Exercise Sheet No. 6
– with solutions –

Exercise 26:
Consider the polynomial $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := \frac{1}{8}x^3 + \frac{3}{8}x^2 - \frac{9}{8}x + \frac{5}{8}$.

(a) Expand f about the expansion points $x_1 = 1$ and $x_2 = -3$. Use this representation to discuss the behavior of f on the interval $[1, \infty)$.

(b) Use a sketch of f to find intervals on which f has an inverse function. Also sketch the inverse.

Solution 26:

(a) We show the expansion about the point $x_1 = 1$ in detail. We are looking for a representation of f of the form

$$f(x) = \sum_{k=0}^{3} a_k (x-1)^k \qquad \text{with } a_k \in \mathbb{R} \text{ for } k = 0, .., 3$$

We start by replacing x by $(x - 1 + 1)$:

$$f(x) = \frac{1}{8}(x-1+1)^3 + \frac{3}{8}(x-1+1)^2 - \frac{9}{8}(x-1+1) + \frac{5}{8}$$

Now we use the binomial theorem $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ with $a = x - 1$ and $b = 1$:

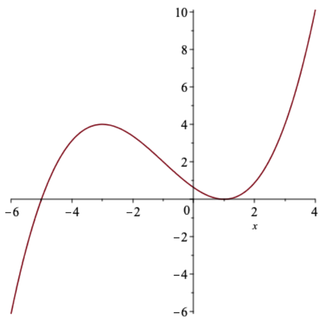
$$f(x) = \frac{1}{8}\left[(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1\right] + \frac{3}{8}\left[(x-1)^2 + 2(x-1) + 1\right] - \frac{9}{8}\left[(x-1) + 1\right] + \frac{5}{8}$$

Writing this as a polynomial in $(x - 1)$ we obtain

$$f(x) = \frac{1}{8}\left[(x-1)^3 + 6(x-1)^2\right].$$

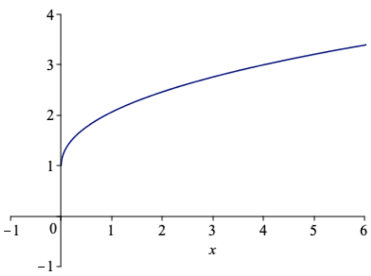
Similarly we obtain the expansion of f about the point $x_2 = -3$:

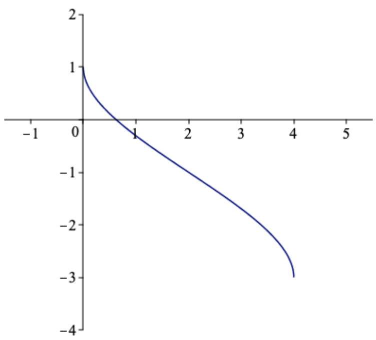
$$\begin{aligned} f(x) &= \frac{1}{8}(x+3-3)^3 + \frac{3}{8}(x+3-3)^2 - \frac{9}{8}(x+3-3) + \frac{5}{8} \\ &= \frac{1}{8}\left[(x+3)^3 + 3(x+3)^2(-3) + 3(x+3)(-3)^2 + (-3)^3\right] \\ &\quad + \frac{3}{8}\left[(x+3)^2 + 2(x+3)(-3) + (-3)^2\right] - \frac{9}{8}\left[(x+3) - 3\right] + \frac{5}{8} \\ &= \frac{1}{8}\left[(x+3)^3 - 6(x+3)^2 + 32\right] \end{aligned}$$

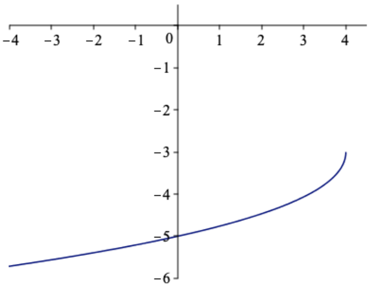


On $[1, \infty)$ the summands $(x-1)^3$ and $(x-1)^2$ are strictly monotonically increasing, thus f is strictly monotonically increasing on $[1, \infty)$.

(b) An inverse function exists exaclty on the intervals $(-\infty, -3]$, $[-3, 1]$ and $[1, \infty)$, since there f is strictly monotone.







