Exercise Sheet No. 4
– with solutions –

Exercise 16:

(a) Give a representation of $\mathrm{i}+1$ in polar coordinates. Use this representation to find all solutions $z \in \mathbb{C}$ of the equality

$$z^2 = \mathrm{i} + 1.$$

(b) Find all solutions $z \in \mathbb{C}$ of the following equation by means of completing the square

$$z^2 + (3+5\mathrm{i})z - \left(3 - \tfrac{17}{2}\mathrm{i}\right) = 0.$$

Solution 16:

(a) $i+1$ is in the first quadrant, thus, we obtain $|1+i| = \sqrt{2}$, $\mathrm{Arg}(1+i) = \arctan(\frac{1}{1}) = \frac{\pi}{4}$, and thus, in polar coordinates,

$$1 + i = 2^{\frac{1}{2}} e^{\frac{\pi}{4} i + 2\pi i k}, \quad k \in \mathbb{Z}.$$

If $z^2 = 1+i$, we have

$$z = \left(2^{\frac{1}{2}}\right)^{\frac{1}{2}} \cdot e^{i\frac{\pi}{8} + \pi i k}, \quad k \in \mathbb{Z}.$$

We only need to consider $k=0$ and $k=1$, for which we obtain

$$z = 2^{\frac{1}{4}} \cdot e^{i\frac{\pi}{8}} \qquad \text{or} \qquad z = 2^{\frac{1}{4}} \cdot e^{i\frac{\pi}{8} + i\pi} = 2^{\frac{1}{4}} \cdot e^{-i\frac{7\pi}{8}}.$$

(b) Completing the square yields

$$\left(z + \frac{3+5\mathrm{i}}{2}\right)^2 - \frac{(3+5\mathrm{i})^2}{4} - (3 - \frac{17}{2}\mathrm{i}) = 0,$$

which can be simplified to $(z + \frac{3+5i}{2})^2 = -(\mathrm{i}+1)$. After substituting $w = z + \frac{3+5\mathrm{i}}{2}$, where $w = a + b\mathrm{i}$ with $a, b \in \mathbb{R}$, we have to solve the equality

$$(\mathrm{i}w)^2 = \mathrm{i} + 1.$$

Using part (a) we obtain

$$\mathrm{i}w = 2^{\frac{1}{4}} \cdot e^{i\frac{\pi}{8}} \qquad \text{or} \qquad \mathrm{i}w = 2^{\frac{1}{4}} \cdot e^{-i\frac{7\pi}{8}},$$

thus, we have

$$w = 2^{\frac{1}{4}} \cdot e^{i(\frac{\pi}{8} - \frac{\pi}{2})} = 2^{\frac{1}{4}} \cdot e^{-i\frac{3\pi}{8}} \qquad \text{or} \qquad w = 2^{\frac{1}{4}} \cdot e^{i(\frac{-7\pi}{8} - \frac{\pi}{2})} = 2^{\frac{1}{4}} \cdot e^{i\frac{5\pi}{8}},$$

and resubstituting yields

$$z = w - \frac{3+5\mathrm{i}}{2} = 2^{\frac{1}{4}}\left(\cos(-\tfrac{3\pi}{8}) - \frac{3}{2^{5/4}} + i(\sin(-\tfrac{3\pi}{8}) - \frac{5}{2^{5/4}})\right)$$

or

$$z = 2^{\frac{1}{4}}\left(\cos(\tfrac{5\pi}{8}) - \frac{3}{2^{5/4}} + i(\sin(\tfrac{5\pi}{8}) - \frac{5}{2^{5/4}})\right).$$

Exercise 17: Consider the sequence $(a_n)_n$ with $a_n = \frac{n-1}{n+1}$, $n \in \mathbb{N}$. Find an index N such that $|a_n - 1| \leq \varepsilon$ for every $n \geq N$, when

(a) $\varepsilon = \dfrac{1}{10}$, (b) $\varepsilon = \dfrac{1}{1000}$, (c) $\varepsilon > 0$ is arbitrary.

(d) Does the sequence $(a_n)_n$ converge? If so, what is the limit?

Solution 17:

(a) We have

$$|a_n - 1| = |\frac{n-1}{n+1} - 1| = |\frac{n-1-(n+1)}{n+1}| = |\frac{-2}{n+1}| = \frac{2}{n+1}$$

which is at most $\frac{1}{10}$ iff

$$20 \leq n+1 \text{ i.e. } 19 \leq n.$$

Thus, chosing $N \geq 19$, (a) holds for all $n \geq N$.

(b) Now we want $\frac{2}{n+1} \leq \frac{1}{1000}$ to hold, which holds iff

$$2000 \leq n+1 \text{ i.e. } 1999 \leq n.$$

Thus, chosing $N = 1999$ (b) holds for all $n \geq N$.

(c)

$$\frac{2}{n+1} \leq \varepsilon \text{ für alle } n \geq N$$

holds iff $\frac{2}{N+1} \leq \varepsilon$. This is equivalent to $N \geq \frac{2}{\varepsilon} - 1$.

If we would have started with part (c), we would have been able to obtain the N for (a) and (b) from this formula. E.g. for $\varepsilon = 1/10$ the formula gives $N \geq 19$.

(d) As shown in (c), the sequence converges to the limit 1.

Exercise 18: Consider the sequence

$$a_n = \frac{1}{2} + (-1)^n \left(1 - \frac{1}{n}\right).$$

(a) Is the sequence bounded? If so, give a value for r for which $|a_n| \leq r$.

(b) Give the smallest possible such r; justify your answer.

