Exercise Sheet No. 9
– with solutions –

Exercise 41:
For which $x \in \mathbb{R}$ does the power series

$$\sum_{n=1}^{\infty} (-2)^n \frac{n^2+2}{n^3+n} x^{3n}$$

converge?

Solution 41:
For $x = 0$ the series converges . For $x \neq 0$ we have $a_n := (-2)^n \frac{n^2+2}{n^3+n} x^{3n} \neq 0$ and the ratio test yields:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1} \frac{(n+1)^2+2}{(n+1)^3+(n+1)} x^{3n+3}}{(-2)^n \frac{n^2+2}{n^3+n} x^{3n}}\right| = \frac{(n+1)^2+2}{(n+1)^3+n+1} \cdot \frac{n^3+n}{n^2+2} |x|^3 \overset{n\to\infty}{\longrightarrow} 2|x|^3 \overset{!}{<} 1$$

$\Rightarrow |x|^3 < \frac{1}{2} \Rightarrow$ So the series converges for $x \in \left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$, and diverges for $|x| > \frac{1}{\sqrt[3]{2}}$.
Boundary points:

- $x = \frac{1}{\sqrt[3]{2}}$. We have $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2}{n^3+n}$ converges if $\frac{n^2+2}{n^3+n}$ monotone by Leibniz'. Indeed the sequence $\left|\frac{n^2+2}{n^3+n}\right| \leq \frac{n^2+n^2}{n^3} = \frac{2}{n}$, tends to 0 since it is majorized by $2/n$. Let $X = [(n+1)^3+n+1] \cdot [n^3+n]$. $\frac{(n+1)^2+2}{(n+1)^3+n+1} - \frac{n^2+2}{n^3+n} = \frac{(n^3+n)(n^2+2n+3)-(n^2+2)(n^3+3n^2+4n+2)}{\text{X}} = \frac{-n^4-2n^3-6n^2-5n-4}{\text{X}} < 0$, for all n, so the sequence is monotone.

- $x = -\frac{1}{\sqrt[3]{2}}$. The series $\sum_{n=1}^{\infty} \frac{n^2+2}{n^3+n} \geq \sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n} \geq \sum_{n=1}^{\infty} \frac{(n^2+1)}{n(n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

So the series converges for $x \in \left(-\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right]$.

Exercise 42:
For which $x \in \mathbb{R}$ does the power series

$$\sum_{n=0}^{\infty} \frac{n+1}{\sqrt{8^n}} \binom{n}{2} (x-2)^{3n}$$

converge?

Solution 42:
First note that we have $(n+1)\binom{n}{2} = \frac{(n+1)!}{2!(n-2)!}$. We apply the ratio test to $a_n := \frac{n+1}{\sqrt{8^n}} \binom{n}{2} (x-2)^{3n}$.

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(n+2)! \cdot 2! \cdot (n-2)! \cdot \sqrt{8^n} \cdot |(x-2)^{3n+3}|}{(n+1)! \cdot 2! \cdot (n-1)! \cdot \sqrt{8^{n+1}} \cdot |(x-2)^{3n}|} \\ &= \lim_{n\to\infty} |(x-2)^3| \frac{n+2}{n-1} \frac{1}{\sqrt{8}} \\ &= \frac{|(x-2)^3|}{\sqrt{8}} \overset{!}{<} 1 \end{aligned}$$

