Introduction to Origamis in Teichmüller Space

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Contents

1 Motivation

An *origami* is in the first place a combinatorial object. We shall give in Section 2 four different characterizations, not all of them combinatorial.
Step by step we shall discover more and more structure on these objects:

- An origami determines a Riemann surface and even a surface with a translation structure.

- The translation structure can be varied in a natural way; this yields a complex one-parameter family of Riemann surfaces.

- The parameter space can be identified with the upper half plane, and the Riemann surfaces come along with a natural marking.

- Thus from an origami, we obtain a map from the upper half plane into a Teichmüller space; this map turns out to be an isometric and holomorphic embedding.

- The stabilizer in the mapping class group of this embedded upper half plane is a Fuchsian group, more precisely a lattice in $\mathrm{PSL}_2(\mathbb{R})$. It is called the *Veech group* of the origami and can also be characterized by affine diffeomorphisms.

- The image of the embedded upper half plane in the moduli space of Riemann surfaces is an affine algebraic curve, possibly with singularities; it is called the *origami curve* to the given origami.

- Every origami is defined over a number field. Thus the (absolute) Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on origami curves, and hence ultimately on origamis.

Because of the last property, P. Lochak considered these objects as one-dimensional analogs of dessins d'enfants; in [13], he proposed the name "origami" for them. A *dessin d'enfants* also is a combinatorial object, namely a graph with certain properties on a surface. It determines a Riemann surface which is defined over a number field, and thus a zero-dimensional arithmetic subvariety of the moduli space.
For dessins d'enfants as well as for origamis, the dream would be to understand the Galois action on the combinatorial objects well enough to obtain new structural insights of the Galois group. In particular, one would like to understand how close the Galois group is to the so called Grothendieck-Teichmüller group. So far, this is but a dream, but at least there are several interesting relations between origamis and dessins d'enfants. Some of them are described in [11].

Although not under this name, origamis have been known since the pioneering work of Thurston and the seminal paper of Veech [19] in the late 1980s. They were – and still are – often studied in the context of Teichmüller disks and Teichmüller curves. In Veech's work, they provide examples of polygonal billiard tables where the dynamical behaviour is "optimal", i. e. the same as on the classical rectangle.

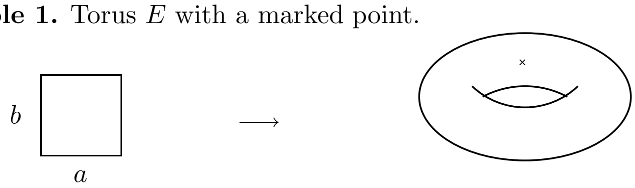
2 Definition and characterization of origamis

2.1 Combinatorial definition

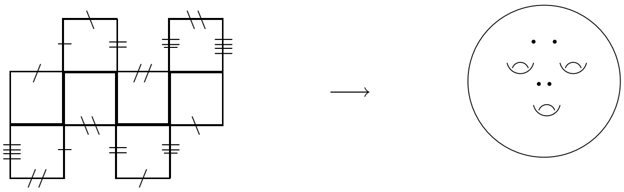
Take finitely many euclidean unit squares and glue them in such a way that

- each left edge is glued to a right edge;
- each upper edge is glued to lower edge;
- the resulting closed surface X is connected.

Example 1. Torus E with a marked point.



Example 2. Quaternion origami.



Note that the genus g of X can be calculated with Euler's formula: for this we need the number of squares, edges and vertices on the surface obtained by the gluing. If d denotes the number of squares, we have $2d$ edges on X since each square has 4 edges, and each edge belongs to exactly 2 squares. The number of vertices is $4d$ before gluing, but their number on X depends on the way the squares are glued: we call it n and obtain

$$2 - 2g = d - 2d + n = n - d \quad \text{or} \quad g = \frac{d-n}{2} + 1.$$

In Example 2 we find $g = 3$.

2.2 Coverings of the punctured torus

An origami as above comes with a map $p : X \to E$ from the surface X to a torus E: p maps each square to the torus as in Example 1. The identifications that were made to construct the origami X are compatible with the identifications

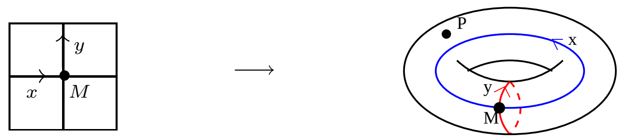
made to construct the torus E. Indeed a right edge is glued to a left edge and an upper edge is glued to a lower edge.
The map p is unramified outside the vertices of the squares, hence ramified over at most one point on E which we call ∞.
Conversely, given a covering $p : X \to E$ unramified outside ∞, we get an origami as follows: every connected component of $p^{-1}(E - (a \cup b))$ is an open square, and the gluing of the squares is given by taking the closure of the components in X. We take this characterization as our official definition:

Definition 2.1. a) An *origami* O of genus g and degree d is a covering $p : X \to E$ from a connected closed oriented surface X of genus g to the torus E which is ramified at most over one point ∞ in E.

b) Origamis $O_1 = p_1 : X_1 \to E$ and $O_2 = p_2 : X_2 \to E$ are called *equivalent* if there is a homeomorphism $\varphi : X_1 \to X_2$ such that $p_1 = p_2 \circ \varphi$.

2.3 Monodromy

Recall that the fundamental group of the punctured torus is a free group on two generators. To be more precise, let $E^\star = E - \{\infty\}$. Then with notation as in the following picture, we have $\pi_1(E^\star, M) = F_2(x, y)$. Note that the commutator $xyx^{-1}y^{-1}$ is a loop around ∞.