Solution 18:
(a) We have

$$|a_n| \leq \frac{1}{2} + \left|1 - \frac{1}{n}\right| \leq \frac{1}{2} + 1 = \frac{3}{2}.$$

Thus, (a_n) is bounded with $r = \frac{3}{2}$.

(b) We show that $r_0 = \frac{3}{2}$ is also the smallest r possible. For the sake of contradiction assume there is $r < \frac{3}{2}$ with the property $|a_n| \leq r$,. Then we have

$$\frac{3}{2} - a_n = \left(\frac{3}{2} - r\right) + (r - a_n) \geq \frac{3}{2} - r > 0.$$

But for even n we have:

$$\frac{3}{2} - a_n = \frac{3}{2} - \frac{1}{2} - \left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

This is a contradiction for even n with $n > 1/\left(\frac{3}{2} - r\right)$, thus, such an r cannot exist.

Exercise 19: Test the following sequences for convergence:

$$a_n = \frac{n}{n^2-2}, \qquad b_n = \frac{n^2-2}{n}, \qquad c_n = n - 1,$$

$$d_n = b_n - c_n, \qquad e_n = \frac{2 + (-1)^{n-1}}{n+7}.$$

Solution 19:
We have

$$a_n = \frac{\frac{1}{n}}{1-\frac{2}{n^2}} \text{ und } \lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty}\frac{1}{n}}{1-\lim\limits_{n\to\infty}\frac{2}{n^2}} = \frac{0}{1-0} = 0 \text{ (convergent);}$$

.

$$b_n = \frac{1}{a_n} = n - \frac{2}{n} \to \infty \text{ (unbounded, thus divergent);}$$

.

$$c_n = n - 1 \to \infty \text{ (unbounded, divergent);}$$

.

$$d_n = b_n - c_n = 1 - \frac{2}{n} \text{ und } \lim_{n\to\infty} d_n = 1 - \lim_{n\to\infty}\frac{2}{n} = 1 - 0 = 1 \text{ (convergent).}$$

For (e_n) we have

$$\frac{1}{n+7} \leq e_n \leq \frac{3}{n+7}.$$

Since the two sequences $\left(\frac{1}{n+7}\right)$ and $\left(\frac{3}{n+7}\right)$ tend to zero, by the sandwich theorem (e_n) also converges to 0.

Exercise 20: Determine the limit of the sequence $(a_n)_n$, where

(a) $a_n = \sqrt[n]{4 + \frac{n-1}{n+1}}$, (b) $a_n = \frac{n^4-2}{n^2+4} + \frac{n^3(3-n^2)}{n^3+1}$,

(c) $a_n = \sqrt[n]{34^n + 118^n} \cdot \left[\frac{(n+4)^4}{n^3} - n + 1\right] + 3$.

Solution 20:

(a) We estimate:

$$\sqrt[n]{5} \geq \sqrt[n]{4 + \frac{n-1}{n+1}} = a_n \geq 1$$

From the tutorial we know that $\lim\limits_{n\to\infty} \sqrt[n]{5} = 1$. Thus, we have

$$(x_n)_n \leq (a_n)_n \leq (y_n)_n$$

with $(x_n) = \sqrt[n]{5}$ and $(y_n) = 1$. Since we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$, by the sandwich theorem we have

$$\lim_{n\to\infty} a_n = 1.$$

(b) We have

$$\begin{aligned} a_n &= \frac{n^4-2}{n^2+4} - \frac{n^3(n^2-3)}{n^3+1} = \frac{n^2\left(n^2+4\right) - 4n^2 - 2}{n^2+4} - \frac{\left(n^3+1\right)\left(n^2-3\right) - n^2 + 3}{n^3+1} \\ &= n^2 - \frac{4n^2+2}{n^2+4} - n^2 + 3 + \frac{n^2-3}{n^3+1} = 3 - \frac{4+\frac{2}{n^2}}{1+\frac{4}{n^2}} + \frac{\frac{1}{n}-\frac{3}{n^3}}{1+\frac{1}{n^3}}. \end{aligned}$$

According to the computational rules we obtain

$$\lim_{n\to\infty} a_n = 3 - \frac{\lim\limits_{n\to\infty}\left(4+\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty}\left(1+\frac{4}{n^2}\right)} + \frac{\lim\limits_{n\to\infty}\frac{1}{n} - \lim\limits_{n\to\infty}\frac{3}{n^3}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n^3}\right)} = 3 - \frac{4}{1} + \frac{0}{1} = -1.$$

Here we have used $\lim\limits_{n\to\infty} \frac{c}{n^p} = 0$ for all $c \in \mathbb{R}$ and $p \in \mathbb{N}$.

(c) We look at the individual factors: For the root we can estimate

$$118 = \sqrt[n]{118^n} \leq \sqrt[n]{34^n + 118^n} \leq \sqrt[n]{2 \cdot 118^n} = 118\sqrt[n]{2} \to 118 \quad (n \to \infty),$$

and thus, by the sandwich theorem, the limit is 118.

For the second factor we obtain

$$\frac{(n+4)^4}{n^3} - n + 1 = \frac{n^4 + 16n^3 + 96n^2 + 256n + 256 - n^4 + n^3}{n^3} = \frac{n^3(17 + 96\frac{1}{n} + 256\frac{1}{n^2})}{n^3} \to 17 \quad (n \to \infty).$$

Since the limits of both factors exist, we can calculate the limit of the product as the limit of the limits (i.e. $\lim(b_n \cdot c_n) = \lim b_n \cdot \lim c_n$). Thus, we have

$$\lim_{n\to\infty} \sqrt[n]{34^n + 118^n} \cdot \left[\frac{(n+4)^4}{n^3} - n + 1\right] + 3 = 118 \cdot 17 + 3 = 2006 + 3 = 2009\,.$$