Karlsruher Institut für Technologie

Institute for Algebra and Geometry
JProf. Dr. Gabriela Weitze-Schmithüsen
Dipl.-Inf. Tobias Columbus

Selected Topics in Geometric Group Theory

Problem Sheet 6

Problem 1 $\mathrm{Mod}(S)$ *and outer automorphisms of* $\pi_1(S)$

Give an example demonstrating that the natural map $\Psi\colon\ \mathrm{Mod}(S) \to \mathrm{Out}\big(\pi_1(S)\big)$ is not necessarily surjective[1].

Problem 2 *The Birman exact sequence*

Denote by S a compact, connected surface without punctures and let (S, x) denote the surface obtained from S by adding an additional puncture $x \in \operatorname{int} S$. The fibre bundle

$$\mathrm{Homeo}(S,x) \xrightarrow{i_*} \mathrm{Homeo}(S) \xrightarrow{p_*} S$$

induces a long exact sequence

$$\cdots \longrightarrow \pi_1\big(\mathrm{Homeo}(S)\big) \xrightarrow{p_*} \pi_1(S,x) \xrightarrow{\partial} \mathrm{Mod}(S,x) \xrightarrow{i_*} \mathrm{Mod}(S) \xrightarrow{p_*} \pi_0(S,x)$$

of homotopy groups that is – especially in the case that $\pi_1\big(\mathrm{Homeo}(S)\big)$ vanishes – known as *Birman exact sequence*. Show that the kernel of ∂ is contained in the center of $\pi_1(S,x)$.

Problem 3 *The graph of curves of the torus, again*

Let $\Gamma = \Gamma_C(T)$ denote the graph of curves of the torus T and consider the usual metric d on Γ assigning to each edge of Γ a length of one.

(a) Prove that Γ is a planar graph and that the removal of any edge xy of Γ separates Γ into two connected components.

(b) Fix two vertices $x, y \in \Gamma$ and consider the set $S(x,y)$ of edges whose removal separates x from y. Find a necessary and sufficient condition for $S(x,y)$ to be empty.

(c) Now suppose $S(x,y) \neq \emptyset$. Prove that there is some constant $\delta > 0$, independent of x and y, such that any geodesic[2] $\overline{xy}$ is contained in a δ-neighbourhood of $S(x,y)$.

(d) Prove that there exists some constant $\delta > 0$, such that $\overline{xz}$ is contained in some δ-neighbourhood of $\overline{xy} \cup \overline{yz}$ for any choice of vertices $x, y, z \in \Gamma$. This property is also known as *δ-hyperbolicity*.

[1] Hint: Consider surfaces with punctures.
[2] The layman might also speak of a *shortest path* between x and y.

Problem 4 *Free differential calculus – part 1*

In this problem, we introduce the so-called *free differential calculus* – a theory developed by Ralph H. Fox in 1952. For this purpose, we fix a free group G on n generators $x_1, \dots, x_n$ and consider the *group ring* $\mathbb{Z}[G]$. Recall that elements of $\mathbb{Z}[G]$ may be considered as maps $G \to \mathbb{Z}$ with finite support whose addition and multiplication is given by

$$(f+g)(u) = f(u) + g(u)$$
$$(f \cdot g)(u) = \sum_{v \in G} f(uv^{-1}) \cdot g(v) \,.$$

One may also think of elements of $\mathbb{Z}[G]$ as formal linear combinations $\sum_{u \in G} f(u) \cdot u$. Recall that the *augmentation* $\epsilon \colon \mathbb{Z}[G] \to \mathbb{Z}$ given by $f \mapsto \sum_{u \in G} f(u)$ is a ring homomorphism.

A map $D \colon \mathbb{Z}[G] \to \mathbb{Z}[G]$ is called a $\mathbb{Z}[G]$*-derivation* if $D(f+g) = Df + Dg$ and $D(fg) = Df \cdot \epsilon(g) + f \cdot Dg$. The collection of all $\mathbb{Z}[G]$-derivations becomes a right $\mathbb{Z}[G]$-module by letting $(D_1 + D_2)f = D_1 f + D_2 f$ and $(D \cdot f)g = Dg \cdot f$.

Recall that a word u on the letters $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$ is called *reduced* if no generator x_i and its inverse x_i^{-1} occur at adjacent positions in u. For $1 \le j \le n$ and $u \in G$, define $\langle j, u \rangle \in \mathbb{Z}$ by

$$\langle j, u \rangle = \begin{cases} 1 & \text{if the reduced word } u \text{ begins with } x_j \\ 0 & \text{otherwise} \end{cases}$$

and extend this definition by $\langle j, \sum_{u \in G} a_u u \rangle = \sum_{u \in G} a_u \langle j, u \rangle$ to a map $\mathbb{Z}[G] \to \mathbb{Z}$. Finally, for $f \in \mathbb{Z}[G]$ define

$$\langle j, u, f \rangle = \langle j, u^{-1} f \rangle - \langle j, u^{-1} \rangle \cdot \epsilon(f)$$

and let

$$\frac{\partial}{\partial x_j} \colon \mathbb{Z}[G] \to \mathbb{Z}[G]$$
$$f \mapsto \sum_{u \in G} \langle j, u, f \rangle \cdot u \,.$$

(a) Prove any assertion in the preceding paragraph that looks suspicious to you.

(b) Show that $\frac{\partial}{\partial x_j}$ is a $\mathbb{Z}[G]$-derivation that furthermore satisfies $\frac{\partial x_i}{\partial x_j} = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta.

(c) Show that there is a unique $\mathbb{Z}[G]$-derivation mapping the generators x_j of G to prescribed elements $f_j \in \mathbb{Z}[G]$.

(d) Show that any $f \in \mathbb{Z}[G]$ satisfies the equality

$$f = \epsilon(f) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_j - 1) \,.$$

This equality is also known as *fundamental formula*.

As the title of this problem suggests, we will continue to study the free differential calculus on the next problem sheet – also in relation to the mapping class group $\mathrm{Mod}(S)$ of a surface S.