For an origami $O = (p : X \to E)$ of degree d, let $p^{-1}(M) = \{M_1, \dots, M_d\}$. We get a *monodromy map* $m : \pi_1(E^\star, M) \to S_d = \mathrm{Sym}\{M_1, \dots, M_d\}$ as follows: for $\gamma \in \pi_1(E^\star, M)$ we set $m(\gamma)(i) = j$ if the lift of γ, that starts in M_i, ends in M_j.
In this way, the origami O induces a homomorphism $m = m_O : F_2 \to S_d$. Let

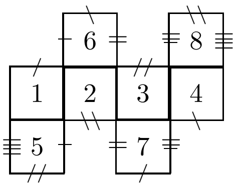
$$\sigma_a = m(x), \ \sigma_b = m(y).$$

The fact that X is connected implies that the subgroup of S_d generated by σ_a and σ_b acts transitively on the set $\{1, \dots, d\}$. If we number the elements of $p^{-1}(M)$ in a different way, σ_a and σ_b are replaced by their conjugates by the permutation that describes the renumbering.
If O is given by squares, each square contains exactly one of the M_i. The lift

of x that starts in M_i joins this point to the corresponding point in the next square to the right. Thus σ_a (resp. σ_b) is the permutation of the labels given by passing to the right (resp. upper) neighbouring square. Conversely, given σ_a and σ_b in S_d, generating a transitive subgroup, we obtain an origami of degree d by labeling d squares with the numbers $1, \dots, d$ and gluing the right edge of square i to the left edge of square $\sigma_a(i)$, and its upper edge to the lower edge of square $\sigma_b(i)$. Summing up we have shown

Proposition 2.2. *There is a bijection between equivalence classes of origamis of degree d and conjugacy classes of pairs (σ_a, σ_b) in S_d that generate a transitive subgroup.*

Example. Taking up the second example in Section 2.1 and labelling the squares as follows



we find $\sigma_a = (1\,2\,3\,4)\,(5\,6\,7\,8)$ and $\sigma_b = (1\,7\,3\,5)\,(2\,6\,4\,8)$.

Exercise. Determine the subgroup of S_8 generated by σ_a and σ_b.

We shall clarify the meaning of this group in general at the end of the next subsection.

2.4 Subgroups of F_2

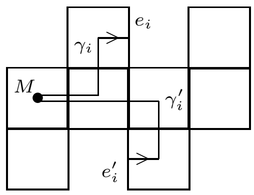
An origami $O = (p : X \to E)$ restricts to an unramified covering $p : X^\star \to E^\star$ where $E^\star = E - \{\infty\}$ as above and $X^\star = X - p^{-1}(\infty)$. By the universal property of the universal covering $u : \tilde{E} \to E^\star$, there is a unique unramified covering $q : \tilde{E} \to X^\star$ such that $u = p \circ q$.

The group of deck transformations of the universal covering is in a natural way identified with the fundamental group, thus we have $\mathrm{Deck}(\tilde{E}/E^\star) \cong \pi_1(E^\star) \cong F_2$. Under this identification, $U = \mathrm{Deck}(\tilde{E}/X^\star) \cong \pi_1(X^\star)$ is a subgroup of $\mathrm{Deck}(\tilde{E}/E^\star)$ of index $d = \mathrm{degree}(p)$.

Conversely, any subgroup U of finite index d of F_2 determines an unramified covering $p : X^\star \to E^\star$ of degree d. This map can be extended in a unique way to a ramified covering $p : X \to E)$, where X is a closed surface. We have

Proposition 2.3. *There is a bijection between equivalence classes of origamis of degree d and conjugacy classes of subgroups of F_2 of index d.*

By the rank-index formula for subgroups of a free group, any subgroup of F_2 of index d is free of rank $d+1$. On the other hand, the fundamental group of $X^\star$ is free of rank $2g+n-1$, where g is the genus of X and n the number of punctures, i. e. $n = |p^{-1}(\infty)|$. Thus we have $d+1 = 2g+n-1$, which confirms the Euler characteristic count of Section 2.1: $g = \frac{d-n}{2} + 1$.
An explicit set of free generators for $\pi_1(X^\star)$ as a subgroup of $\pi_1(E^\star) = F_2(x, y)$ can be found as follows: represent the origami by a simply connected rectangular polygon P (which need not be planar); this can be achieved by beginning with an arbitrary square and then inductively gluing new squares to one of the free edges of the polygon obtained so far (according to the gluing of the origami). The final polygon P has 2(d+1) free edges $e_1, \ldots, e_{d+1}, e'_1, \ldots, e'_{d+1}$, and the origami is obtained by gluing each e_i to e'_i.
Now fix a base point M inside P, e. g. the midpoint of a square. For each i, there is a unique path γ_i in P from M to the midpoint of e_i which is composed of horizontal and vertical connections of midpoints of squares, and a unique path γ'_i of the same type from the midpoint of e'_i to M. Together they represent a closed path on $X^\star$, i. e. an element u_i in $\pi_1(X^\star, M)$. Clearly $u_1, \ldots, u_{d+1}$ generate $\pi_1(X^\star, M)$. As a word in x and y, u_i is obtained by replacing each horizontal piece of γ_i and γ'_i by x or x^{-1}, depending on the orientation of the piece, and vertical pieces by y or y^{-1}. In the following example, the element u_i is given by the word $xyxyx^{-2}$:



The relation to the description in Section 2.3 is as follows: Let $m = m_O$ be the monodromy homomorphism to O and let γ be an element of $\pi_1(E^\star)$. Then $m(\gamma)(1) = 1$ if and only if γ lifts to a closed path in $X^\star$, i. e. if and only if $\gamma \in \pi_1(X^\star) = U$. It follows that the squares of the origamis correspond bijectively to the cosets of U in F_2. Under this correspondence, σ_a is the right multiplication by x on the cosets, and σ_b the right multiplication by y.

Definition 2.4. An origami O is called *normal* (or *Galois*) if U is a normal subgroup of F_2.

Eqivalently, an origami $O = (p : X \to E)$ is normal if and only if p is a normal covering.

Remark 2.5. For a normal origami,

$$\mathrm{Gal}\,(X/E) = \mathrm{Deck}\,(X^\star/E^\star) \cong F_2/U \cong <\sigma_a, \sigma_b> \subseteq S_d.$$

Example. The origami W is normal. The Galois group is the quaternion group of order 8. This can be checked e. g. by solving the exercise at the end of Section 2.3. A careful analysis of this example can be found in [8].

