = 1 - \mathrm{i}\sqrt{2}, \qquad v = -7 - \mathrm{i}(8 + 2\sqrt{2}).$$

Then we complete the square to

$$z^2 + 2uz + u^2 - u^2 + v = (z + u)^2 - (u^2 - v) = 0 \quad \Longleftrightarrow \quad (z + u)^2 = u^2 - v.$$

Thus, we need to solve $u^2 - v$ for u:

$$\begin{aligned} u^2 &= 1^2 - \left(-\sqrt{2}\right)^2 + \mathrm{i}2 \cdot 1 \cdot \left(-\sqrt{2}\right) = -1 - \mathrm{i}2\sqrt{2}, \\ u^2 - v &= -1 - \mathrm{i}2\sqrt{2} + 7 + \mathrm{i}(8 + 2\sqrt{2}) = 6 + \mathrm{i}8. \end{aligned}$$

Here the left-hand side $u^2 - v$ is not real, so let $z + u = a + \mathrm{i}b$. Then $(z + u)^2 = a^2 - b^2 + \mathrm{i}2ab$, which gives

$$\begin{aligned} a^2 - b^2 &= 6 = \mathrm{Re}(6 + \mathrm{i}8) \\ ab &= 4 = \tfrac{1}{2}\,\mathrm{Im}(6 + \mathrm{i}8) \quad \Longleftrightarrow \quad b = \tfrac{4}{a}. \end{aligned}$$

We insert $b = \frac{4}{a}$ in the first identity and multiply both sides with a^2. Then we obtain a real quadratic equation for a^2 which we can solve by completing the square, i.e.

$$0 = (a^2)^2 - 6a^2 - 16 = (a^2)^2 - 6a^2 + 9 - 9 - 16 = (a^2 - 3)^2 - 25.$$

The solutions are $a^2 = 3 \pm 5$, but since $a^2 \geq 0$ we have $a^2 = 3 + 5 = 8$ and thus, $a = 2\sqrt{2}$ or $a = -2\sqrt{2}$. The conditions $b = 4/a$ and $z = -u + a + \mathrm{i}b$ yield

$$z = -2\sqrt{2} - 1 \quad \text{or} \quad z = 2\sqrt{2} - 1 + \mathrm{i}2\sqrt{2}\,.$$

Exercise 15: Determine all complex solutions z of the following equations:

(a) $z^2 + (2i + 2)z + 4i = 0,$

(b) $z^4 + (2i + 2)z^2 + 4i = 0.$

Solution 15:

(a) By completing the square we obtain

$$0 = (z + (\mathrm{i} + 1))^2 - (\mathrm{i} + 1)^2 + 4\mathrm{i} = (z + (\mathrm{i} + 1))^2 + 2\mathrm{i}.$$

Setting $w := z + (\mathrm{i} + 1)$ we obtain the equation $w^2 = -2i$. With $x := \mathrm{Re}(w)$ and $y := \mathrm{Im}(w)$ we have $w = x + \mathrm{i}y$, thus, $w^2 = (x + \mathrm{i}y)^2 = x^2 - y^2 + 2\mathrm{i}xy \overset{!}{=} -2\mathrm{i}$. Comparing real and imaginary parts gives us $x^2 = y^2$ und $xy = -1$.

From the second equation we obtain $y = -1/x$, inserting this in the first equation we obtain $x = \pm y = \pm 1/x$, thus $x^2 = \pm 1$. Since x is real, only $x^2 = 1$ is possible and thus, we have $|x| = 1$, so $x = \pm 1$ and $y = \mp 1$ accordingly. Thus, we obtain either $w = 1 - \mathrm{i}$ or $w = -1 + \mathrm{i}$.

Applying the definition of w, we obtain $z = w - \mathrm{i} - 1$, thus, $z = -2\mathrm{i}$ or $z = -2$. Hence, the solutions are $\{-2, -2\mathrm{i}\}$.

(b) We substitute $w = z^2$ and obtain the equation $w^2 + (2\mathrm{i} + 2)w + 4\mathrm{i} = 0$. This equation was solved in part (a): either $w = -2$ or $w = -2\mathrm{i}$.

If $z^2 = -2 = 2\mathrm{i}^2$, we have $z = \pm\sqrt{2}\mathrm{i}$. For $z^2 = -2\mathrm{i}$ the solutions $z = 1 - \mathrm{i}$ or $z = -1 + \mathrm{i}$ were also already calculated in part (a). Thus, the solution set is $\{\sqrt{2}\mathrm{i}, -\sqrt{2}\mathrm{i}, 1 - \mathrm{i}, -1 + \mathrm{i}\}$.