Problem 1  Euler characteristic of the connected sum
Give a formula relating the Euler characteristic $\chi(X \# Y)$ of the connected sum $X \# Y$ of two surfaces $X$ and $Y$ and prove your formula to be correct.

Problem 2  Glueing of surfaces along circles
Let $X$ and $Y$ be surfaces with boundary components $S^1 = c_1 \subseteq \partial X$ and $S^1 = c_2 \subseteq \partial Y$. Furthermore, let $f: c_1 \to c_2$ be a homeomorphism and let $Z$ be the surface obtained from $X$ and $Y$ by glueing them along $f$, that is, $Z$ is the quotient of $X \sqcup Y$ by the equivalence relation generated by $x \sim f(x)$ for $x \in c_1$.

(a) Construct a natural structure of a surface with boundary on $Z$.

(b) Prove that $Z$ depends only up to homeomorphism on the choice of $f$ and is orientable if and only if both $X$ and $Y$ are orientable.

Now consider a surface $X$ with two distinct boundary components $c_1, c_2 \subseteq \partial X$ both homeomorphic to $S^1$. Again, let $f: c_1 \to c_2$ be a homeomorphism and let $Z$ be the quotient of $X$ by the equivalence relation generated by $x \sim f(x)$ for $x \in c_1$.

(c) Prove that $Z$ depends only on the degree\(^1\) $\deg(f)$ of $f$.

(d) Show that $Z$ is orientable if and only if $\deg(f) = -1$.

\(^1\)The degree of a map $S^1 \to S^1$ already occurred on the second problem sheet.
Problem 3  Some explicit computations of mapping class groups

In this problem, we are concerned with some fundamental computations of mapping class groups.

(a) Prove that any homeomorphism $\Phi: S^2 \to S^2$ may be deformed into a homeomorphism $\Psi: S^2 \to S^2$ that fixes at least one point $x \in S^2$.

(b) Let $\Phi: \overline{D} \to \overline{D}$ be a homeomorphism of the closed unit disk $\overline{D} \subseteq \mathbb{R}^2$ that induces the identity on the boundary $S^1$ of $\overline{D}$. Construct an isotopy $I \times \overline{D} \to \overline{D}$ from $\Phi$ to the identity of $\overline{D}$ that keeps the boundary $S^1$ of $\overline{D}$ fixed for all $t \in I$.

(c) Does your argument from part (b) generalise to the once- or twice-punctured disc?

(d) What immediate conclusions about mapping class groups can you draw from the previous parts?