Exercise Sheet No. 5
– with solutions –

Exercise 21: Which of the following assertions are true? Justify correct ones and give a counter example for each incorrect assertion.

(a) If a sequence is monotone and bounded, then it converges.

(b) If a sequence converges, then it is monotone and bounded.

(c) If a sequence is not bounded, then it is not convergent.

(d) If a sequence is not monotone, then it is not convergent.

(e) If a sequence has exactly one accumulation point, then it converges.

(f) If a sequence converges, then it has exactly one accumulation point.

Solution 21:

(a) Correct. (This is the Monotonicity Criterion.)

(b) Incorrect. The sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ converges to 0, but is not monotone. This is not a contradiction to (a), since (a) does *not* say anything about sequences which are *not* monotone and bounded!

(c) Correct. Every convergent sequence is bounded (see Theorem 4.4 lecture notes).

(d) Incorrect. See counter example in (b).

(e) Incorrect. For the sequence $(a_n)_n$ with $a_n = 2^n + (-2)^n$ we have for even n $a_n = 2 \cdot 2^n$, and for odd n we have $a_n = 0$. Thus, $(a_n)_n$ contains exactly 2 convergent subsequences whose limits are 0. Hence the only accumulation point of (a_n) is 0. But (a_n) does not converge, since it is unbounded.

(f) Correct. In this case the limit is the only accumulation point (see Theorem 4.15 lecture notes).

Exercise 22:

Let the sequence (a_n) be given by a starting value $a_0 \in [0, 2]$ and the recursion

$$a_{n+1} = \frac{a_n\,(a_n^2 + 3)}{3a_n^2 + 1}, \qquad n = 0, 1, 2, \ldots.$$

(a) Show that

$$a_{n+1} - 1 = \frac{(a_n - 1)^3}{3a_n^2 + 1}, \qquad n = 0, 1, 2, \ldots.$$

Also prove the following two statements:

$$0 < a_0 < 1 \implies 0 < a_n < 1 \text{ for all } n \in \mathbb{N},$$
$$1 < a_0 < 2 \implies 1 < a_n < 2 \text{ for all } n \in \mathbb{N}.$$

(b) Show that the sequence is strictly montonically increasing for $0 < a_0 < 1$ and strictly monotonically decreasing for $1 < a_0 < 2$.

(c) For which $a_0 \in [0, 2]$ does the sequence converge? If so, determine the limit.

Solution 22:

(a) We have

$$a_{n+1} - 1 = \frac{a_n\,(a_n^2 + 3)}{3a_n^2 + 1} - 1 = \frac{a_n^3 - 3a_n^2 + 3a_n - 1}{3a_n^2 - 1} = \frac{(a_n - 1)^3}{3a_n^2 + 1}.$$

The 2nd part we show by induction.
Base $n = 1$: For $0 < a_0 < 1$ we have

$$-1 < \frac{(a_0 - 1)^3}{3a_0^2 + 1} < 0.$$

Thus, $0 < a_1 < 1$.
For $1 < a_0 < 2$ we have

$$0 < \frac{(a_0 - 1)^3}{3a_0^2 + 1} < 1.$$

Thus, $1 < a_0 < 2$.

Step: Assume the statements hold for some $n \in \mathbb{N}$ (I.H.). For $a_0 \in (0,1)$ we have $0 < a_n < 1$ which gives

$$-1 < \frac{(a_n - 1)^3}{3a_n^2 + 1} < 0,$$

which shows $a_{n+1} \in (0,1)$.
For $a_0 \in (1,2)$ we have $1 < a_n < 2$ which gives

$$0 < \frac{(a_n - 1)^3}{3a_n^2 + 1} < 1,$$

which shows $a_{n+1} \in (1,2)$.

(b) We have

$$\frac{a_{n+1}}{a_n} = \frac{a_n^2 + 3}{3a_n^2 + 1} = 1 + 2\,\frac{1 - a_n^2}{3a_n^2 + 1}.$$

According to (a) for $0 < a_0 < 1$ we have $0 < a_n < 1$. Thus, the last fraction in the equality above is positiv. Hence, we obtain $a_{n+1}/a_n > 1$, which shows that the sequence is strictly monotonically increasing.

Similarly for $1 < a_n < 2$ the fraction is negativ and it follows that $a_{n+1}/a_n < 1$. Thus, the sequence is strictly monotonically decreasing.

