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Selected Topics in Geometric Group Theory

Problem Sheet 11

Problem 1 *Complex of Curves*
Let X be a surface. Extending our definition of the graph of curves $\Gamma_C(X)$, we define the *complex of curves* $C(X)$ as the (abstract) simplicial complex whose vertices are isotopy classes of essential curves and whose p-simplices are $p+1$ distinct isotopy classes that have mutually non-intersecting representatives.

(a) Is $C(X)$ of finite dimension? If so, can you give a formula for $\dim C(X)$ in terms of invariants of X?

(b) Let us call a subcomplex $D \subseteq C(X)$ *rigid*, if any locally injective[1] simplicial map $j\colon D \to C(X)$ is the restriction of a map $g\colon C(X) \to C(X)$.

Give an example of a surface X and a finite subcomplex $D \subseteq C(X)$ that is not rigid in the above sense.

Problem 2 *Groups and quasi-isometries*
Let (X,d) and (Y,e) be metric spaces. A map $f\colon X \to Y$ is a *quasi-isometry*, if there are positive constants $a, b, c \in \mathbb{R}$, such that

$$\frac{1}{a} \cdot d(x,y) - b \le e(f(x), f(y)) \le a \cdot d(x,y) + b$$

for all $x, y \in X$ and such that for all $y \in Y$ there is some $x \in X$ with $e(y, f(x)) < c$.

(a) Show that the Cayley graph $\Gamma = \Gamma(G,S)$ of an infinite but finitely generated group G depends only up to quasi-isometry on the finite set $S \subseteq G$ of generators.

(b) Let $G \subseteq H$ be a subroup of finite index in a finitely generated group H. Prove that $\Gamma(G)$ and $\Gamma(H)$ are quasi-isometric.

Problem 3 *The upper halfplane*
Prove that the upper halfplane $\mathbb{H}$ equipped with the Poincaré-metric is hyperbolic.

[1] For a simplicial complex D, the star $\operatorname{st}(x)$ of a vertex x is the set of faces that contain x. A map $f\colon D \to C$ of simplicial complexes is called *locally injective*, if its restriction to the star $\operatorname{st}(x)$ of any vertex $x \in D$ is injective.

Problem 4 *Fricke coordinates*

Recall that we have defined a map $\iota\colon \mathcal{T}(S_g) \to \operatorname{Rep}^*\big(\pi_1(S_g), \operatorname{PSL}_2(\mathbb{R})\big)$. Let α_i and β_i denote the usual generators of $\pi_1(S_g)$. We have thus associated to each point $[(X, f)] \in \mathcal{T}(S_g)$ a family of $2g$ Moebius transformations

$$\iota(\alpha_i) = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \qquad\qquad \iota(\beta_i) = \begin{pmatrix} a_i' & b_i' \\ c_i' & d_i' \end{pmatrix},$$

which we assume to be normalised, such that the following conditions are satisfied:

(i) $\iota(\beta_g)$ has its attracting fixed point at 0 and its repelling fixed point at ∞.

(ii) $\iota(\alpha_g)$ has its attracting fixed point at 1.

(iii) $a_i d_i - b_i c_i = 1$ and $a_i' d_i' - b_i' c_i' = 1$ for all i.

Assure yourself that such a normalisation is indeed possible. and prove that

$$[(X, f)] \mapsto (a_1, c_1, d_1, a_1', c_1', d_1', \ldots, a_{g-1}, c_{g-1}, d_{g_1}, a_{g-1}', c_{g-1}', d_{g-1}'),$$

defines an injective map $F\colon \mathcal{T}(S_g) \to \mathbb{R}^{6g-6}$.