Recall that we are given three smooth simple closed curves $\alpha$, $\beta$ and $\alpha'$, such that $\alpha$ and $\alpha'$ are freely homotopic, $\alpha \pitchfork \beta$ and $\alpha' \pitchfork \beta$ are transversal intersections.

Furthermore we have that $\alpha'$ and $\beta$ bound a bigon and $$\#(\alpha \cap \beta) = \#(\alpha' \cap \beta).$$

We want to show that $\alpha$ and $\beta$ bound a bigon as well.

As before consider $H^{-1}(\beta)$ in the annulus $A := S^1 \times I$. If $H^{-1}(\beta)$ contains a component of Type 3, then we are done.

Suppose that not, then we already saw in the lecture that all connected components of $H^{-1}(\beta)$ are of Type 1 or Type 2.

Since $\alpha'$ and $\beta$ bound a bigon, we have a subarc $v$ of $\alpha'$ and a subarc $w$ of $\beta$ with common starting point $P_1$ and common endpoint $P_2$ such that $vw$ is null-homotopic.

Recall that $H|_{\delta_1} = \alpha'$ and $\alpha'$ is simple. Let $s$ be the preimage of $v$ in $\delta_1$ and $Q_1$ and $Q_2$ the starting and the endpoint, respectively. Since $\alpha'$ and $\beta'$ are transverse, we have an arc $p$ starting in $Q_1$ and an arc $q$ starting in $Q_2$ in the preimage $H^{-1}(\beta)$. Let $R_1$ and $R_2$ be the end points of these arcs which have to lie in $\delta_0 = S^1 \times \{0\}$. Let finally $r$ be the simple curve from $R_1$ to $R_2$ in $\delta_0$.

We have:

$$prq$$

is null-homotopic.

Recall that $H(s) = v$ and $vw$ is null-homotopic in $X$. Hence we have:

$$H(p)H(r)H(q)H(s)v\overline{w} = H(p)H(r)H(q)\overline{w}$$

is null-homotopic and thus $H(r)H(q)\overline{w}H(p)$ is null-homotopic. But $H(q)\overline{w}H(p)$ is a subarc of $\beta$ and $H(r)$ is a subarc of $\alpha$, thus by Corollary 4.10 from the lecture, we have that $\alpha$ and $\beta$ bound a bigon.