(c) For $a_0 \in (0,1) \cup (1,2)$ the sequence is bounded by (a) and monotone by (b). Thus, by the Monotonicity Criterion it converges. The limit has to be a solution of

$$a = \frac{a\,(a^2 + 3)}{3a^2 + 1}.$$

Thus, we have either $a = 0$ or we have

$$a^2 + 3 = 3a^2 + 1.$$

This last equality has solutions $a = \pm 1$. In the case where $a_0 \in (0,1)$, the sequence is positive and increasing, so the limit has to be 1. For $a_0 \in (1,2)$ the limit can also only be 1.

For $a_0 = 0$ the sequence is constant (=0), so it converges to 0. For $a_0 = 1$ the sequence is constant (=1) and thus, converges to 1. For $a_0 = 2$ we have $a_1 = 14/13$, thus, $1 < a_1 < 2$. The remainder of the sequence (i.e. shifting the index by one) is by (a) and (b) bounded and strictly decreasing. By the Monotonicity criterion the sequence is convergent, and the limit also has to be 1.

Exercise 23:
Split the sequence $(a_n)_n$, given by

$$a_n = \frac{1 + 2^n}{1 + 2^n + (-2)^n}, \qquad n \in \mathbb{N},$$

into appropriate subsequences and test those for monotonicity, boundedness and convergence. Does the sequence (a_n) converge?

Solution 23:
We split $(a_n)_n$ into the two subsequences

$$a_n = \frac{1 + 2^n}{1 + 2^n + (-2)^n} = \begin{cases} \frac{1+2^n}{1+2^{n+1}}, & n \text{ even,} \\ 1 + 2^n, & n \text{ odd.} \end{cases}$$

We check monotonicity for even n:

$$\begin{aligned} a_{n+2} - a_n &= \frac{1 + 2^{n+2}}{1 + 2^{n+3}} - \frac{1 + 2^n}{1 + 2^{n+1}} = \frac{8 \cdot 2^{2n} + 4 \cdot 2^n + 2 \cdot 2^n + 1 - 8 \cdot 2^{2n} - 8 \cdot 2^n - 2^n - 1}{(1 + 2^{n+3})(1 + 2^{n+1})} \\ &= \frac{-3 \cdot 2^n}{(1 + 2^{n+3})(1 + 2^{n+1})} < 0 \end{aligned}$$

Thus, the subsequence (a_{2k}) is strictly monotonically decreasing and thus bounded from above by $a_2 \geq a_{2k}$. Since both numerator and denominator are always positive, we also have $a_{2k} > 0$. Thus, the subsequence is also bounded from below and hence, by the Monotonocity Criterion, converges.
For odd n we obtain from

$$a_{n+2} - a_n = 1 + 2^{n+2} - 1 - 2^n = 4 \cdot 2^n - 2^n = 3 \cdot 2^n > 0\,,$$

that the subsequence is strictly monotonically increasing. Additionally we always have $a_{2k-1} > 0,\ k \in \mathbb{N}$. But (2^n) cannot be bounded from above, thus, this subsequence diverges.
Thus, $(a_n)_n$ has only 1 accumulation point, but is still not convergent, i.e. it diverges.

Exercise 24: Let f be a real-valued function defined by

$$f(x) = \frac{x^3 - 3x + 2}{x^3 - 7x + 6}.$$

Determine the maximal domain D and the range $f(D)$ of f.

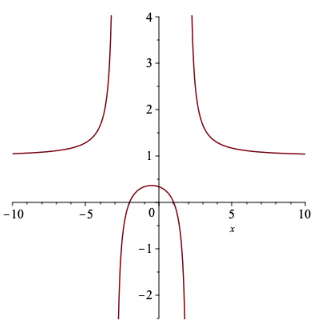
Solution 24:
We start by dividing the numerator and denominator into linear factors. It is easy to see that 1 is a zero of both. By long division we obtain

$$(x^3 - 3x + 2) : (x - 1) = x^2 + x - 2 \qquad \text{and} \qquad (x^3 - 7x + 6) : (x - 1) = x^2 + x - 6$$

and by means of completing the square we obtain the zeros of the quadratic expressions so that we can write f as

$$f(x) = \frac{(x-1)(x^2 + x - 2)}{(x-1)(x^2 + x - 6)} = \frac{(x-1)^2(x+2)}{(x-1)(x-2)(x+3)}.$$

