Karlsruher Institut für Technologie

Institute for Algebra and Geometry
JProf. Dr. Gabriela Weitze-Schmithüsen
Dipl.-Inf. Tobias Columbus

Selected Topics in Geometric Group Theory

Problem Sheet 5

Problem 1 *Outer Automorphisms of free groups*
Consider the free group F on $n \geq 1$ generators.

(a) Prove the abelianisation of F to be isomorphic to $\mathbb{Z}^n$.

(b) Prove the natural map $\mathrm{Out}(F) \to \mathrm{GL}_n(\mathbb{Z})$ to be surjective.

(c) Show that $\mathrm{Out}(F) \to \mathrm{GL}_n(\mathbb{Z})$ is not injective if $n \geq 3$.

Problem 2 *Extension of homeomorphisms*
Let X be a punctured surface. Prove that any homeomorphism $f\colon X \to X$ extends to a homeomorphism of $\overline{X}$.

Problem 3 *Curves suitable for the Alexander method*
Generalise the construction of a family of curves suitable for the Alexander method that was presented in the lectures from surfaces to punctured surfaces.

Problem 4 *The graph of curves again*
Consider the graph $\Gamma_C(S_2)$ of curves on a surface of genus 2. We suppose $\Gamma_C(S_2)$ to carry the usual graph-theoretic metric. Construct an example of a pair a, b of simple closed curves, such that a and b have distance 3 in $\Gamma_C(S_2)$.

Problem 5 *Fibrations – part two*

We continue our excursion to fibrations that we have begun on the preceding problem sheet. For this purpose, let $p\colon E \to B$ be a fibration, $b \in B$ and $e \in F$, where F denotes the fibre of p over b.

(a) Use the fibre transport maps f_α constructed on the last problem sheet to define a map $\partial\colon \pi_1(B, b) \to \pi_0(F, e)$ that makes the sequence

$$\pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\partial} \pi_0(F, e) \xrightarrow{i_*} \pi_0(E, e)$$

exact.

Having mastered all this abstract theory of fibrations, let us consider some examples. For the first example, consider a sufficiently nice topological space X and the associated spaces

$$\Omega X = \left\{\omega\colon I \to X \,\middle|\, \omega(0) = \omega(1) = x\right\}$$
$$PX = \left\{\omega\colon I \to X \,\middle|\, \omega(0) = x\right\}$$

of loops and paths based in the fixed point $x \in X$.

(b) Prove that $p\colon PX \to X$ given by $\omega \mapsto \omega(1)$ is a fibration with fibre ΩX.

(c) Show that PX is contractible and deduce $\pi_1(X) \cong \pi_0(\Omega X)$.

If X happens to be a surface, the above example does not only work for the loop and path space, but we can also consider spaces of homeomorphisms. That is, we consider the space $\mathrm{Homeo}_*(X)$ of homeomorphisms of X that fix some basepoint x and the space $\mathrm{Homeo}(X)$ of all homeomorphisms. Actually, this is not only a fibration but even a fibre bundle[1]. We will silently make use of the fact that any sufficiently nice fibre bundle is also a fibration.

(d) Prove that $p\colon \mathrm{Homeo}(X) \to X$ given by $f \mapsto f(x)$ is a fibre bundle with fibre $\mathrm{Homeo}_*(X)$.

(e) Prove that in this example, $\partial\colon \pi_1(B, b) \to \pi_0(F, e)$ and $i_*\colon \pi_0(F, e) \to \pi_0(E, e)$ are group homomorphisms.

(f) Describe the map ∂ in geometric terms in the case that X is a surface.

Finally, consider a homeomorphism $f\colon X \to X$ and the mapping torus T_f of f given by the quotient

$$T_f = \frac{X \times I}{(x, 0) \sim (f(x), 1)} \,.$$

(g) Construct a bundle $p\colon E \to B$ having T_f as total space. What are the fibre and the base?

(h) What is the mapping torus T_f of the map $f\colon S^1 \to S^1$ given by $z \mapsto z^{-1}$?

[1] Look up the definition of a fibre bundle!