Karlsruher Institut für Technologie

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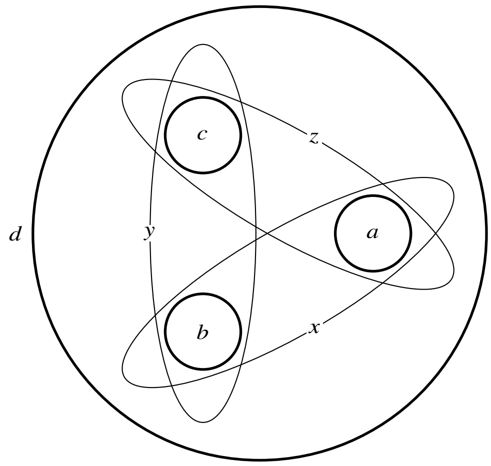
Selected Topics in Geometric Group Theory

Problem Sheet 8

Problem 1 *The lantern relation*
Consider the surface S_0^4 of genus $g = 0$ with four boundary components a, b, c and d. Let x, y and z denote the curves shown in the following picture:



Prove that the Dehn twists T_x, T_y and T_z around the curves x, y and z and the Dehn twists T_a, T_b, T_c and T_d around the boundary components satisfy the equality

$$T_x T_y T_z = T_a T_b T_c T_d \, .$$

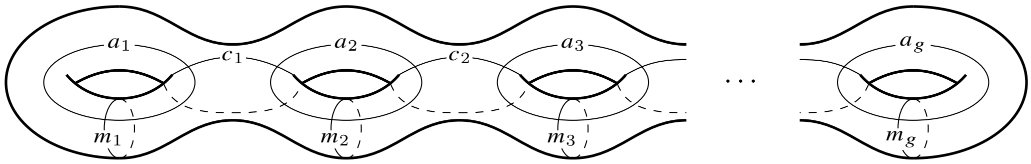
This relation is known as *lantern relation*.

Problem 2 *The mapping class group of a pair of pants*
Compute the mapping class group $\mathrm{Mod}(S_0^3)$ of the surface S_0^3 of genus zero with three boundary components[1].

[1]This surface is also known as *pair of pants*

Problem 3 *Generators of* Mod(S)

Let S be a surface of genus g without boundary or marked points. The mapping class group Mod(S) is generated by Dehn twists around the curves $a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1$ and m_2 that are shown in the following picture[2]:



Write the Dehn twist around m_3 as word in the Dehn twists around the curves $a_1, \ldots, a_g$ and $c_1, \ldots, c_{g-1}$ and m_1 and m_2.

Problem 4 *Free differential calculus – part 2*

Let $\phi\colon F \to F$ be an endomorphism of a free group F freely generated by $x_1, \ldots, x_n$. Denote the prolongation $R \to R$ of ϕ to the group ring $R = \mathbb{Z}[F]$ by ϕ, too.

(a) Show that the "chain rule"

$$\frac{\partial \phi(f)}{\partial x_j} = \sum_k \frac{\partial f}{\partial x_k} \cdot \frac{\partial \phi(x_k)}{\partial x_j}$$

holds for all $f \in \mathbb{Z}[F]$.

Now consider a surface $X = S_{g,1}$ of genus g with one marked point $x \in X$. Let us consider the action Mod$(X) \to$ Aut$(\pi_1(X, x))$ of the mapping class group on the fundamental group $\pi_1(X, x)$. Again, let $R = \mathbb{Z}[\pi_1(X, x)]$ denote the group ring over $\pi_1(X, x)$ and for $f \in$ Mod(X) define an element

$$f_* = \left(\frac{\partial f(x_i)}{\partial x_j}\right)_{i,j}$$

of Gl$(2g, R)$.

(b) Show that $f \mapsto f_*$ defines a faithful representation Mod$(X) \to$ Gl$(2g, R)$ of the mapping class group. This representation is also known as the *Magnus representation* of Mod(X).

(c) Can you say something about the kernel of the map Mod$(X) \to$ Gl$(2, \mathbb{Z})$ induced by the augmentation map $\epsilon\colon R \to \mathbb{Z}$?

[2]We will see in the lectures that Mod(S) is generated by Dehn twists, even though we will not deal with these generators.