The domain cannot contain any zeros of the denominator, so we have $D = \mathbb{R} \setminus \{1, 2, -3\}$.
To get an idea of the range, we sketch the function. For $x < -3$ f is positive and takes all values in the interval $(1, \infty)$, the same is true for $x > 2$. For $-3 < x < 2$, $x \neq 1$ f takes all values in the interval $(-\infty, \widehat{x}]$, where we still have to determine $\widehat{x}$.
(Note: The singularity $x = 1$ does not split the interval $(-\infty, \widehat{x}]$ into two, since f is continuous on $(-3, 1)$. Even though we do not reach 0 in $x = 1$ since $1 \notin D$, we have $f(-2) = 0$ and $-2 \in D$.)
We have:



$$f(x) = \frac{x^2 + x - 2}{x^2 + x - 6} = 1 + \frac{4}{x^2 + x - 6} = 1 + \frac{4}{(x + \frac{1}{2})^2 - \frac{25}{4}} = 1 - \frac{4}{\frac{25}{4} - (x + \frac{1}{2})^2 \frac{25}{4}}$$

The Denominator of the fraction on the right-hand side is positive for $-3 < x < 2$ and largest for $x = \frac{1}{2}$. Then the fraction is smallest (and therefore f largest) for $x = -1/2$. So we have

$$f(x) = 1 - \frac{4}{\frac{25}{4} - \left(x + \frac{1}{2}\right)^2} \leq 1 - \frac{4}{\frac{25}{4}} = f\left(-\frac{1}{2}\right) = \frac{9}{25} = \widehat{x}, \qquad -3 < x < 2,\ x \neq 1.$$

In conclusion, the range is $f(D) = (-\infty, 9/25] \cup (1, \infty)$.

Exercise 25:
Consider the following polynomial: $f(x) = x^4 + 5x^3 - 8x^2 + 1 - (x - 2)^3$.

(a) Expand f about the expansion points $x_1 = 1$ and $x_2 = -1$.

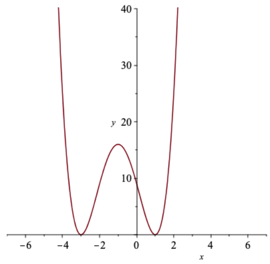
(b) Determine all zeros x_i of f by factoring out factors of the form $(x - x_i)$.

Solution 25:

(a) we have

$$f(x) = x^4+5x^3-8x^2+1-(x^3-6x^2+12x-8) = x^4+4x^3-2x^2-12x+9.$$

For the expansion about $x_1 = 1$ we write



$$f(x) = ((x-1)+1)^4+4((x-1)+1)^3-2((x-1)+1)^2-12((x-1)+1)+9.$$

Using the binomial theorem we obtain

$$\begin{aligned} f(x) = & \left[(x-1)^4+4(x-1)^3+6(x-1)^2+4(x-1)+1\right] \\ & +4\left[(x-1)^3+3(x-1)^2+3(x-1)+1)\right] \\ & -2\left[(x-1)^2+2(x-1)+1\right]-12\left[(x-1)+1\right]+9 \\ = & (x-1)^4+8\,(x-1)^3+16\,(x-1)^2\,. \end{aligned}$$

Similarly for the expansion point $x = -1$ we obtain

$$\begin{aligned} f(x) = & \left[(x+1)^4-4(x+1)^3+6(x+1)^2-4(x+1)+1\right]+4\left[(x+1)^3-3(x+1)^2+3(x+1)-1)\right] \\ & -2\left[(x+1)^2-2(x+1)+1\right]-12\left[(x+1)-1\right]+9 \\ = & (x+1)^4-8\,(x+1)^2+16. \end{aligned}$$

(b) From the expansion about $x = 1$ we obtain

$$\begin{aligned} f\,(x) &= (x-1)^4+8\,(x-1)^3+16\,(x-1)^2 = (x-1)^2\left[(x-1)^2+8(x-1)+16\right] \\ &= (x-1)^2\left[(x-1)+4\right]^2 = (x-1)^2\,(x+3)^2\,. \end{aligned}$$

That shows that the polynomial has two zeros of multiplicity 2, $x_1 = -3$ und $x_2 = 1$.