Exercise Sheet No. 7
– with solutions –

Exercise 31:
Decide for each of the following subsets of $\mathbb{R}$ if they are bounded, open, closed and compact:

(a) $A = (-13, 3]$, (b) $B = ([-4, 7] \cup [10, \infty)) \cap (-12, 15]$, (c) $C = [-\frac{7}{2}, \infty) \setminus \mathbb{N}$.

Solution 31:

(a) The set A is bounded since for all $x \in A$ we have $|x| \leq 13$. It is not open, since we have $3 \in A$ but there exists no $\varepsilon > 0$ s.t. $(3 - \varepsilon, 3 + \varepsilon) \in A$. To see that A is not closed consider the sequence $(-13 + \frac{1}{n})_n$. It is contained completely in A but its limit -13 is not in A. Since A is not closed, it cannot be compact.

(b) The set B is a subset of the bounded set $(-12, 15]$ and thus itself bounded. We further have

$$B = ([-4, 7] \cap (-12, 15]) \cup ([10, \infty) \cap (-12, 15]) = [-4, 7] \cup [10, 15],$$

so B is the union of two closed sets and hence itself closed (see tutorial). Thus, B is compact. B is not open since e.g. the boundary point -4 is in B (see (a)).

(c) The set C is not closed, since for example the sequence $(2 + \frac{1}{n+1})_n$ is contained completely in C but for the limit 2 we have $2 \notin C$. C is also not bounded since for example the points $a_n = n + \frac{1}{2}$, $n \in \mathbb{N}$, are all in C. Thus, C cannot be compact. Since C has a boundary point $-\frac{7}{2} \in C$, C is not open.

Exercise 32: Consider the function $f : [-2, 2] \to \mathbb{R}$ given by

$$f(x) = 1 - 2x - x^2.$$

Show that there exist a maximum x_+ and a minimum x_-. Determine x_+ and x_-.

Solution 32: As a polynomial the function f is continuous and the set $[-2, 2]$ is bounded and closed, hence compact. Thus, according to Theorem 5.17., f has a minimum and a maximum. To determine the extrema, we rewrite f as

$$f(x) = 2 - (x + 1)^2.$$

Since we have $(x+1)^2 \geq 0$, f is largest at $x_1 = -1$, since we have $(x_1+1)^2 = 0$. We have $f(x_1) = 2-(x_1+1)^2 = 2 - 0$, so we have $x_+ = f(-1) = 2$.
We have

$$f(x) - f(y) = (y + 1)^2 - (x + 1)^2 = (y - x)(y + x + 2).$$

For $-2 \leq y < x \leq -1$ both factors are negative, i.e. $f(x) - f(y) > 0$. Thus, on the interval $[-2, -1]$ is strictly increasing and the smallest value is $f(-2) = 1$.
Similarly we see that f is strictly decreasing on $[-1, 2]$ and the smallest value there is $f(2) = -7$. Thus, we have $x_- = f(2) = -7$.

Exercise 33:

(a) By means of Bolzano's intermediate value theorem show that the set $M = \{\sqrt{(1 - x^2)} : x \in [-1, 1/2)\} \subset \mathbb{R}$ is compact.

(b) How many solutions does the equation

$$2x^5 - 6x^3 + 2x = 4x^4 - 6x^2 + 1$$

have in the interval $I = [-2, 2]$? Justify your answer!
Hint: Find an appropriate function f s.t. the equality can be written as $f(x) = 0$ and evaluate f at the points -2, -1, $\frac{1}{2}$, 1, 2, 3.

Solution 33:

(a) The function $f : [-1, 1/2)$, $f(x) := \sqrt{(1-x^2)}$ is minimal at $x = -1$ with $f(-1) = 0$ and maximal at $x = 0$ with $f(0) = 1$. Since f is continuous as a composition of 2 continuous functions, by the intermediate value theorem it takes all values between $f(-1)$ and $f(0)$. Thus, $M = [0, 1]$, i.e. bounded and closed, and thus compact.

(b) We define the polynomial

$$f(x) = 2x^5 - 4x^4 - 6x^3 + 6x^2 + 2x - 1.$$

By the intermediate value theorem and the values

$$\begin{aligned} f(-2) &= -61 \\ f(-1) &= 3 \\ f(0) &= -1 \\ f(1/2) &= 9/16 \end{aligned} \qquad \begin{aligned} f(1) &= -1 \\ f(2) &= -21 \\ f(3) &= 59 \end{aligned}$$

There exists at least 1 root in each of the intervals $(-2, -1)$, $(-1, 0)$, $(0, 1/2)$, $(1/2, 1)$ and $(2, 3)$. Since there exist at most 5 zeros of this function (Fundamental Theorem of Algebra), the equality has exactly 4 zeros in the interval $[-2, 2]$.

