Problem 1  Additive subgroups of $\mathbb{C}$
Recall that a subgroup $H$ of a topological group $G$ is discrete if the subspace topology on $H \subseteq G$ is the discrete topology on $H$, that is, if each subset $U \subseteq H$ is open with respect to this topology.

Let $A \subseteq \mathbb{C}$ be a subgroup of the additive group $(\mathbb{C}, +)$. Show that $A$ is discrete if and only if $A$ is generated by at most two elements that are linearly independent over $\mathbb{R}$.

Problem 2  Moebius transformations
(a) Let $A, B \in \text{SL}_2(\mathbb{C})$ and denote by $\gamma_A$ and $\gamma_B$ their associated Moebius transformations $\mathbb{P}^1 \mathbb{C} \to \mathbb{P}^1 \mathbb{C}$.
Show that $\gamma_A$ and $\gamma_B$ commute if and only if they have the same fixed points.

(b) Now let $A, B \in \text{SL}_2(\mathbb{R})$ and let $\gamma_A$ and $\gamma_B$ denote their associated Moebius transformations $\mathbb{H} \to \mathbb{H}$.
Suppose that $\gamma_A$ and $\gamma_B$ have either one or two fixed points in common. Prove that $A$ and $B$ generate a discrete subgroup if and only if they have a common power.

Problem 3  Rigidity of surface groups
Let $X$ and $Y$ denote two connected surfaces, such that $\pi_1(X) \cong \pi_1(Y)$. Are $X$ and $Y$ homeomorphic? If not, give criteria for when you can conclude that they are.

Problem 4  Teichmüller space and the representation variety of $\pi_1(S_g)$
In the lectures we made use of a map

$$\iota: T(S_g) \to \text{Rep}^*(\pi_1(S_g), \text{PSL}_2(\mathbb{R}))$$

$$[X, f] \mapsto [\rho_{X,f}],$$

where $\rho_{X,f}$ was given as the composition

$$\pi_1(S_g, *) \xrightarrow{f_*} \pi_1(X, *) \cong \text{Deck}(\mathbb{H}/X) \hookrightarrow \text{PSL}_2(\mathbb{R}).$$

Prove that $\iota$ is indeed well-defined and injective.