Exercise 36:
Test the following series for convergence. Determine the value of the series in part (a).

(a) \( \sum_{k=0}^{\infty} \left( \frac{3 + 4i}{6} \right)^k \), (b) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k + 1}} \), (c) \( \sum_{k=1}^{\infty} \frac{1}{k(k + 1)} \).

Solution 36:
(a) We have a geometric series. Since
\[ \left| \frac{3 + 4i}{6} \right| = \frac{\sqrt{9 + 16}}{6} = \frac{5}{6} < 1, \]
it converges. The value of the series is
\[ \sum_{k=0}^{\infty} \left( \frac{3 + 4i}{6} \right)^k = \frac{1}{1 - \frac{3 + 4i}{6}} = \frac{6(3 + 4i)}{25}. \]

(b) We show that the sequence \( \left( \frac{1}{\sqrt{k + 1}} \right) \) converges to 1, i.e. it does not converge to 0. The series \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k + 1}} \) is hence divergent.

The claims follows by the sandwich theorem for limits of sequences, since
\[ \frac{1}{\sqrt{k + 1}} \leq \frac{1}{\sqrt{2k}} = \frac{1}{\sqrt{2}} \rightarrow 1 \quad \text{and} \quad \frac{1}{\sqrt{k + 1}} \geq \frac{1}{\sqrt{2k}} \rightarrow 1 \quad (k \rightarrow \infty). \]

(c) We have
\[ \sqrt{\frac{1}{k(k + 1)}} = \frac{1}{\sqrt{k} \sqrt{k + 1}} \rightarrow 1 \quad \text{and} \quad \frac{1}{k(k + 1)} = \frac{k}{k + 2} \rightarrow 1 \]
for \( k \rightarrow \infty \). Thus, we cannot use the ratio or the root test, to discuss convergence of this series.

But since we have
\[ \frac{1}{k(k + 1)} = \frac{1}{k^2 + k} \leq \frac{1}{k^2} \]
and the series \( \sum \frac{1}{k^2} \) converges, by the comparison test the given series also converges.

Exercise 37:
Test the following series for convergence using the ratio test:

(a) \( \sum_{k=0}^{\infty} \frac{2^k \cdot 3^k}{k! \cdot (2k + 1)} \), (b) \( \sum_{k=1}^{\infty} \frac{k!}{2^k + 1} \), (c) \( \sum_{k=0}^{\infty} \frac{k^k}{k! \cdot t^k} \) for fixed \( t \in \mathbb{N} \).

Solution 37:
(a) With \( a_k = \frac{2^k \cdot 3^k}{k! \cdot (2k + 1)} \) we have
\[ \left| \frac{a_{k+1}}{a_k} \right| = \frac{2^{k+1} \cdot 3^{k+1}}{(k + 1)! \cdot (2k + 3)} \cdot \frac{k! \cdot (2k + 1)}{2^k \cdot 3^k} = \frac{2 \cdot 3 \cdot (2k + 1)}{(k + 1) \cdot (2k + 3)} \rightarrow 0. \]

By the ratio test the series converges.
(b) With \( a_k = \frac{k^1}{2^k+1} \) we have

\[
\frac{|a_{k+1}|}{a_k} = \frac{(k+1)!}{2^{k+1}+1} \frac{k!(k+1)}{k!} = \frac{(k+1)(2^k+1)}{2^{k+1}+1} = \frac{k2^k+k+2^k+1}{2^{k+1}+1} = \frac{k + \frac{k}{2} + 1 + \frac{1}{2^k}}{2 + \frac{1}{2^k}} \to \infty \geq 1
\]

By the ratio test the series diverges.

(c) With \( a_k = \frac{k^k}{k!} \) we have

\[
\frac{|a_{k+1}|}{a_k} = \frac{(k+1)^{k+1}}{(k+1)!} \frac{k!}{k^k} = \frac{(k+1)^{k+1}}{k^k} \frac{1}{(k+1)!} \frac{k!}{k^k} = \frac{1}{t} \left( \frac{k+1}{k} \right)^k = \frac{1}{t} \left( 1 + \frac{1}{k} \right)^k \to e^t.
\]

Here \( e \approx 2.72 \) denotes Euler’s constant. Thus, the series converges if \( t > e \) (i.e. \( t \geq 3 \)) and diverges for \( t < e \), i.e. \( t = 1 \) and \( t = 2 \).

Exercise 38:

Test the following series for convergence using the root test:

\[
\text{(a) } \left( \sum_{k=1}^{\infty} \left( \frac{3}{4} + \frac{1}{k} \right)^k \right), \quad \text{(b) } \left( \sum_{k=1}^{\infty} \left( 1 + \frac{1}{k} \right)^{k^2} \frac{1}{2^k} \right), \quad \text{(c) } \left( \sum_{k=1}^{\infty} (-1)^k \frac{k^3}{3^k} \right).
\]

Solution 38:

(a) \( \sqrt[k]{\left( \frac{3}{4} + \frac{1}{k} \right)^k} = \left| \frac{3}{4} + \frac{1}{k} \right| = \sqrt[k]{\frac{9}{16}} + \frac{1}{k^2} \to \frac{3}{4} < 1 \). By the root test, the series converges (absolutely).

(b) \( \sqrt[k]{\left( 1 + \frac{1}{k} \right)^{k^2} \frac{1}{2^k}} = \left| 1 + \frac{1}{k} \right| \frac{1}{2^k} \to \frac{1}{2} > 1 \). As in Problem 37c), \( e \approx 2.72 \) is Euler’s constant. By the root test, the series diverges.