Exercise 27:
Let $c \in \mathbb{R}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 - 2x - x^2, & x < 0, \\ c(x-2)^2, & x \geq 0. \end{cases}$$

(a) Find a value of c such that f is continuous and give a sketch of $f(x)$ for this c on the interval $[-3, 4]$.

For the remaining exercise study f for this fixed c.

(b) Find all maximal intervals I of $\mathbb{R}$ where f is invertible.
Note: Here maximal means that there is no interval I' with $I \subsetneq I'$ and f is invertible on I'.

For each of these intervals give the inverse of f and the domain of the inverse on this interval.

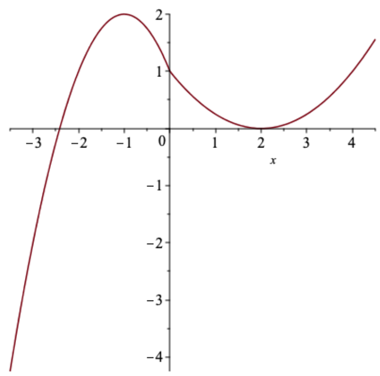
(c) Find a maximal domain $D \subseteq \mathbb{R}$ such that the function $f : D \to \mathbb{R}$ is bijective.

Solution 27:

(a) The function f is continuous on $\mathbb{R} \setminus \{0\}$ for every choice of c, since for $x < 0$ and $x \geq 0$ f it is given by polynomials, which are always continuous.
It remains to check continuity at $x = 0$. Consider $x < 0$. There we have $f(x) = p(x)$ for the polynomial $p : \mathbb{R} \to \mathbb{R}, x \mapsto 1 - 2x - x^2$. Since polynomials are continuous, for every sequence (x_n) in $(-\infty, 0)$ with $\lim_{n\to\infty} x_n = 0$ we have $f(x_n) = p(x_n) \to p(0)$. Thus, f is continuous if and only if

$$\lim_{x\to 0,\ x>0} f(x) = f(0) = p(0) = 1.$$

By definition we have $f(0) = c(0-2)^2 = 4c$. Thus, we must choose $c = \frac{1}{4}$.



(b) By means of completing the square we obtain

$$f(x) = 1 - 2x - x^2 = 2 - 1 - 2x - x^2 = 2 - (1+x)^2, \qquad \text{für } x \in (-\infty, 0).$$

That shows that f is strictly monotonically increasing on $(-\infty, -1]$ and strictly monotonically decreasing on $[-1, 0)$. (On $(-\infty, 0)$ the graph of f is a parabola opening to the top with vertex $(-1, 2)$.) We further see that $f(x) = \frac{1}{4}(x-2)^2$ is strictly monotonically decreasing on $[0, 2]$ and strictly monotonically increasing on $[2, \infty)$.

By the continuity of f in $x = 0$ we obtain the 3 intervals $I_1 = (-\infty, -1]$, $I_2 = [-1, 2]$ and $I_3 = [2, \infty)$ on which f is strictly monotone. There f is invertible if we restrict the codomain to the range. The ranges are $f(I_1) = (-\infty, 2]$, $f(I_2) = [0, 2]$ and $f(I_3) = [0, \infty)$.

Denote the inverse functions by g_1, g_2, g_3.

To determine $g_1 : (-\infty, 2] \to (-\infty, -1]$ we solve $y = 2 - (1+x)^2$ for x, where $x \in (-\infty, -1]$ and $y \in (-\infty, 2]$. This leads to $(1+x)^2 = 2 - y$ and since $1 + x \leq 0$, we have

$$-1 - x = |1 + x| = \sqrt{2-y} \quad \Longleftrightarrow x = -1 - \sqrt{2-y},$$

so we have

$$g_1 : (-\infty, 2] \to (-\infty, -1], \quad g_1(y) = -1 - \sqrt{2-y}.$$

For $x \in [-1, 2]$ and $y \in [0, 2]$ we solve

$$\begin{cases} y = 2 - (1+x)^2, & y \in (1, 2], \\ y = \frac{1}{4}(x-2)^2, & y \in [0, 1] \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 1 + x = |1+x| = \sqrt{2-y}, & y \in (1, 2], \\ 2 - x = |x - 2| = 2\sqrt{y}, & y \in [0, 1] \end{cases}$$

for y (we used $x + 1 \geq 0$ and $x - 2 \leq 0$). This leads to

$$g_2 : [0, 2] \to [-1, 2], \quad g_2(y) = x = \begin{cases} \sqrt{2-y} - 1, & y \in (1, 2], \\ 2 - 2\sqrt{y}, & y \in [0, 1] \end{cases}$$