Thus, we habe absolute convergence for $|(x-2)^3| < \sqrt{8}$ which is equivalent to $|x-2| < \sqrt{2}$, and divergence for $|x-2| > \sqrt{2}$.
It remains to check the two points $x = 2 \pm \sqrt{2}$: For $x = 2+\sqrt{2}$ we have $\frac{(x-2)^{3n}}{\sqrt{8^n}} = \frac{\sqrt{2}^{3n}}{\sqrt{8^n}} = 1$, at $x = 2-\sqrt{2}$ we have $\frac{(x-2)^{3n}}{\sqrt{8^n}} = \frac{(-\sqrt{2})^{3n}}{\sqrt{8^n}} = (-1)^n$. The remainder of the series at $x = 2+\sqrt{2}$ is

$$\left(\sum_{n=0}^{\infty} \frac{(n-1) \cdot n \cdot (n+1)}{2}\right).$$

Since the sequence over which we sum does not converge to 0, the series is not convergent. Similarly the series diverges for $x = 2 - \sqrt{2}$, since the sequence there also does not converge to 0.
In conclusion, the power series converges exactly for $x \in (2 - \sqrt{2}, 2 + \sqrt{2})$.

Exercise 43:
Consider the function $f : \left\{ z \in \mathbb{C} : \ |z| \leq \sqrt{2} \right\} \to \mathbb{C}$ given by:

$$f(z) = \frac{z - 1}{z^2 + 2}.$$

Express f as a power series $\sum\limits_{n=0}^{\infty} a_n z^n$. Moreover, find the radius of convergence of this power series.

Solution 43:
We have

$$\begin{aligned} z - 1 = (z^2 + 2) \sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} a_n z^{n+2} + \sum_{n=0}^{\infty} 2 a_n z^n \\ &= \sum_{n=2}^{\infty} a_{n-2} z^n + \sum_{n=0}^{\infty} 2 a_n z^n = 2a_0 + 2a_1 z + \sum_{n=2}^{\infty} \left(a_{n-2} + 2a_n \right) z^n. \end{aligned}$$

Now by comparing the coefficients we obtain:

$$a_0 = -\frac{1}{2}, \qquad a_1 = \frac{1}{2}, \qquad a_{n-2} + 2a_n = 0 \Leftrightarrow a_n = -\frac{1}{2}\, a_{n-2}, \quad n \geq 2.$$

By induction one can obtain the representation

$$a_{2k} = \left(-\frac{1}{2} \right)^{k+1}, \qquad a_{2k+1} = -\left(-\frac{1}{2} \right)^{k+1}, \qquad k \in \mathbb{N}_0.$$

To obtain the radius of convergence, we use the root test. Consider $\sqrt[n]{|a_n z^n|}$:

$$\begin{aligned} \sqrt[2k]{|a_{2k}\, z^{2k}|} &= \frac{1}{\sqrt[2k]{2^{k+1}}} \, |z| = 2^{-\frac{k+1}{2k}} |z| = 2^{-\frac{1+1/k}{2}} |z| \longrightarrow \frac{1}{\sqrt{2}} |z| \quad (k \to \infty), \\ \sqrt[2k+1]{|a_{2k+1}\, z^{2k+1}|} &= \frac{1}{\sqrt[2k+1]{2^{k+1}}} \, |z| = 2^{-\frac{k+1}{2k+1}} |z| = 2^{-\frac{1+1/k}{2+1/k}} |z| \longrightarrow \frac{1}{\sqrt{2}} |z| \quad (k \to \infty). \end{aligned}$$

Thus, the sequence $(\sqrt[n]{|a_n z^n|})$ is convergent with limit $|z|/\sqrt{2}$. For absolute convergence this limit has to be smaller than 1, i.e. $|z| < \sqrt{2}$. Thus, the radius of convergence is $\sqrt{2}$.

Observation: As a function on $\mathbb{R}$, f is defined everywhere. Why does the power series only have radius of convergence $\sqrt{2}$? This is due to the complex zeros of the denominator!

Exercise 44:
Determine the limit

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{\exp\left(x^4\right) - 1}$$

by substituting the respective power series for cos and exp into the given expression.