3 Teichmüller disks

3.1 Translation structures

Let $O = (p : X \to E)$ be an origami. We can use the squares that constitute O as chart maps in the following way:

- for every $x \in X$ which is in the interior of a square, we take this (open) square as a chart around x
- for every $x \in X$ that lies on an edge e of a square, but is not a vertex, we take as chart an open disk in the union of the two squares adjacent to e that contains x but no vertex.

For the charts defined so far, the transition maps between different charts are translations in the plane. Considering the plane as the complex plane $\mathbb{C}$, they are (very special!) holomorphic functions. Thus we have defined a structure of Riemann surface on $X^\star$. It is known from the general theory of Riemann surfaces that it can be extended in a unique way to a structure of Riemann surface on the whole of X.

In our situation, this construction is very explicit: Let $v \in X$ be a vertex of a square, i. e. $v \in p^{-1}(\infty)$. Consider a small loop around v on X starting in horizontal direction, say. Since it also ends in horizontal direction, the number of squares that the loop crosses must be a multiple of 4, hence of the form $4k$ for some $k \geq 1$. This number k is also the ramification index $e_p(v)$ of the covering p in the point v. Thus in suitable local coordinates, p is given by $z \mapsto z^k$ in a neighbourhood of v.
Summing up we have:

Proposition 3.1. *An origami $O = (p : X \to E)$ determines a structure of Riemann surface on X, and a translation structure on $X^\star$.*

Definition 3.2. We call a surface S together with an atlas a *translation surface* if outside a finite subset $\Sigma \subset S$, the transition maps are translations, and for each $v \in \Sigma$, the chart maps are of the form $z \mapsto z^{k_v}$ for some positive integer k_v.

Note that a translation structure on a surface X^* can be lifted in a unique way to any unramified covering $Y^* \to X^*$. Thus in particular, the universal covering $\tilde{E}$ of $E^\star$ is endowed with a translation structure.
More generally, let $p : S' \to S$ be a ramified covering, where S is a translation surface. Then S' inherits a structure of translation surface: Let $\Sigma_{S'} = p^{-1}(\Sigma_S) \cup \{\text{ramification points of } p\}$. Then the restriction $p : S' - \Sigma_{S'} \to S - \Sigma_S$ is unramified, and for every $v \in \Sigma_{S'}$ the exponent k_v in Definition 3.2 can be taken to be $k_v = e_p(v) \cdot k_{p(v)}$.

3.2 Variation of the translation structure

We have seen in the last section how to define, for an origami $O = (p : X \to E)$, a structure of translation surface on X, and that this structure is completely determined by (p and) the translation structure on E. For the latter, we considered E as a square with opposite sides glued. We get a similar, but in general different translation structure on E, if we replace the square by an arbitrary parallelogram (say of area 1, to keep the volume of E fixed).

Such parallelograms correspond bijectively to the points in the upper half plane $\mathbb{H}$. Thus any $\tau \in \mathbb{H}$ induces a structure of Riemann surface on X. Moreover there is a natural marking coming from the identity map on X: Recall that a *marked Riemann surface* is an equivalence class of pairs (X, f) where X is a Riemann surface and $f : X_0 \to X$ is a diffeomorphism from a fixed reference surface X_0 of the same topological type as X; two pairs (X, f) and (Y, g) are considered equivalent, if $g \circ f^{-1} : X \to Y$ is homotopic to a biholomorphic map. The set of all marked Riemann surfaces of a fixed genus g and a fixed number $n \geq 0$ of distinguished points on it is the *Teichmüller space* $T_{g,n}$.

From the above considerations we obtain

Remark 3.3. Every origami $O = (p : X \to E)$ induces a map $\iota_O : \mathbb{H} \to T_{g,n}$, where g is the genus of X and $n = |p^{-1}(\infty)|$.

A proof of the following theorem can be found e.g. in [11], Sect. 3.2. The arguments given there were essentially known already to Teichmüller himself:

Theorem 3.4. *For an origami O, the map $\iota_O : \mathbb{H} \to T_{g,n}$ is*

a) *injective,*

b) *isometric (w.r.t. the hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on $T_{g,n}$),*

c) *holomorphic (w.r.t the natural complex structure on $T_{g,n}$).*

Definition 3.5. A map with the properties of Theorem 3.4 is called a *Teichmüller embedding.* The image

$$\Delta_O = \iota_O(\mathbb{H})$$

is called a *Teichmüller disk* (or *complex geodesic*) in $T_{g,n}$.

Proof. a) follows from b).
For b), we interpret ι_O as follows: Any parallelogram of area 1 in the upper half plane with 0 as bottom left corner is the image of the unit square under an $\mathbb{R}$-linear map of determinant 1, i. e. given by a matrix $A \in \mathrm{SL}_2(\mathbb{R})$. Two matrices A and B give the same complex structure if and only if they differ by a conformal map, i. e. a rotation, in other words if $AB^{-1} \in \mathrm{SO}_2(\mathbb{R})$. Thus we can consider ι_O equivalently as a map $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) \to T_{g,n}$. The identification of $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$ with $\mathbb{H}$ is given by $A \mapsto -\bar{A}^{-1}(i)$, where a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ in the usual way as fractional linear transformation: $z \mapsto \frac{az+b}{cz+d}$.
Now recall that the Teichmüller distance between two marked Riemann surfaces (X, f) and (Y, g) in $T_{g,n}$ is defined as $\log K$, where K is the minimal dilatation of a quasiconformal map $h : X \to Y$ which is isotopic to $g \circ f^{-1}$. It is a well known exercise in quasiconformal maps that an affine map has minimal dilatation within its isotopy class. Therefore the Teichmüller distance between $\iota_O(A)$ and $\iota_O(B)$ is equal to $\log K$ for the dilatation K of $h = AB^{-1}$. Explicitly, $K = \frac{1+\kappa}{1-\kappa}$, where $\kappa = \frac{f_{\bar{z}}}{f_z}$.
If e. g. AB^{-1} is the matrix $M = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix}$ for some $K > 1$, a short calculation shows $\kappa = \frac{K-1}{K+1}$, and the dilatation is $\frac{1+\frac{K-1}{K+1}}{1-\frac{K-1}{K+1}} = K$. On the other hand, $-\overline{M^{-1}(i)} = Ki$, and the hyperbolic distance between i and Ki also equals $\log K$.

