Karlsruher Institut für Technologie

Institute for Algebra and Geometry
JProf. Dr. Gabriela Weitze-Schmithüsen
Dipl.-Inf. Tobias Columbus

Selected Topics in Geometric Group Theory

Problem Sheet 3

Problem 1 *Homeomorphisms of the universal cover and deck transformations*

Let B be a path connected topological space and let $p\colon E \to B$ be a cover of B. Denote by $\mathrm{Homeo}(E/B)$ the group of homeomorphisms of E that descend to homeomorphisms of B and let $D = \mathrm{Aut}(p)$ denote the group of deck transformations.

(a) Give a precise definition of $\mathrm{Homeo}(E/B)$.

(b) Prove that $f.\sigma = f \circ \sigma \circ f^{-1}$ defines an action of $\mathrm{Homeo}(E/B)$ on the group $\mathrm{Aut}(p)$ of deck transformations.

(c) Now, let $p\colon E \to B$ be the universal cover of B and let $\Psi\colon \mathrm{Aut}(D) \to \mathrm{Out}(\pi_1(B))$ denote the natural surjection. Prove that the action of $\mathrm{Homeo}(E/B)$ on $\mathrm{Aut}(D)$ is compatible with the natural outer action $\psi\colon \mathrm{Homeo}(B) \to \mathrm{Out}(\pi_1(B))$ in the sense that the diagram

$$\begin{array}{ccc} \mathrm{Homeo}(E/B) & \longrightarrow & \mathrm{Aut}(D) \\ \downarrow & & \downarrow \Psi \\ \mathrm{Homeo}(B) & \underset{\psi}{\longrightarrow} & \mathrm{Out}(\pi_1(B)) \end{array}$$

commutes.

Problem 2 *Coverings, graphs and free groups*

Let $p\colon E \to B$ be a cover with path connected base B and path connected total space E. Let $F = p^{-1}(b)$ be the fibre of p over b and let $e \in F$.

(a) Show that there is a bijection $F \cong \pi_1(B,b)/p_*\pi_1(E,e)$ between the fibre F and the set of cosets of $p_*\pi_1(E,e)$ in $\pi_1(B,b)$.

(b) Prove that the natural action of $\mathrm{Aut}(p)$ on F is transitive if and only if $p_*\pi_1(E,e)$ is a normal subgroup of $\pi_1(B,b)$.

Let us now consider the universal cover of the graph consisting of a single vertex and n loops only.

(c) Using one of the equivalent characterisations of the universal cover, prove that E is a $2n$-regular infinite tree, i.e. a tree whose vertices have valency $2n$ each.

(d) Give an explicit description of the isomorphism between $\mathrm{Aut}(p)$ and the fundamental group $\pi_1(B,b)$.

(e) Prove that any non-trivial normal subgroup N of a free group F that has infinite index is not finitely generated.

Problem 3 *Homeomorphisms of the circle*

Fix a generator of $\pi_1(S^1) \cong \mathbb{Z}$. Define the degree $\deg(f)$ of a continuous map $f\colon S^1 \to S^1$ to be the unique integer $d \in \mathbb{Z}$, such that $f_*\colon \pi_1(S^1) \to \pi_1(S^1)$ is given by multiplication with d.

(a) Prove that $\deg(f) \in \{\pm 1\}$ for all homeomorphisms $f\colon S^1 \to S^1$. Let us call $f \in \mathrm{Homeo}(S^1)$ orientation preserving[1] if $\deg(f) = 1$. This gives rise to the notion of a mapping class group $\mathrm{Mod}(S^1)$ of S^1.

(b) As usual, denote by $\mathrm{Homeo}^+(S^1)$ the space of orientiation preserving homeomorphisms $S^1 \to S^1$. Each $\lambda \in S^1 \subseteq \mathbb{C}$ induces a homeomorphism f_λ of S^1 given by $z \mapsto \lambda z$. Prove that $\lambda \to f_\lambda$ defines an injection $S^1 \to \mathrm{Homeo}^+(S^1)$ of topological groups[2].

(c) Denote by Homeo^+_* the subspace of $\mathrm{Homeo}^+(S^1)$ containing those homeomorphisms of S^1 that fix the basepoint $1 \in S^1$. Prove that there is an isomorphism $S^1 \times \mathrm{Homeo}^+_*(S^1) \to \mathrm{Homeo}^+(S^1)$ of topological groups.

(d) Prove that $\mathrm{Homeo}^+_*(S^1)$ is contractible and deduce that $S^1 \to \mathrm{Homeo}^+(S^1)$ is a homotopy equivalence. What conclusions can you draw about the mapping class group $\mathrm{Mod}(S^1)$?
Hint: Consider S^1 as a quotient of the unit interval $[0, 1]$.

(e) Finally, prove that $\mathrm{Homeo}(S^1)$ consists of two connected components, each of them homotopy equivalent to S^1, and that $\mathrm{Homeo}(S^1)$ is homotopy equivalent to the orthogonal group $\mathrm{O}(2)$. What conclusions can you draw about $\mathrm{Mod}^\pm(S^1)$?

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[1] We will probably discuss this notion of orientation in the problem sessions
[2] Look up the notion of a topological group if necessary