(c) \( \sqrt[k]{\left| (-1)^k \frac{k^3}{3^k} \right|} = \left( \frac{\sqrt[k]{k^3}}{3} \right)^3 \to \frac{1}{3} < 1 \). By the root test, the series converges (absolutely).

Exercise 39:

Use Leibniz’ test to show that the following series converge. Moreover find an index \( N \) such that for all \( n \geq N \) the \( n^{th} \) partial sum differs from the value of the series by at most \( \frac{1}{100} \).

\[
\text{(a) } \left( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k^2+3k+2} \right), \quad \text{(b) } \left( \sum_{k=1}^{\infty} (-1)^{k+1} \left( \sqrt{2k+2} - \sqrt{2k} \right) \right).
\]

Solution 39:

(a) Let \( a_k = \frac{k+1}{k^2+2k+2} = \frac{k+1}{(k+1)(k+2)} = \frac{1}{k+2} \). Then we have \( a_k > 0 \) for all \( k \in \mathbb{N} \) and \( a_k \to 0 \) for \( k \to \infty \). Further the sequence \( (a_k) \) is monotonically decreasing, since

\[
a_{k+1} - a_k = \frac{1}{k+3} - \frac{1}{k+2} = \frac{k+2 - k - 3}{(k+3)(k+2)} = \frac{-1}{(k+3)(k+2)} < 0.
\]

Thus, by Leibniz’ test the series converges.

Further for every \( n \in \mathbb{N} \), for the \( n^{th} \) partial sum \( s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k \) and the value \( A = \sum_{k=1}^{\infty} a_k \) we have

\[
|s_n - A| \leq a_{n+1} = \frac{1}{n+3}.
\]

Thus, for all \( n \geq N = 97 \), \( s_n \) differs from \( A \) by at most \( \frac{1}{100} \).
(b) Since \(\sqrt{2k} - \sqrt{2k+2}\) is negative, we cannot use Leibniz’ test. Consider

\[
b_k = -\left(\sqrt{2k} - \sqrt{2k+2}\right) = -\left(\sqrt{2k} - \sqrt{2k+2}\right) \frac{\sqrt{2k} + \sqrt{2k+2}}{\sqrt{2k} + \sqrt{2k+2}} = \frac{2}{\sqrt{2k} + \sqrt{2k+2}}.
\]

One can see that \(b_k > 0\) for all \(k \in \mathbb{N}\) and that \((b_k)\) is monotonically decreasing and converges to 0. The monotonicity follows from

\[
b_{k+1} - b_k = \frac{2}{\sqrt{2k} + \sqrt{2k+2} + \sqrt{2k+4}} - \frac{2}{\sqrt{2k} + \sqrt{2k+2}} = 2 \left[\frac{\sqrt{2k} + \sqrt{2k+2} - \sqrt{2k+2} - \sqrt{2k+4}}{(\sqrt{2k} + \sqrt{2k+2})(\sqrt{2k} + \sqrt{2k+4})}\right] < 0.
\]

Thus, the series \(\sum_{k=1}^{\infty} (-1)^{k+1}b_k\) converges by Leibniz’ test and we have

\[
\sum_{k=1}^{\infty} (-1)^{k+1} \left(\sqrt{2k} - \sqrt{2k+2}\right) = -\sum_{k=1}^{\infty} (-1)^{k+1}b_k.
\]

We further have for all \(n \in \mathbb{N}\) for the \(n\)th partial sum \(s_n = \sum_{k=1}^{n} (-1)^{k+1}b_k\) and the value \(B = \sum_{k=1}^{\infty} (-1)^{k+1}b_k\):

\[
|s_n - B| \leq b_{n+1} = \frac{2}{\sqrt{2k} + \sqrt{2k+2} + \sqrt{2k+4}}.
\]

Thus, for \(n \geq N = 5000\) we have

\[
|s_n - B| \leq b_{n+1} \leq b_N = \frac{2}{\sqrt{10002} + \sqrt{10004}} \leq \frac{2}{\sqrt{10000} + \sqrt{10000}} = \frac{2}{200} = \frac{1}{100}.
\]

This estimate also holds for the series under consideration.

**Exercise 40:**

(a) Show that the series

\[
\left(\sum_{k=0}^{\infty} \left(\frac{x-1}{x+1}\right)^k\right), \text{ for fixed } x \in \mathbb{R}_{>0}
\]

converges and determine the value.

(b) For which \(q \in \mathbb{R}\) does the series \(\sum_{n=0}^{\infty} (n+1)q^n\) converge?

**Solution 40:**

(a) We have a geometric series with \(q = \frac{x-1}{x+1}\). For \(x > 0\) we have \(-2x < 2x\), which is equivalent to \(1 - 2x + x^2 < 1 + 2x + x^2\); i.e. \((x - 1)^2 < (1 + x)^2\) and thus, \(\left(\frac{x-1}{x+1}\right)^2 < 1\) and \(|\frac{x-1}{x+1}| < 1\). Thus, this geometric series converges and the value is

\[
1 - q = \frac{1}{1 - \frac{x-1}{x+1}} = \frac{x + 1}{2}.
\]

(b) Root test: \(\sqrt[n]{|(n+1)q^n|} = \sqrt[n]{n + 1|q|^n} \xrightarrow{n \to \infty} |q|\). Absolute convergence for \(|q| < 1\), divergence for \(|q| > 1\). For \(|q| = 1\) the sequence \((n+1)q^n\) does not converge to 0, so the series is divergent.