Step 2 in the proof of the bigon theorem

Recall that we are given three smooth simple closed curves α, β and α', such that α and α' are freely homotopic, $\alpha \pitchfork \beta$ and $\alpha' \pitchfork \beta$ are transversal intersections.
Furthermore we have that α' and β bound a bigon and

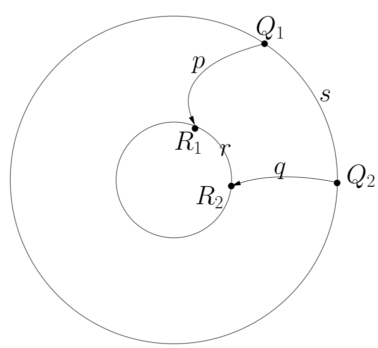
$$\sharp(\alpha \cap \beta) = \sharp(\alpha' \cap \beta).$$

We want to show that α and β bound a bigon as well.

As before consider $H^{-1}(\beta)$ in the annulus $A := S^1 \times I$.
If $H^{-1}(\beta)$ contains a component of Type 3, then we are done.
Suppose that not, then we already saw in the lecture that all connected components of $H^{-1}(\beta)$ are of Type 1 or Type 2.

Since α' and β bound a bigon, we have a subarc v of α' and a subarc w of β with common starting point P_1 and common endpoint P_2 such that $v\bar{w}$ is null-homotopic.

Recall that $H|_{\delta_1} = \alpha'$ and α' is simple. Let s be the preimage of v in δ_1 and Q_1 and Q_2 the starting and the endpoint, respectively. Since α' and β' are transverse, we have an arc p starting in Q_1 and an arc q starting in Q_2 in the preimage $H^{-1}(\beta)$. Let R_1 and R_2 be the end points of these arcs which have to lie in $\delta_0 = S^1 \times \{0\}$. Let finally r be the simple curve from R_1 to R_2 in δ_0.



We have:

$$pr\overline{q}\,\overline{s} \text{ is null-homotopic.}$$

Recall that $H(s) = v$ and $v\overline{w}$ is null-homotopic in X. Hence we have:

$$H(p)H(r)H(\overline{q})H(\overline{s})v\overline{w} = H(p)H(r)H(\overline{q})\overline{w}$$

is null-homotopic and thus $H(r)H(\overline{q})\overline{w}H(p)$ is null-homotopic. But $H(\overline{q})\overline{w}H(p)$ is a subarc of β and $H(r)$ is a subarc of α, thus by Corollary 4.10 from the lecture, we have that α and β bound a bigon.