For c), we use the property of $T_{g,n}$ of being a moduli space: The construction at the beginning of this section provides us, for every $\tau \in \mathbb{H}$, with a marked Riemann surface X_τ. The union of all the X_τ is in a natural way a complex manifold $\mathcal{X}_O$ such that the projection $\mathcal{X}_O \to \mathbb{H}$, which sends all points of X_τ to τ, is a proper holomorphic map. The universal property of a moduli space is that in such a situation, where we are given a holomorphic family of marked Riemann surfaces, the map from the base $\mathbb{H}$ of the family to the moduli space $T_{g,n}$, which maps the point $\tau \in \mathbb{H}$ to the point in $T_{g,n}$ which represents the isomorphy class of the fibre X_τ, is holomorphic. $\square$

3.3 Teichmüller disks

The definition of the Teichmüller embedding ι_O for an origami O is a special case of a more general construction:

Let X be a compact Riemann surface of genus g and ω a holomorphic quadratic differential on X. This means that ω is a global section of the square of the canonical bundle on X, i. e. $\omega \in H^0(X, \Omega_X^{\otimes 2})$. Equivalently, we can consider ω as an element of the Riemann-Roch vector space $L(2K_X)$, where K_X is a canonical divisor on X. In a local coordinate z, ω can be described as $\omega = f\,dz^2$ with a holomorphic function f; the transition functions between local coordinates z and $\tilde{z}$ are given by $f\,dz^2 = \tilde{f}\,d\tilde{z}^2$ with $\tilde{f} = f \cdot (\frac{d\tilde{z}}{dz})^2$.

Note that for $g = 0$, i. e. for $X = \mathbb{P}^1(\mathbb{C})$, there are no nonzero holomorphic quadratic differentials. If $g = 1$, Ω_X is isomorphic to the structure sheaf $\mathcal{O}_X$, hence also $\Omega_X^{\otimes 2} \cong \mathcal{O}_X$, and there is, up to multiplication by a scalar, only one holomorphic quadratic differential. Explicitly, if X is the elliptic curve with equation $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \neq 0, 1$, the 1-form $\frac{dx}{y}$ is holomorphic, and therefore ω can be taken to be $\frac{dx^2}{y^2}$.

For $g \geq 2$, it follows from the Riemann-Roch theorem that $H^0(X, \Omega_X^{\otimes 2})$ is a $3g - 3$ dimensional complex vector space.

Now let Σ be the (finite!) set of zeroes of ω in X. For $P \in X^* = X - \Sigma$ choose a simply connected neighbourhood U_P contained in X^*. The map $\varphi : U_P \to \mathbb{C}$, $Q \mapsto \int_P^Q \sqrt{\omega} = \int_P^Q \sqrt{f} dz$ is a chart; here the integration is along an arbitrary path from P to Q in U_P. Since U_P is simply connected and does not contain any zero of ω, $\sqrt{\omega}$ is well defined in U_P up to sign. The transition map between U_P and $U_{P'}$ is obtained by adding $c = \int_P^{P'} \sqrt{\omega}$ for a fixed path from P to P', and possibly a change of sign. Thus the transition maps are of the form $\varphi' = \pm\varphi + c$. Such an atlas is called a *flat structure* on X^*, and X endowed with this structure is called a *flat surface*.

Remark 3.6. If $\omega = \eta^2$ is the square of a holomorphic 1-form, the associated flat structure on X is a translation structure.

Example 3.7. 1. If $X = E_{-1}$ is the elliptic curve with equation $y^2 = x(x-1)(x+1)$, we find back the translation structure induced on the torus by the Euclidean unit square. This can be seen using the universal covering $E_{-1} = \mathbb{C}/\Lambda$ with the standard lattice $\Lambda = \mathbb{Z} + \mathbb{Z}i$: the holomorphic differential dz is invariant under Λ and hence descends to $\frac{dx}{y}$ on E_{-1}. Therefore the translation structure is induced by the fundamental domain for Λ, which is the unit square.

2. On an origami $O = (p : X \to E)$, the translation structure is pulled back from E via p. Hence it corresponds to the differential $(p^*(\eta_E))^2$, where $\eta_E = \frac{dx}{y}$.

The flat structure induced by a holomorphic quadratic differential on a Riemann surface X can be varied in essentially the same way as for an origami: For a matrix $A \in \mathrm{SL}_2(\mathbb{R})$, we obtain a new flat structure on the surface underlying X. The same argument as above gives

Theorem 3.8. *Let ω be a holomorphic quadratic differential on a compact Riemann surface of genus $g \geq 1$. The construction just described gives a map*

$$\iota_\omega : \mathbb{H} = \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2\mathbb{R} \to T_{g,n},$$

where n is the number of zeroes of ω.
The map ι_ω is a Teichmüller embedding.

4 Veech groups

4.1 Teichmüller and moduli space

Recall that an origami $O = (p : X \to E)$ defines a Teichmüller embedding $\iota_O : \mathbb{H} \to T_{g,n}$ and a Teichmüller disk $\Delta_O = \iota_O(\mathbb{H}) \subset T_{g,n}$. We want to study the image $C(O)$ of Δ_O in $M_{g,n}$. Recall from Section 3.2 that a point in $T_{g,n}$ represents a pair (X, f), where X is a Riemann surface of genus g with n distinguished points on it, and f is a diffeomorphism from the reference surface X_0 to X. The *mapping class group*

$$\Gamma_{g,n} = \mathrm{Diffeo}^+(X_0)/\mathrm{Diffeo}_0^+(X_0)$$

acts on $T_{g,n}$ by $\alpha(X, f) = (X, f \circ \alpha)$. Note that this is well defined: if α and β represent the same element in $\Gamma_{g,n}$, $\alpha \circ \beta^{-1}$ is homotopic to the identity, and thus $(X, f \circ \alpha)$ and $(X, f \circ \beta)$ are equivalent pairs, hence define the same point in $T_{g,n}$.
It is a crucial fact that the action of $\Gamma_{g,n}$ on $T_{g,n}$ is properly discontinuous. The orbit space is the moduli space $M_{g,n}$ of Riemann surfaces of genus g with n distinguished points. The points of $M_{g,n}$ correspond bijectively to the classes under biholomorphic maps of such Riemann surfaces.

