Finite Group Schemes – Exercise Sheet 1

Problem 1
Let $k$ be a ring and $X$ a $k$-functor. We define

$$|X| := \lim_{\rightarrow l} X(l)$$

where $l$ runs over all field extensions of $k$. In the colimit we identify two points $x_1 \in X(l_1)$ and $x_2 \in X(l_2)$ whenever there is a field $l_3$ and embeddings $l_1, l_2 \to l_3$ which send $x_1$ and $x_2$ to the same element $x_3 \in X(l_3)$ after application of $X$. We assume that $|X|$ is a set.

(a) Show that for any subfunctor $Y \subseteq X$ there is a canonical inclusion $|Y| \subseteq |X|$.

(b) Set

$$\mathcal{O} := \{|U| \mid U \to X \text{ is an open subfunctor}\}.$$ 

Prove that $(|X|, \mathcal{O})$ is a topological space, which depends functorially on $X$.

(c) For any $X = \text{Sp}_k A$, we have a canonical identification $|X| = \text{Spec} A$, the latter denoting the set of prime ideals in $A$ together with the Zariski topology.

(d) Let $\mathcal{O}_X$ denote the sheaf on $|X|$ associated to the presheaf $|U| \mapsto \mathcal{O}(U)$. If $X$ is a scheme, then $(|X|, \mathcal{O}_X)$ is a locally ringed space, locally isomorphic to $(\text{Spec} \mathcal{O}(U), \mathcal{O}_U)$ for $U \to X$ open affine.

Problem 2
Let $l/k$ denote a finite extension of fields.

(a) Prove that the base change functor

$$- \otimes_k l : \widehat{\text{Fun}}(k) \to \widehat{\text{Fun}}(l)$$

on formal functors induces a well defined base change functor

$$- \otimes_k l : \widehat{\text{Sch}}(k) \to \widehat{\text{Sch}}(l)$$

on formal schemes.

(b) Describe the base change functor in terms of profinite $k$-algebras and also in terms of $k$-coalgebras.

(c) Show that the base change functor has a canonical extension to arbitrary field extensions $l/k$. 