Exercise Sheet No. 8
– with solutions –

Exercise 36:
Test the following series for convergence. Determine the value of the series in part (a).

$$\text{(a)}\quad \left(\sum_{k=0}^{\infty}\left(\frac{3+4i}{6}\right)^k\right),\qquad \text{(b)}\quad \left(\sum_{k=1}^{\infty}\frac{1}{\sqrt[k]{k+1}}\right),\qquad \text{(c)}\quad \left(\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\right).$$

Solution 36:

(a) We have a geometric series. Since

$$\left|\frac{3+4i}{6}\right| = \frac{\sqrt{9+16}}{6} = \frac{5}{6} < 1,$$

it converges. The value of the series is

$$\sum_{k=0}^{\infty}\left(\frac{3+4i}{6}\right)^k = \frac{1}{1-\frac{3+4i}{6}} = \frac{6}{3-4i} = \frac{6(3+4i)}{25}.$$

(b) We show that the sequence $\left(\frac{1}{\sqrt[k]{k+1}}\right)$ converges to 1, i.e. it does not converge to 0. The series $\left(\sum_{k=1}^{\infty}\frac{1}{\sqrt[k]{k+1}}\right)$ is hence divergent.

The claims follows by the sandwich theorem for limits of sequences, since

$$\frac{1}{\sqrt[k]{k+1}} \geq \frac{1}{\sqrt[k]{2k}} = \frac{1}{\sqrt[k]{2}}\,\frac{1}{\sqrt[k]{k}} \longrightarrow 1 \quad\text{and}\quad \frac{1}{\sqrt[k]{k+1}} \leq \frac{1}{\sqrt[k]{k}} \longrightarrow 1 \quad (k\to\infty).$$

(c) We have

$$\sqrt[k]{\frac{1}{k(k+1)}} = \frac{1}{\sqrt[k]{k}\,\sqrt[k]{k+1}} \longrightarrow 1 \qquad\text{and}\qquad \frac{\frac{1}{(k+1)(k+2)}}{\frac{1}{k(k+1)}} = \frac{k}{k+2} \longrightarrow 1$$

for $k\to\infty$. Thus, we cannot use the ratio or the root test, to discuss convergence of this series.

But since we have

$$\frac{1}{k(k+1)} = \frac{1}{k^2+k} \leq \frac{1}{k^2}$$

and the series $(\sum \frac{1}{k^2})$ converges, by the comparison test the given series also converges.

Exercise 37:
Test the following series for convergence using the ratio test:

$$\text{(a)}\quad \left(\sum_{k=0}^{\infty}\frac{2^k\cdot 3^k}{k!\cdot(2k+1)}\right),\qquad \text{(b)}\quad \left(\sum_{k=1}^{\infty}\frac{k!}{2^k+1}\right),\qquad \text{(c)}\quad \left(\sum_{k=0}^{\infty}\frac{k^k}{k!\cdot t^k}\right)\quad \text{for fixed } t\in\mathbb{N}.$$

Solution 37:

(a) With $a_k = \frac{2^k\cdot 3^k}{k!\cdot(2k+1)}$ we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{k+1}\cdot 3^{k+1}}{(k+1)!\cdot(2k+3)}\cdot\frac{k!\cdot(2k+1)}{2^k\cdot 3^k} = \frac{2\cdot 3\cdot(2k+1)}{(k+1)\cdot(2k+3)} \longrightarrow 0.$$

By the ratio test the series converges.

(b) With $a_k = \frac{k!}{2^k+1}$ we have

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\frac{(k+1)!}{2^{k+1}+1}}{\frac{k!}{2^k+1}} = \frac{\frac{k!(k+1)}{2^{k+1}+1}}{\frac{k!}{2^k+1}} = \frac{(k+1)(2^k+1)}{2^{k+1}+1} = \frac{k2^k+k+2^k+1}{2^{k+1}+1} = \frac{k+\frac{k}{2^k}+1+\frac{1}{2^k}}{2+\frac{1}{2^k}} \longrightarrow \infty > 1$$

By the ratio test the series diverges.

(c) With $a_k = \frac{k^k}{k!\cdot t^k}$ we have

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^{k+1}}{(k+1)!\cdot t^{k+1}} \cdot \frac{k!\cdot t^k}{k^k} = \frac{(k+1)^{k+1}}{k^k\cdot(k+1)\cdot t} = \frac{1}{t}\left(\frac{k+1}{k}\right)^k = \frac{1}{t}\left(1+\frac{1}{k}\right)^k \longrightarrow \frac{e}{t}.$$

Here $e \approx 2.72$ denotes Eulers's constant. Thus, the series converges if $t > e$ (i.e. $t \geq 3$) and diverges for $t < e$, i.e. $t = 1$ and $t = 2$.

Exercise 38:
Test the following series for convergence using the root test:

$$\text{(a)} \quad \left(\sum_{k=1}^{\infty}\left(\frac{3}{4}+\frac{1}{k}\mathrm{i}\right)^k\right), \qquad \text{(b)} \quad \left(\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^2}\frac{1}{2^k}\right), \qquad \text{(c)} \quad \left(\sum_{k=1}^{\infty}(-1)^k\frac{k^3}{3^k}\right).$$

Solution 38:

(a) $\sqrt[k]{\left|\left(\frac{3}{4}+\frac{1}{k}\mathrm{i}\right)^k\right|} = \left|\frac{3}{4}+\frac{1}{k}\mathrm{i}\right| = \sqrt{\frac{9}{16}+\frac{1}{k^2}} \longrightarrow \frac{3}{4} < 1$. By the root test, the series converges (absolutely).