Fact. $M_{g,n}$ is a quasiprojective variety.

This means that there is a projective variety of which $M_{g,n}$ is an open subvariety (for the Zariski topology). The Deligne-Mumford moduli space $\overline{M}_{g,n}$ of stable curves is such a compactification of $M_{g,n}$, and Knudsen and Mumford proved that $\overline{M}_{g,n}$ is in fact projective; see [6] for an introduction to these results.

A well-known example of this fact is $M_{1,1}$ which is isomorphic to the (complex) affine line $\mathbb{A}^1 \cong \mathbb{C}$. The isomorphism is given by the famous *j-invariant*: every (complex) elliptic curve can be represented by a Weierstrass equation $y^2 = x^3 + Ax + B$ with complex numbers A and B satisfying $\Delta = 4A^3 + 27B^2 \neq 0$; the parameters A and B define isomorphic elliptic curves if and only if their j-invariants $j(A, B) = 1728\frac{4A^3}{\Delta}$ agree.

The map from Δ_O to its image $C(O)$ in $M_{g,n}$ clearly factors through the quotient by its stabilizer

$$\mathrm{Stab}(O) = \mathrm{Stab}_{\Gamma_{g,n}}(\Delta_O)$$

in the mapping class group. The following proposition tells us that $C(O)$ is almost determined by $\mathrm{Stab}(O)$:

Proposition 4.1. *For an origami O, the map $q_O : \Delta_O/\mathrm{Stab}(O) \to C(O)$ is birational.*

In fact, this result holds for arbitrary Teichmüller disks. It is due to the fact that $\Gamma_{g,n}$ acts properly discontinuously on $T_{g,n}$.

For origamis, this result has recently been strengthened by G. Schmithüsen [17]. To explain her result, note that the covering $p : X \to E$, which defines the origami O, can be considered as a family of coverings of elliptic curves (through the variation of the translation structure). It can be shown that this family induces a holomorphic map $p_O : C(O) \to M_{1,1}$; p_O maps the Riemann surface defined by a particular translation structure μ to the elliptic curve with the translation structure $p_*(\mu)$. On $M_{1,1}$ there are two special points E_0 and E_{1728} corresponding to the elliptic curves $y^2 = x^3 - 1$ (for $j = 0$) and $y^2 = x^3 - x$ (for $j = 1728$). They are the only elliptic curves with nontrivial automorphisms. Now the result is

Theorem 4.2 (Schmithüsen [17]). *The map q_O is an isomorphism outside $p_0^{-1}(E_0)$ and $p_0^{-1}(E_{1728})$.*

4.2 The affine group

Recall that an origami $O = p : X \to E$ defines a translation structure on X^*.

Definition 4.3. Let X and Y be translation surfaces. A diffeomorphism $f : X \to Y$ is called *affine* if there are coverings of X by charts (U_i, z_i) and of Y by charts (V_i, w_i) such that $f(U_i) \subseteq V_i$ and on U_i, f is given by an affine map $z_i \mapsto A_i z_i + b_i$, where $A_i \in \mathrm{GL}_2(\mathbb{R})$ and $b_i \in \mathbb{R}^2$.

Remark 4.4. If $f : X \to Y$ is an affine diffeomorphism of translation surfaces, the matrix A_i is independent of i.

Definition 4.5. Let X be a translation surface.

a) $\mathrm{Aff}^+(X)$ is defined to be the set of orientation preserving affine diffeomorphisms $f : X \to X$

b) $D : \mathrm{Aff}^+(X) \to \mathrm{GL}_2^+(\mathbb{R}), f \mapsto A$, is a well defined group homomorphism

c) $\mathrm{Trans}(X) = \ker(D)$ is called the group of *translations* of X

d) $\Gamma(X) = D(\mathrm{Aff}^+(X)) \subseteq \mathrm{GL}_2^+(\mathbb{R})$ is called the *Veech group* of X.

For an origami O, we define the Veech group $\Gamma(O)$ to be the Veech group $\Gamma(X^*)$ of the translation surface $X^* = X - p^{-1}(\infty)$. Note that every affine diffeomorphism of X^* has a unique extension to a diffeomorphism of X that maps the set $p^{-1}(\infty)$ of marked points to itself.
The compact surface X and therefore also X^* has finite area. Since the area has to be preserved by the affine diffeomorphisms, the Veech group $\Gamma(O) = \Gamma(X^*)$ is contained in $\mathrm{SL}_2(\mathbb{R})$.

Example 4.6. For the "baby" (or trivial) origami $E = \mathrm{id} : E \to E$, the Veech group is $\Gamma(E) = \mathrm{SL}_2(\mathbb{Z}) = \mathrm{Aff}^+(E^*)$.

To see this, consider the universal covering $\mathbb{R}^2$ of E: The translation structure on $\mathbb{R}^2$ coming from the "square" one on E is the usual euclidean structure. The affine maps for this translation structure are the familiar euclidean affine maps. Such an affine map descends to E if and only if it preserves the lattice $\mathbb{Z} + \mathrm{i}\mathbb{Z}$, i. e. if and only if the matrix part is in $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 4.7 (Earle/Gardiner [3]). *For an origami $O = p : X \to E$ we have*

$$\mathrm{Aff}^+(X^*) \cong \mathrm{Stab}(O).$$

For the proof recall that every $f \in \mathrm{Aff}^+(X^*)$ uniquely extends to a diffeomorphism of X. Thus we obtain a homomorphism $\rho : \mathrm{Aff}^+(X^*) \to \Gamma_{g,n}$. It is clear that its image is contained in $\mathrm{Stab}(O)$ and not very difficult to show that ρ is injective. The hardest part of the proof is to show that ρ is surjective onto the stabilizer $\mathrm{Stab}(O)$.

