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Selected Topics in Geometric Group Theory

Problem Sheet 7

Problem 1 *Uniqueness of decomposition into Dehn twists*

Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_m\}$ be two sets of nontrivial[1] isotopy classes of simple closed curves on a surface X, such that $i(a_j, a_k) = 0$ and $i(b_j, b_k) = 0$ for all suitable choices of indices j and k. Furthermore, let $e_1, \ldots, e_n$ and $d_1, \ldots, d_m$ be arbitrary, non-zero integers. Prove that

$$T_{a_1}^{e_1} \cdots T_{a_n}^{e_n} = T_{b_1}^{d_1} \cdots T_{b_m}^{d_m}$$

implies $n = m$, $a_j = b_{\sigma(j)}$ and $e_j = d_{\sigma(j)}$ for all $1 \leq j \leq n$ and some permutation $\sigma \in \mathfrak{S}_n$.

Problem 2 *The centre of the mapping class group*

It has been shown in the lectures that the centre $Z(\mathrm{Mod}(S_g))$ of the mapping class group of a surface S_g of genus $g \geq 3$ is trivial.
What happens in the case that $g < 3$? What can you say about surfaces $S_{g,n}$ with punctures?

[1] Nontrivial means that the curves are neither null-homotopic nor homotopic to a puncture.

Problem 3

(a) In the lecture we saw:

(1) The pushing map $\mathrm{Push}\colon F_2 \cong \pi_1(S_{0,3}) \to \mathrm{PMod}(S_{0,4})$ is an isomorphism.

(2) The map $\Psi\colon \mathrm{PMod}(S_{0,4}) \hookrightarrow \mathrm{Out}(\pi_1(S_{0,4})) \cong \mathrm{Out}^+(F_3)$ is an embedding.

Determine the map $\Psi \circ \mathrm{Push}$ explicitly.

(b) Consider $S_{0,4} = S_{0,3} \setminus \{P\}$ and let the mapping class $f = [\varphi]$ be an element of the image of the pushing map $\mathrm{Push}\colon \pi_1(S_{0,3}) \to \mathrm{PMod}(S_{0,4})$ from above. Let $\beta\colon X^* \to S_{0,3}$ be an unramified cover of finite degree. This induces an unramified cover

$$X^{**} := X^* \setminus \beta^{-1}(P) \to S_{0,4} \tag{$*$}$$

which we also call β. Show that $\varphi \in \mathrm{Homeo}^+(S_{0,4})$ can be lifted via this cover to X^{**}.

(c) Fix an identification $S_1 \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})$. The map $z \mapsto -z$ descends to a map ι of order 2 on S_1. Denote by P_1, P_2, P_3 and P_4 the four fixed points of ι. The quotient S_1 / ι is a sphere and if we remove the fixed points and their images on the quotient we obtain an unramified cover

$$p\colon S_{1,4} \cong (\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\mathrm{i})) \setminus \{P_1, \dots, P_4\} \to S_{0,4}\,.$$

Right or wrong?

(i) Any homeomorphism of $S_{0,4}$ can be lifted to a homeomorphism of $S_{1,4}$.
Hint: What is $p_*(\pi_1(S_{1,4}))$?

(ii) Any homeomorphism of $S_{1,4}$ descends to a homeomorphism of $S_{0,4}$.

(iii) The epimorphism $q\colon \mathrm{Homeo}^+(S_{1,1}) \to \mathrm{Mod}(S_{1,1})$ splits, that is, there exists some homomorphism $s\colon \mathrm{Mod}(S_{1,1}) \to \mathrm{Homeo}^+(S_{1,1})$ with $q \circ s = \mathrm{id}$.

(d) Let $U_1 := p_*(\pi_1(S_{1,4}))$ and $U_2 := \beta_*(\pi_1(X^{**}))$ with β defined as in $(*)$. Then by the theorem of the universal cover $U_1 \cap U_2$ defines unramified covers q_1 and q_2 of $S_{1,4}$ and X^{**}. What are the degrees of these covers? What do we know about $p \circ q_1$ and $\beta \circ q_2$? Is there some lift of φ from (b) via $p \circ q_1$?