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Selected Topics in Geometric Group Theory

Problem Sheet 3

Problem 1 *Euler characteristic of the connected sum*

Give a formula relating the Euler characteristic $\chi(X \# Y)$ of the connected sum $X \# Y$ of two surfaces X and Y and prove your formula to be correct.

Problem 2 *Glueing of surfaces along circles*

Let X and Y be surfaces with boundary components $S^1 \cong c_1 \subseteq \partial X$ and $S^1 \cong c_2 \subseteq \partial Y$. Furthermore, let $f\colon c_1 \to c_2$ be a homeomorphism and let Z be the surface obtained from X and Y by glueing them along f, that is, Z is the quotient of $X \amalg Y$ by the equivalence relation generated by $x \sim f(x)$ for $x \in c_1$.

(a) Construct a natural structure of a surface with boundary on Z.

(b) Prove that Z depends only up to homeomorphism on the choice of f and is orientable if and only if both X and Y are orientable.

Now consider a surface X with two distinct boundary components $c_1, c_2 \subseteq \partial X$ both homeomorphic to S^1. Again, let $f\colon c_1 \to c_2$ be a homeomorphism and let Z be the quotient of X by the equivalence relation generated by $x \sim f(x)$ for $x \in c_1$.

(c) Prove that Z depends only on the degree[1] $\deg(f)$ of f.

(d) Show that Z is orientable if and only if $\deg(f) = -1$.

[1] The degree of a map $S^1 \to S^1$ already occurred on the second problem sheet.

Problem 3 *Some explicit computations of mapping class groups*

In this problem, we are concerned with some fundamental computations of mapping class groups.

(a) Prove that any homeomorphism $\Phi\colon S^2 \to S^2$ may be deformed into a homeomorphism $\Psi\colon S^2 \to S^2$ that fixes at least one point $x \in S^2$.

(b) Let $\Phi\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a homeomorphism of the closed unit disk $\overline{\mathbb{D}} \subseteq \mathbb{R}^2$ that induces the identity on the boundary S^1 of $\overline{\mathbb{D}}$. Construct an isotopy $I \times \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ from Φ to the identity of $\overline{\mathbb{D}}$ that keeps the boundary S^1 of $\overline{\mathbb{D}}$ fixed for all $t \in I$.

(c) Does your argument from part (b) generalise to the once- or twice-punctured disc?

(d) What immediate conclusions about mapping class groups can you draw from the previous parts?