Under the isomorphism of the proposition, the translations correspond to the pointwise stabilizer $\mathrm{Stab}^{pw}(O)$ of Δ_O.

Remark 4.8. For an origami O, $\mathrm{Trans}(O) = \mathrm{Trans}(X^*) \cong \mathrm{Stab}^{pw}(O)$ is a finite group. For every point $(Y, h) \in \Delta_O$, it is contained in the automorphism group of the Riemann surface Y.

Corollary 4.9.

$$\Gamma(O) \cong \mathrm{Stab}(O)/\mathrm{Stab}^{pw}(O).$$

In view of Proposition 4.1 this corollary states that the Riemann surface $\mathbb{H}/\Gamma(O)$ is birationally equivalent to the algebraic curve $C(O)$ in $M_{g,n}$. Again, this statement holds in greater generality, namely for all Teichmüller disks Δ whose image $C(\Delta)$ in $M_{g,n}$ is closed. By a theorem of Smillie (cf. [18] §5 for a proof), this happens if and only if the Veech group of Δ is a lattice.

4.3 Veech groups of origamis

Using the definition of an origami by gluing squares, and also Example 4.6, one finds that the Veech group of an origami is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Less obvious is the fact that this subgroup is "large":

Proposition 4.10 (Veech, Gutkin/Judge [5]). *For an origami O, the Veech group $\Gamma(O)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.*

The following is a more precise result, which is the basis for an explicit calculation of Veech groups of origamis, since it allows for an algorithmic approach.

Theorem 4.11 (Schmithüsen [14]). *For an origami O, let $U \cong \pi_1(X^*)$ be the subgroup of $F_2 \cong \pi_1(E^*)$ induced by the covering $p : X \to E$. Then*

$$\Gamma(O) = \beta\,(\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U)),$$

where $\beta : \mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$ is the canonical homomorphism.

Sketch of proof. Consider $\mathbb{H}$ as the universal covering of E^* (and hence of X^*, too), and endow it with the translation structure induced by the square(s). Clearly $\Gamma(O) \subseteq \Gamma(\mathbb{H})$.
A crucial step now is to show that $\Gamma(\mathbb{H}) = \mathrm{SL}_2(\mathbb{Z})$ (and not larger!). The idea for this is as follows: Let $\mathbb{C} \to E$ be the universal covering. Its restriction to E^* is an unramified covering $w : \mathbb{C} - \Lambda \to E^*$, where $\Lambda = \mathbb{Z} + \mathrm{i}\,\mathbb{Z}$. Hence the universal covering $\mathbb{H} \to E^*$ factors through an unramified covering $h : \mathbb{H} \to \mathbb{C} - \Lambda$. Schmithüsen shows that h is the "developing map" for the translation structure on $\mathbb{H}$. As a consequence, $\Gamma(O)$ consists of all matrices $A \in \mathrm{SL}_2(\mathbb{Z})$ which are of the form $A = D(\hat{f})$ for some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ that descends to X^*.
To characterize those A, identify $F_2 = \pi_1(E^*)$ with the group $\mathrm{Gal}(\mathbb{H}/E^*)$ of deck transformations of the universal covering. Then $\mathrm{Aff}^+(\mathbb{H})$ can be identified with $\mathrm{Aut}^+(F_2)$ by sending $\hat{f}$ to the automorphism $\sigma \mapsto \hat{f}_*(\sigma) = \hat{f} \circ \sigma \circ \hat{f}^{-1}$.
The other crucial step in the proof is to show that $\hat{f}$ descends to X^* if and only if $\hat{f}_*(U) = U$. □

Note that Proposition 4.10 is a corollary of this theorem.

4.4 Characteristic origamis

Recall that a subgroup U of a group G is called *characteristic* if it is fixed by all automorphisms of G, i. e. $\varphi(U) = U$ for all $\varphi \in \mathrm{Aut}(G)$. Note that free groups have many characteristic subgroups in the following sense:

Remark 4.12. Let U be a subgroup of F_n of finite index (for some $n \geq 1$). Then U contains a characteristic subgroup of finite index.

Proof. Clearly the intersection of all $\varphi(U)$, where φ runs through all automorphisms of F_n, is characteristic. Since $\mathrm{Aut}\, F_n$ is finitely generated, this intersection is finite, and a finite intersection of finite index subgroups of a finitely generated group has finite index itself. □

We call an origami O characteristic if the corresponding subgroup $U(O) = \pi_1(X^*)$ of F_2 is characteristic. As a consequence of Theorem 4.11, the Veech group of a characteristic origami is $\mathrm{SL}_2(\mathbb{Z})$. Together with Remark 4.12 we obtain the following surprising fact:

Corollary 4.13. *There are infinitely many origamis of genus $g \geq 2$ whose Veech group is equal to* $\mathrm{SL}_2(\mathbb{Z})$.

Perhaps even more surprising, there are also examples of non-normal origamis with Veech group $\mathrm{SL}_2(\mathbb{Z})$.

One of the first characteristic origamis discovered, and the smallest nontrivial one, is the quaternion origami W. That W is characteristic can be seen using the following remark.

Remark 4.14. Let $U \subseteq F_2$ be a normal subgroup of finite index, and let $G = F_2/U$. Then U is characteristic if, for any two pairs (a, b) and (a', b') of generators of G, there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(a) = a'$ and $\sigma(b) = b'$.