In the case $x \in [2,\infty)$ and $y \in [0,\infty)$ we proceed similarly and obtain

$$y = \tfrac{1}{4}(x-2)^2 \iff 2\sqrt{y} = x-2 \iff x = 2+2\sqrt{y} = g_3(y),$$

since here we have $x-2 \geq 0$. Note that $g_3 : [0,\infty) \to [2,\infty)$.

Now we need to check that the identitites (5.1) (page 32 lecture notes) are satisfied:
For $x \in (\infty, -1]$:

$$f\left(g_1(y)\right) = 2 - (1+g_1(y))^2 = 2 - \left(1 - 1 - \sqrt{2-y}\right)^2 = 2 - \left(-\sqrt{2-y}\right)^2 = 2-(2-y) = y,$$

$$g_1\left(f(x)\right) = -1 - \sqrt{2-f(x)} = -1 - \sqrt{(1+x)^2} = -1 - |1+x| \overbrace{=}^{(*)} -1-(-1-x) = x$$

$(*)$: since $x \in (\infty,-1]$, we have $1+x \leq 0$ and thus, $|1+x| = -(1+x) = -1-x$.

we can do the same for g_2 and g_3.

(c) Clearly f is surjektiv but not injective. Thus, we have to restrict the domain of f such that f becomes injective (without losing surjectivity). This can be obtained for example by choosing $D = (-\infty, -1-\sqrt{2}) \cup [0,\infty)$, since $-1-\sqrt{2}$ is the zero of f in the interval $(-\infty, 2)$.

Exercise 28:
Show that the function

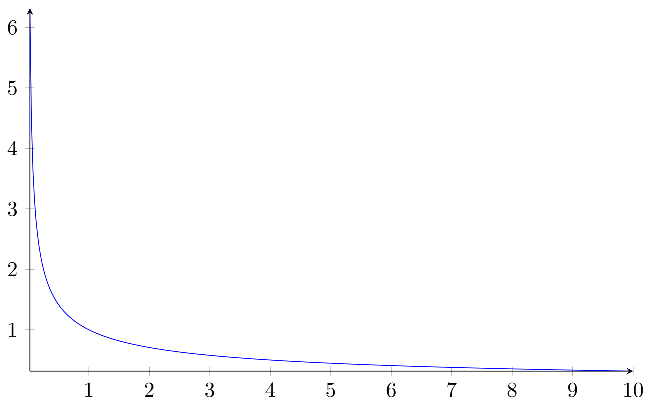
$$x \mapsto \frac{1}{\sqrt{x}}$$

is continuous on the domain $D = (0,\infty)$. Sketch the graph of the function.

Solution 28: let $(x_n)_n$ be a positive sequence with limit x. Since $\sqrt{x} \geq 0$ we have

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x_n}}\right| = \left|\frac{\sqrt{x_n}-\sqrt{x}}{\sqrt{x}\sqrt{x_n}}\right| = \left|\frac{x_n - x}{\sqrt{x}\sqrt{x_n}(\sqrt{x_n}+\sqrt{x})}\right|$$
$$\leq \left|\frac{x_n - x}{\sqrt{x}\sqrt{x_n}\sqrt{x_n}}\right| = \frac{|x-x_n|}{|x_n\sqrt{x}|} \to 0,$$

since $|x-x_n| \to 0$ and $x_n\sqrt{x} \to x\sqrt{x}$ for $n \to \infty$.