Solution 44:
We obtain the power series with centre of expansion 0 for denominator and numerator:

$$\begin{aligned} \cos x - 1 + \frac{x^2}{2} &= -1 + \frac{x^2}{2} + \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!} = \sum_{n=2}^{\infty} (-1)^n \, \frac{x^{2n}}{(2n)!} \overset{m=n-2}{=} x^4 \sum_{m=0}^{\infty} (-1)^m \, \frac{x^{2m}}{(2m+4)!}, \\ \exp(x^4) - 1 &= -1 + \sum_{n=0}^{\infty} \frac{\left(x^4\right)^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{4n}}{n!} \overset{m=n-1}{=} x^4 \sum_{m=0}^{\infty} \frac{x^{4m}}{(m+1)!}. \end{aligned}$$

For the quotient we then have

$$\frac{\cos x - 1 + \frac{x^2}{2}}{\exp(x^4) - 1} = \frac{\sum_{m=0}^{\infty} (-1)^n \frac{x^{2m}}{(2m+4)!}}{\sum_{m=0}^{\infty} \frac{x^{4m}}{(m+1)!}} = \frac{\frac{1}{4!} + \sum_{m=1}^{\infty} (-1)^m \frac{x^{2m}}{(2m+4)!}}{\frac{1}{1!} + \sum_{m=1}^{\infty} \frac{x^{4m}}{(m+1)!}}$$

for every x in the disc of convergence of both series. By the ratio test it is easy to see, that both series have radius of convergence ∞. According to Thm 6.13 (lecture notes), both functions are continuous and thus, we can interchange $\sum$ and lim. In particular, for $x = 0$ we obtain

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{\exp(x^4) - 1} = \frac{\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}}{\lim_{x\to 0} \frac{\exp(x^4)-1}{x^4}} = \frac{\frac{1}{4!}}{\frac{1}{1!}} = \frac{1}{2\cdot 3\cdot 4} = \frac{1}{24}.$$

Exercise 45:
Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \cos(z) - 2.$$

Determine all zeros of f.

Solution 45: We want to find solutions of the equation $\cos(z) - 2 = 0$. With $\cos(z) = \frac{1}{2}(e^{\mathrm{i}z} + e^{-\mathrm{i}z})$ we have

$$\cos(z) - 2 = 0 \Longleftrightarrow e^{\mathrm{i}z} + e^{-\mathrm{i}z} = 4.$$

We substitute $u = e^{\mathrm{i}z}$:

$$u + u^{-1} - 4 = 0 \Leftrightarrow u^2 - 4u + 1 = 0 \Leftrightarrow (u-2)^2 - 3 = 0 \Leftrightarrow (u-2)^2 = 3.$$

With $w := u - 2$ we obtain

$$\operatorname{Re} w = \pm\sqrt{3}, \quad \operatorname{Im} w = 0.$$

and $u = 2 \pm \sqrt{3}$. For $z = x + \mathrm{i}y$ we have

$$2 \pm \sqrt{3} = e^{\mathrm{i}z} = e^{\mathrm{i}(x+\mathrm{i}y)} = e^{-y+\mathrm{i}x} = e^{-y}(\cos x + \mathrm{i}\sin x) \Rightarrow \begin{cases} e^{-y}\cos x = 2 \pm \sqrt{3} \\ e^{-y}\sin x = 0 \end{cases}$$

$$\Rightarrow \sin x = 0 \Rightarrow x = k\pi \quad (k \in \mathbb{Z}).$$

If k is even, we have $\cos(k\pi) = 1$. In this case we have

$$e^{-y} = 2 \pm \sqrt{3} \Rightarrow y = -\ln(2 \pm \sqrt{3}),$$

where ln denotes the natural logarithm for positive real numbers.
If k is odd, we have $\cos(k\pi) = -1$. In this case we obtain

$$e^{-y} = -2 \mp \sqrt{3} < 0.$$

This can never hold (there is no solution y to that equation). Thus, k must be even, i.e. $k = 2m$ for $m \in \mathbb{Z}$. Thus, we have the solution set

$$\{2m\pi - \ln(2 \pm \sqrt{3})\mathrm{i}\,,\ m \in \mathbb{Z}\},$$

which is just the set of zeros of the function f.