Proof. Giving a pair (a, b) of generators is equivalent to giving a surjective homomorphism $h : F_2 \to G$, namely $h(x) = a$, $h(y) = b$. Then for any $\varphi \in \mathrm{Aut}(F_2)$, $h_\varphi = h \circ \varphi : F_2 \to G$ is also surjective. Hence by assumption, there exists $\sigma \in \mathrm{Aut}(G)$ such that $h_\varphi = \sigma \circ h$. It follows that

$$U = \ker(h) = \ker(\sigma \circ h) = \ker(h_\varphi) = \ker(h \circ \varphi) = \varphi^{-1}(U).$$

□

In the case of the quaternion origami W, the quotient group G is the classical quaternion group $Q_8 = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$. Except for 1 and -1, all elements of Q_8 have order 4, and any two of them, that are not inverse to each other, generate Q_8. Clearly every such pair can be mapped to (i, j) by an automorphism of Q_8.

In [7] we give an explicit construction of a characteristic subgroup contained in a given (normal) finite index subgroup. Applied to the smallest origami of genus > 1, the $L_{2,2}$, this construction first yields the stairlike origami of degree 6 as normal origami; the corresponding characteristic origami has degree 108 and was studied in detail in [1].

4.5 Congruence groups

It is a very natural question to ask which subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index are Veech groups of origamis.

The complete answer to this question is still open, but there are substantial partial results. Most of them have to do with congruence groups:

Definition 4.15. A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called a *congruence group* if Γ contains $\Gamma(n)$ for some $n \geq 1$, where

$$\Gamma(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n\} \subseteq \mathrm{SL}_2(\mathbb{Z})$$

is the kernel of the projection $p_n : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.
The smallest n such that $\Gamma(n) \subseteq \Gamma$ is called the *level* of the congruence group.

In her thesis [15], G. Schmithüsen proved that most congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ arise as Veech groups of origamis. The precise result is

Theorem 4.16. *Let $n \geq 1$ and $B = B_1 \dot{\cup} \ldots \dot{\cup} B_k$ any partition of $(\mathbb{Z}/n\mathbb{Z})^2$ satisfying $B_1 = \{(0,0)\}$. Denote by $\overline{\Gamma}_B \subseteq \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ the stabilizer of B, and let $\Gamma_B = p_n^{-1}(\overline{\Gamma}_B)$.*
Then there exists an origami O_B with $\Gamma(O_B) = \Gamma_B$.

The proof uses cleverly chosen coverings of the "trivial" $n \times n$ origami. From this theorem, Schmithüsen deduces

Corollary 4.17. *For any prime $p > 11$, any congruence group of level p is the Veech group of an origami. For each of the primes $2, 3, 5, 7, 11$, the same holds with one possible exception.*

For subgroups of $\Gamma(2)$, Ellenberg and McReynolds showed in [4]:

Theorem 4.18. *Every subgroup of $\Gamma(2)$ of finite index that contains $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Veech group of an origami.*

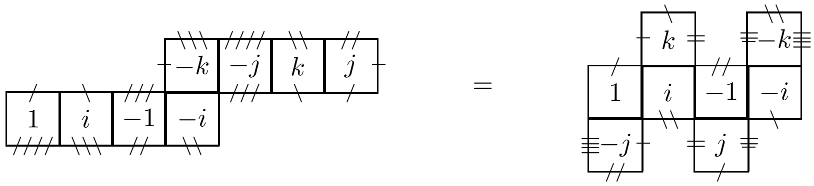
There are also results for noncongruence groups:

Proposition 4.19 (Schmithüsen; Hubert/Lelièvre [12]). *The group* $\Gamma(L_{2,n})$ *is a noncongruence group for* $n \geq 3$.

For the first example $n = 3$ in this series, the Veech group $\Gamma(L_{2,3})$ has index 9 in $\mathrm{SL}_2(\mathbb{Z})$.
There are also other origamis whose Veech group is known to be noncongruence. see [16].
On the other hand, there is not a single subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, for which it would be known that it cannot be the Veech group of an origami.

5 An example: The origami W

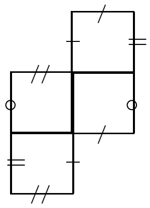
In this section, we take up the discussion of the quaternion origami W that was first introduced in Section 2.1 and later mentioned at several places. For more details we refer to [8].



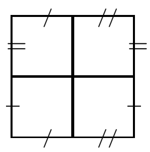
Recall that the corresponding (Riemann) surface X_W has genus 3 and that the Galois group of the covering $p : X_W \to E$ is the quaternion group Q_8. Since $\{\pm 1\}$ is a (normal) subgroup of Q_8, p factors through $q : X_W \to X_W/\{\pm 1\}$.

Remark 5.1. a) $X_W/\{\pm 1\} \cong E$.
b) $p = [2] \circ q$, where $[2]$ is the multiplication by 2 on the elliptic curve E.

This remark follows from the fact that the left half of the above figure is a fundamental domain for the action of $\{\pm 1\}$ on X_W:



which is equivalent to



The next observation is

Remark 5.2. The elliptic involution $[-1]$ on E lifts to an automorphism σ on X_W of order 2.

This remark is equivalent to saying that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is contained in the Veech group $\Gamma(W)$.

Note that σ is obtained by rotating each square by π, and then regluing the squares. It has 4 fixed points on X_W, namely the centers of the squares labelled 1, -1, k and -k. Thus σ can also be described as rotation by an angle of π around one (in fact, any) of these points.