Exercise 29:
For each of the following functions $f_j : \mathbb{R} \to \mathbb{R}$ find all points $x \in \mathbb{R}$ where f_j is continuous

$$\text{(a)} \quad f_1(x) := \begin{cases} \dfrac{x^3+4x^2+x-6}{x^3-3x+2}, & x \in \mathbb{R}\setminus\{1,-2\}, \\ 0, & x = 1, \\ -\frac{1}{3}, & x = -2, \end{cases} \qquad \text{(b)} \quad f_2(x) := \begin{cases} x, & x \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Solution 29:

(a) For $x \notin \{1, -2\}$ we have $f_1(x) = \frac{(x+3)(x-1)(x+2)}{(x-1)^2(x+2)} = \frac{x+3}{x-1}$. Since this is the quotient of two continuous functions and $x \neq 1$, it follows that f_1 is continuous on $\mathbb{R} \setminus \{1, -2\}$. Now let (x_k) an arbitrary sequence with $\lim_{k\to\infty} x_k = -2$. Then

$$f_1(x_k) = \frac{x_k + 3}{x_k - 1} \longrightarrow \frac{-2+3}{-2-1} = -\frac{1}{3} = f_1(-2),$$

thus, f_1 is also continuous in $x = -2$.

Now choose $x_k = 1 + 1/k$. Then we have $\lim_{k\to\infty} x_k = 1$, but

$$f_1(x_k) = \frac{x_k + 3}{x_k - 1} = \frac{4 + \frac{1}{k}}{\frac{1}{k}} = 1 + 4k.$$

This sequence is unbounded, so it definitely does not converge to $0 = f_1(1)$. Thus, f_1 is not continuous in $x = 1$.

(b) For $x \notin \mathbb{Z}$ f_2 is constant in a neighbourhood of x, thus also in x. The same argument holds for $x = 0$.

Now let $x \in \mathbb{Z} \setminus \{0\}$. Set $x_k := x + 1/k$. Then we have $\lim_{k\to\infty} x_k = x$, but $f_2(x_k) = 0$ for all $k \in \mathbb{N}$. It follows that $\lim_{k\to\infty} f_2(x_k) = 0 \neq x = f_2(x)$, thus, the function f_2 is not continuous in x.

Exercise 30:
Prove that the function

$$x \mapsto \sqrt[3]{x}$$

is continuous on the domain $D = [0, \infty)$.
Hint: For the case $x \neq 0$ use the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for $a, b \in \mathbb{R}$.

Solution 30:
We have to show that for $x_n \to x$ we also have $|\sqrt[3]{x} - \sqrt[3]{x_n}| \to 0$ for $n \to \infty$.
We distinguish two cases:
Case 1: $x = 0$. Let $(x_n)_n$ be a non-negative sequence that tends to 0. Then

$$\left|\sqrt[3]{x} - \sqrt[3]{x_n}\right| = \sqrt[3]{x_n},$$

since $x = 0$ and $x_n \geq 0$. Since $(x_n)_n$ converges to 0, by definition there exsists for all $\varepsilon' > 0$ some $N = N(\varepsilon') \in \mathbb{N}$ s.t. $|x_n| \leq \varepsilon'$ for all $n \geq N$. Thus, $\sqrt[3]{x_n}$ also converges to 0, since $\sqrt[3]{x_n} \leq \varepsilon$ is equivalent to $x_n \leq \varepsilon^3$. Set $\varepsilon' = \varepsilon^3$. Then we have $\sqrt[3]{x} \leq \varepsilon$ for all $n \geq N(\varepsilon')$, so $|\sqrt[3]{x} - \sqrt[3]{x_n}|$ converges to 0.
Case 2: $x > 0$. Let $a = \sqrt[3]{x}$ and $b = \sqrt[3]{x_n}$. Then by the hint we have

$$\left|\sqrt[3]{x} - \sqrt[3]{x_n}\right| = \left|\frac{x - x_n}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{x_n} + \sqrt[3]{x_n}^2}\right|.$$

Since $\sqrt[3]{n} \geq 0$ for $n \geq 0$, we have $\sqrt[3]{x}^2 + \sqrt[3]{x x_n} + \sqrt[3]{x_n}^2 \geq \sqrt[3]{x}^2$, so we obtain the estimate

$$\left|\frac{x - x_n}{\sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{x_n} + \sqrt[3]{x_n}^2}\right| \leq \left|\frac{x - x_n}{\sqrt[3]{x}^2}\right| = \frac{|x - x_n|}{\sqrt[3]{x}^2} \to 0 \text{ for } n \to \infty,$$

so it converges to 0.