(b) $\sqrt[k]{\left|\left(1+\frac{1}{k}\right)^{k^2}\frac{1}{2^k}\right|} = \left(1+\frac{1}{k}\right)^k\frac{1}{2} \longrightarrow \frac{e}{2} > 1$. As in Problem 37c), $e \approx 2.72$ is Euler's constant. By the root test, the series diverges.

(c) $\sqrt[k]{\left|(-1)^k\frac{k^3}{3^k}\right|} = \frac{\left(\sqrt[k]{k}\right)^3}{3} \longrightarrow \frac{1}{3} < 1$. By the root test, the series converges (absolutely).

Exercise 39:
Use Leibniz' test to show that the following series converge. Moreover find an index N such that for all $n \geq N$ the n^{th} partial sum differs from the value of the series by at most $\frac{1}{100}$.

$$\text{(a)} \quad \left(\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{k^2+3k+2}\right), \qquad \text{(b)} \quad \left(\sum_{k=1}^{\infty}(-1)^{k+1}\left(\sqrt{2k+2}-\sqrt{2k}\right)\right).$$

Solution 39:

(a) Let $a_k = \frac{k+1}{k^2+3k+2} = \frac{k+1}{(k+1)(k+2)} = \frac{1}{k+2}$. Then we have $a_k > 0$ for all $k \in \mathbb{N}$ and $a_k \to 0$ for $k \to \infty$. Further the sequence (a_k) is monotonically decreasing, since

$$a_{k+1} - a_k = \frac{1}{k+3} - \frac{1}{k+2} = \frac{k+2-k-3}{(k+3)(k+2)} = \frac{-1}{(k+3)(k+2)} < 0.$$

Thus, by Leibniz' test the series converges.

Further for every $n \in \mathbb{N}$, for the nth partial sum $s_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ and the value $A = \sum_{k=1}^{\infty} a_k$ we have

$$|s_n - A| \leq a_{n+1} = \frac{1}{n+3}.$$

Thus, for all $n \geq N = 97$, s_n differs from A by at most $\frac{1}{100}$.

(b) Since $\sqrt{2k} - \sqrt{2k+2}$ is negative, we cannot use Leibniz' test. Consider

$$b_k = -\left(\sqrt{2k} - \sqrt{2k+2}\right) = -\left(\sqrt{2k} - \sqrt{2k+2}\right) \frac{\sqrt{2k} + \sqrt{2k+2}}{\sqrt{2k} + \sqrt{2k+2}} = \frac{2}{\sqrt{2k} + \sqrt{2k+2}}.$$

One can see that $b_k > 0$ for all $k \in \mathbb{N}$ and that (b_k) is monotonically decreasing and converges to 0. The monotonicity follows from

$$\begin{aligned} b_{k+1} - b_k &= \frac{2}{\sqrt{2k+2} + \sqrt{2k+4}} - \frac{2}{\sqrt{2k} + \sqrt{2k+2}} = 2\left[\frac{\sqrt{2k} + \sqrt{2k+2} - \sqrt{2k+2} - \sqrt{2k+4}}{(\sqrt{2k+2} + \sqrt{2k+4})(\sqrt{2k} + \sqrt{2k+2})}\right] \\ &= 2\left[\frac{\sqrt{2k} - \sqrt{2k+4}}{(\sqrt{2k+2} + \sqrt{2k+4})(\sqrt{2k} + \sqrt{2k+2})}\right] < 0. \end{aligned}$$

Thus, the series $\sum_{k=1}^{\infty} (-1)^{k+1} b_k$ converges by Leibniz' test and we have

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left(\sqrt{2k} - \sqrt{2k+2}\right) = -\sum_{k=1}^{\infty} (-1)^{k+1} b_k.$$

We further have for all $n \in \mathbb{N}$ for the nth partial sum $s_n = \sum_{k=1}^{n} (-1)^{k+1} b_k$ and the value $B = \sum_{k=1}^{\infty} (-1)^{k+1} b_k$:

$$|s_n - B| \leq b_{n+1} = \frac{2}{\sqrt{2k+2} + \sqrt{2k+4}}.$$

Thus, for $n \geq N = 5000$ we have

$$|s_n - B| \leq b_{n+1} \leq b_N = \frac{2}{\sqrt{10002} + \sqrt{10004}} \leq \frac{2}{\sqrt{10000} + \sqrt{10000}} = \frac{2}{200} = \frac{1}{100}.$$

This estimate also holds for the series under consideration.

Exercise 40:

(a) Show that the series

$$\left(\sum_{k=0}^{\infty} \left(\frac{x-1}{x+1}\right)^k\right), \quad \text{for fixed } x \in \mathbb{R}_{>0}$$

converges and determine the value.

(b) For which $q \in \mathbb{R}$ does the series $\sum_{n=0}^{\infty} (n+1) q^n$ converge?

Solution 40:

(a) We have a geometric series with $q = \frac{x-1}{1+x}$. For $x > 0$ we have $-2x < 2x$, which is equivalent to $1 - 2x + x^2 < 1 + 2x + x^2$, i.e. $(x-1)^2 < (1+x)^2$ and thus, $\left(\frac{x-1}{1+x}\right)^2 < 1$ and $\left|\frac{x-1}{1+x}\right| < 1$. Thus, this geometric series converges and the value is

$$\frac{1}{1-q} = \frac{1}{1 - \frac{x-1}{x+1}} = \frac{x+1}{2}.$$

(b) Root test: $\sqrt[n]{|(n+1)q^n|} = \sqrt[n]{n+1}|q| \overset{n\to\infty}{\longrightarrow} |q|$. Absolute convergence for $|q| < 1$, divergence for $|q| > 1$. For $|q| = 1$ the sequence $(n+1)q^n$ does not converge to 0, so the series is divergent.