Karlsruher Institut für Technologie

Institute for Algebra and Geometry
JProf. Dr. Gabriela Weitze-Schmithüsen
Dipl.-Inf. Tobias Columbus

Selected Topics in Geometric Group Theory

Problem Sheet 10

Problem 1 *Additive subgroups of $\mathbb{C}$*

Recall that a subgroup H of a topological group G is discrete if the subspace topology on $H \subseteq G$ is the discrete topology on H, that is, if each subset $U \subseteq H$ is open with respect to this topology.

Let $A \subseteq \mathbb{C}$ be a subgroup of the additive group $(\mathbb{C}, +)$. Show that A is discrete if and only if A is generated by at most two elements that are linearly independent over $\mathbb{R}$.

Problem 2 *Moebius transformations*

(a) Let $A, B \in \mathrm{SL}_2(\mathbb{C})$ and denote by γ_A and γ_B their associated Moebius transformations $\mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C}$. Show that γ_A and γ_B commute if and only if they have the same fixed points.

(b) Now let $A, B \in \mathrm{SL}_2(\mathbb{R})$ and let γ_A and γ_B denote their associated Moebius transformations $\mathbb{H} \to \mathbb{H}$. Suppose that γ_A and γ_B have either one or two fixed points in common. Prove that A and B generate a discrete subgroup if and only if they have a common power.

Problem 3 *Rigidity of surface groups*

Let X and Y denote two connected surfaces, such that $\pi_1(X) \cong \pi_1(Y)$. Are X and Y homeomorphic? If not, give criteria for when you can conclude that they are.

Problem 4 *Teichmüller space and the representation variety of $\pi_1(S_g)$*

In the lectures we made use of a map

$$\begin{aligned} \iota\colon \mathcal{T}(S_g) &\to \mathrm{Rep}^*\Big(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R})\Big) \\ [X, f] &\mapsto [\rho_{X,f}]\,, \end{aligned}$$

where $\rho_{X,f}$ was given as the composition

$$\pi_1(S_g, *) \xrightarrow{f_*} \pi_1(X, *) \xrightarrow{\cong} \mathrm{Deck}(\mathbb{H}/X) \hookrightarrow \mathrm{PSL}_2(\mathbb{R})\,.$$

Prove that ι is indeed well-defined and injective.