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Selected Topics in Geometric Group Theory

Problem Sheet 4

Problem 1 *From covering spaces to fibrations – part one*

A compelling feature of covering maps $p\colon E \to B$ is the fact that homotopies $H\colon X \times I \to B$ may be lifted to homotopies $G\colon X \times I \to E$ starting at a prescribed map $G_0\colon X \to E$. We take this feature of coverings as the definition of a broader class of maps called *Hurewicz fibrations.* That is, $p\colon E \to B$ is a *Hurewicz fibration* – or simply a fibration as there is no danger of confusion in our context – if for any commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "G"] & B \end{tikzcd}$$

of solid arrows, there is a dashed map $G\colon X \times I \to E$ making the whole diagram commute[1].

Let us fix a fibration $p\colon E \to B$.

(a) Let $b, c \in B$ and let F_b and F_c denote the fibre of p over b and c, respectively. Construct for each homotopy class α of paths from b to c a homotopy equivalence $f_\alpha\colon F_b \to F_c$.

(b) Let us say that a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of pointed sets is exact at B if gf maps the whole of A to the basepoint of C and if any $b \in B$ that is mapped to the basepoint of C lies in the image of f.

Let $b \in B$, denote by F the fibre of p over b and let $e \in F$. Finally, let $i\colon F \to E$ denote the inclusion of the fibre. Prove that

$$\pi_0(F, e) \xrightarrow{i_*} \pi_0(E, e) \xrightarrow{p_*} \pi_0(B, b)$$

is exact in the above sense[2].

[1] In one of the lectures, we are going to employ a particular type of fibration – called a fibre bundle. You may look in almost any book on topology to find the definition of a fibre bundle and a proof that sufficiently nice fibre bundles are in fact fibrations.

[2] As a prerequisite, you have to think about the basepoints of $\pi_0(F, e)$, $\pi_0(E, e)$ and $\pi_0(B, b)$, of course.

(c) Again, let F denote the fibre of p over some point $b \in B$ and let $e \in F$. Prove the sequence

$$\pi_1(F, e) \xrightarrow{i_*} \pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b)$$

to be exact.

Even though only a tiny fraction of the story is told, we stop at this point. However, as you might have guessed from the title of this problem, we will continue our excursion to fibrations on the next problem sheet.

Problem 2 *The graph of curves of the torus*
Recall that an element $(p, q) \in \mathbb{Z}^2$ is called *primitive* if it may be extended to a base of $\mathbb{Z}^2$ as a free abelian group. Let us denote by $\mathcal{C}(T)$ the free homotopy classes of simple closed curves on the torus T that are not null-homotopic.

(a) Prove that there is a bijection between $\mathcal{C}(T)$ and the set of primitive elements of $\mathbb{Z}^2$.

(b) Define the graph of curves $\Gamma_{\mathcal{C}}(T)$ to be the graph having $\mathcal{C}(T)$ as its vertices and an edge (a, b) for each two curves a and b whose intersection number is minimal, i.e., whose intersection number actually achieves the minimum $\min\left\{ i(c, d) \mid c, d \in \mathcal{C}(T) \text{ and } c \neq d \right\}$ among the intersection numbers of all the pairs c and d of distinct homotopy classes $c, d \in \mathcal{C}(T)$ of curves. Prove that $\Gamma_{\mathcal{C}}(T)$ coincides with the Farey graph[3].

Problem 3 *Curves, intersection numbers and a second appearance of the graph of curves*
Let X be a surface of genus $g \geq 2$ and let α and β be two non-separating simple curves in X, such that $i(\alpha, \beta) = 0$. Prove that there is a curve γ with $i(\alpha, \gamma) = i(\beta, \gamma) = 1$. What conclusions can you draw about the graph of curves $\Gamma_{\mathcal{C}}(X)$ of X defined in the previous problem?

Problem 4 *Yet another mapping class group*
Let $S_{0,3}$ denote the sphere S^2 with three marked points $p, q, r \in S_{0,3}$.

(a) Prove: Any two simple curves α and β between two points $x, y \in \{p, q, r\}$ are isotopic relative $\{0, 1\}$, that is, there is an isotopy from α to β mapping 0 to x and 1 to y at all points in time.

(b) What conclusions can you draw about $\mathrm{Mod}(S_{0,3})$?

[3] Look up the definition of the Farey graph if necessary.