Exercise 34:
Consider the functions f, $g : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -x^2 - 8x - 4, & x \leq -3, \\ x^2 - 3, & x > -3, \end{cases} \qquad \text{and} \qquad g(x) = -x - 2\,, \quad x \in \mathbb{R}.$$

Show that the graphs fo the two functions have at least 3 intersection points.
Hint: You do not need to determine the intersection points.

Solution 34:
We define $h(x) = f(x) - g(x)$, i.e.

$$h(x) = \begin{cases} -x^2 - 8x - 4 + x + 2 = -x^2 - 7x - 2, & x \leq -3, \\ x^2 - 3 + x + 2 = x^2 + x - 1, & x > -3. \end{cases}$$

Since h is not continuous at -3, we cannot apply Corollary 5.19 globally, but we have to restrict h to intervals on which it is continuous.We evaluate h at the points

$$\begin{aligned} h(-7) &= -2, \\ h(-5) &= 8, \end{aligned} \qquad \begin{aligned} h(-2) &= 1, \\ h(0) &= -1, \\ h(1) &= 2. \end{aligned}$$

Then h is continuous on the intervals $[-7, -5]$ and $[-2, 1]$. By Corollary 5.19 we thus have the existence of at least one zero of h in each of the intervals $(-7, -5)$, $(-2, 0)$, $(0, 1)$ and $(1, 2)$. Thus, the graphs of f and g intersect at least thrice.

Exercise 35:
Consider the following series

$$\text{(a)} \quad \left(\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k \right), \qquad \text{(b)} \quad \left(\sum_{k=2}^{\infty} \frac{4k}{(k^2-1)^2} \right).$$

Determine in each case the first three partial sums and find a general representation for the nth partial sum s_n. Do the series converge? If so, what is the sum?
Hint for (b): You may use the identity $\dfrac{4k}{(k+1)^2\,(k-1)^2} = \dfrac{(k+1)^2 - (k-1)^2}{(k+1)^2\,(k-1)^2}$, $k \in \mathbb{N}$.

Solution 35:

(a)

$$s_0 = \left(\frac{2}{3}\right)^0 = 1,$$
$$s_1 = \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 = 1 + \frac{2}{3} = \frac{5}{3},$$
$$s_2 = \left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 = 1 + \frac{2}{3} + \frac{4}{9} = \frac{19}{9}.$$

We have a geometric progression $\sum\limits_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$. Thus, we obtain

$$s_n = \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = \frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}} = \frac{1-\left(\frac{2}{3}\right)^{n+1}}{\frac{1}{3}} = 3\left(1 - \frac{2^{n+1}}{3^{n+1}}\right) = 3 - \frac{2^{n+1}}{3^n}, \quad n \in \mathbb{N}_0.$$

Using the representation $s_n = 3(1 - \left(\frac{2}{3}\right)^{n+1})$ it is easy to see that $s_n \to 3$ $(n \to \infty)$, thus, the series converges to 3.

(b)

$$s_2 = \frac{8}{(4-1)^2} = \frac{8}{9},$$
$$s_3 = s_2 + \frac{12}{(9-1)^2} = \frac{8}{9} + \frac{3}{16} = \frac{155}{144},$$
$$s_4 = s_3 + \frac{16}{(16-1)^2} = \frac{155}{144} + \frac{16}{255} = \frac{459}{400}.$$

Since
$$\frac{4k}{(k^2-1)^2} = \frac{4k}{(k+1)^2\,(k-1)^2} = \frac{(k+1)^2-(k-1)^2}{(k+1)^2\,(k-1)^2} = \frac{1}{(k-1)^2} - \frac{1}{(k+1)^2},$$
we have for the nth partial sum

$$s_n = \sum_{k=2}^{n} \frac{1}{(k-1)^2} - \sum_{k=2}^{n} \frac{1}{(k+1)^2} = \sum_{k=1}^{n-1} \frac{1}{k^2} - \sum_{k=3}^{n+1} \frac{1}{k^2} = 1 + \frac{1}{4} - \frac{1}{n^2} - \frac{1}{(n+1)^2}, \quad n \in \mathbb{N}_{\geq 2}.$$

The last two summands tend to 0, so the series converges and the sum is 5/4.