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Selected Topics in Geometric Group Theory

Problem Sheet 3

Problem 1 *Euler characteristic and orientations*

Recall that the Euler characteristic $\chi(X)$ of a triangulated surface X with triangles T, edges E and vertices V is the alternating sum

$$\chi(X) = |T| - |E| + |V|.$$

(a) The Klein bottle K is the quotient of $S^1 \times I$ by the relation $(z, 0) \sim (\frac{1}{z}, 1)$, where we consider S^1 as a subset of $\mathbb{C}$. Find suitable triangulations of the sphere S^2, the torus T and the Klein bottle K. Compute the Euler characteristic of the surfaces S^2, T and K.

(b) Construct for each of the above surfaces an orientation as defined in the lectures or prove that there cannot possibly exist such an orientation.

Problem 2 $\pi_1(-)$, *the basepoint and homotopy equivalences*

Let X and Y be path connected topological spaces. Recall that a homotopy equivalence $f\colon X \to Y$ is a map for which there is a homotopy inverse $g\colon Y \to X$, i.e. a map g, such that $f \circ g$ is homotopic to the identity of Y and such that $g \circ f$ is homotopic to the identity of X.

(a) Construct an isomorphism $\pi_1(X, x_1) \cong \pi_1(X, x_2)$ for any two basepoints $x_1, x_2 \in X$. Deduce that the abelianisation[1] $H_1(X, x)$ of $\pi_1(X, x)$ is independent of the basepoint $x \in X$.

(b) Prove that $\pi_1(X, x) \cong \pi_1(Y, y)$ for any two basepoints $x \in X$ and $y \in Y$ whenever there exists a homotopy equivalence $f\colon X \to Y$.

(c) Prove that any homotopy equivalence induces a natural map $H_1(X) \to H_1(Y)$.

(d) Explain, possibly employing an appropriate example, why not even a homeomorphism $f\colon X \to X$ induces a natural automorphism of $\pi_1(X, x)$.

[1] The abelianisation of a group G is the quotient of G by its commutator subgroup.

Theorem

Let X be a topological space covered by open subsets U and V, where U, V and $U \cap V$ are path connected. Furthermore, let $x \in U \cap V$. The square

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \longrightarrow & \pi_1(U, x) \\ \downarrow & & \downarrow \\ \pi_1(V, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

is a push-out in the category of groups.

Problem 3 *The theorem of Seifert and van Kampen*

This problem is concerned with the above theorem by Herbert Seifert and Egbert van Kampen.

(a) Translate the statement of the theorem of Seifert and van Kampen into less fancy language. Hint: Look up the notion of a presentation of a group.

(b) Explain why the requirement $x \in U \cap V$ is necessary.

(c) Give an example that demonstrates the necessity of $U \cap V$ being path connected.

(d) Compute the fundamental group $\pi_1(K, x)$ of the Klein bottle.

(e) Prove that there is a homotopy equivalence from the once-punctured torus $T - \{x\}$ to the graph with only a single vertex and two edges.

(f) Recall from the lecture that a surface X of genus g is the connected sum $S^2 \# T \# \ldots \# T$ of a sphere S^2 and g tori T. Compute a presentation of the fundamental group $\pi_1(X, x)$ of a surface X of genus g. Hint: Use parts of problem 2 and part (e).