Proposition 5.3. *The automorphism group* $\mathrm{Aut}(W)$ *is a group of order* 16*, generated by* Q_8 *and* σ*.*
The center of $\mathrm{Aut}(W)$ *is cyclic of order* 4*, generated by* $c = \mathrm{k} \cdot \sigma$*.*
Geometrically, c is the rotation by an angle of π *around the vertices. In particular, c has* 4 *fixed points.*

The quaternion origami W is one of the rare examples of a Teichmüller curve for which the equation is explicitly known:

Proposition 5.4. *The origami curve* $C(W)$ *is the image in* M_3 *of the family*

$$W_\lambda : \; y^4 = x(x-1)(x-\lambda), \;\; \lambda \in \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}.$$

The map $p_W : C(W) \to M_{1,1}$ *is induced from the fibrewise morphism* $p_\lambda : W_\lambda \to E_\lambda$*,* $(x,y) \mapsto [2] \cdot (x, y^2)$*, where* E_λ *is the elliptic curve with equation* $y^2 = x(x-1)(x-\lambda)$*.*

Idea of proof. Since c has 4 fixed points, the genus g of the quotient surface $X_W/ < c >$ can be calculated by the Riemann-Hurwitz formula: We have

$$2 \cdot 3 - 2 = 4 \cdot (2g-2) + 4 \cdot (4-1), \text{ hence } \; 4 = 8 \cdot (g-1) + 12.$$

Thus $g = 0$, and X_W is a cyclic cover of $\mathbb{P}^1$ of degree 4 with 4 ramification points, each of order 4. Applying an automorphism of $\mathbb{P}^1$, we may assume that the branch points of this covering are $0, 1, \infty$ and some $\lambda \in \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$. Thus W_λ has an equation of the form

$$W_\lambda : \; y^4 = x^{\varepsilon_0}(x-1)^{\varepsilon_1}(x-\lambda)^{\varepsilon_\lambda},$$

where each of the ε_i is either 1 or 3. Looking at the combinatorial descripition of W in terms of squares one sees that the monodromy is the same around each of the 4 branch points. In other words, the ε_i are all equal (and thus can be chosen to be 1). □

Besides σ, $\mathrm{Aut}(W)$ also contains the involutions $\tau = \mathrm{i}\sigma$ and $\rho = \mathrm{j}\sigma$.

Proposition 5.5. *For each of the six involutions $\alpha \in \{\pm\sigma, \pm\tau, \pm\rho\}$, the quotient surface $W_\lambda / < \alpha >$ is isomorphic to the elliptic curve*

$$E_{-1}: \; y^2 = x(x-1)(x+1) = x^3 - x.$$

In particular, the quotient surface is independent of λ.

Proof. Since c commutes with α for each choice of α, it descends to an automorphism of order 4 with 2 fixed points on $W_\lambda / < \alpha >$. The only elliptic curve having such an automorphism is E_{-1}. □

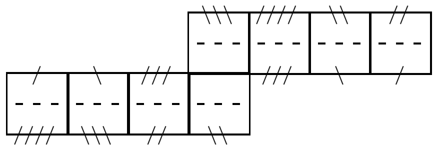
From this proposition, one can deduce the following

Corollary 5.6. *The Jacobian of W_λ is isogenous to $E_\lambda \times E_{-1} \times E_{-1}$, i. e. there is a homomorphism with finite kernel and finite cokernel between the two abelian varieties.*

This is a prominent case of a "fixed part" in a family of Jacobians. On the one hand, there is only a very limited list of 1-parameter families of Jacobians with a fixed part of codimension one. On the other hand, there are not so many origamis known which have a nontrivial fixed part; see [2] for more examples, and also for counter-examples.

Proposition 5.7. a) $C(W) \cong \mathbb{H}/\Gamma(W) = \mathbb{H}/SL_2(\mathbb{Z}) \cong \mathbb{A}^1$.
b) *The closure $\overline{C(W)}$ in $\overline{M}_3$ is isomorphic to $\mathbb{P}^1$.*
c) *The unique point in $\overline{C(W)} - C(W)$ corresponds to the stable curve with two irreducible components, both isomorphic to E_{-1}, that intersect (transversally) in two points.*

Proof of b) and c). As explained in detail in [10], going to the boundary in M_g can be achieved by replacing the square by rectangles that become thinner and thinner. This is equivalent to contracting the center lines of the squares in the original origami:



The contraction results in two irreducible components, both nonsingular of genus 1. The two dotted lines yield two points of intersection of these components. The automorphism c has two fixed points on each component, and acts as an automorphism on both of them. Once again, we now use the fact that the only elliptic curve admitting such an automorphism is E_{-1}. □

Because of its extraordinary properties we called W in german "Eierlegende Wollmilchsau". Perhaps the most important property is the following:

Theorem 5.8. *$C(W)$ intersects infinitely many other origami curves.*

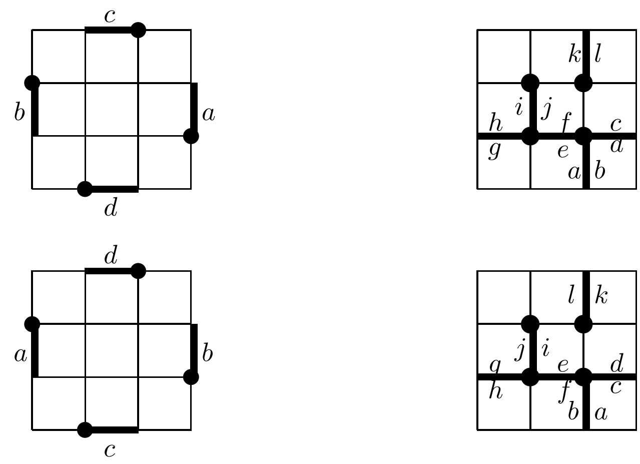
W was the first origami for which this result was discovered. To my knowledge, it is still the only one for which it has been proved.

Sketch of proof. The projection $\kappa_\lambda : W_\lambda \to W_\lambda / {<}\alpha{>}$ (cf. Proposition 5.5) is ramified over 4 points, namely the fixed points of α. If we can find an isogeny $\varphi : E_{-1} \to E_{-1}$ that maps these 4 points all to the same point, $\varphi \circ \kappa$ is an origami.
Any isogeny on E_{-1} is multiplication by a suitable n. Thus it suffices to show that for infinitely many different values of λ, all 4 critical points are torsion points. In fact, the 4 critical points of κ on E_{-1} form an orbit under the automorphism $\bar{c}$ induced by c. Choose one of the two fixed points of $\bar{c}$ as the origin of the group structure on E_{-1}. Then $\bar{c}$ preserves the n-torsion points for each n, and it suffices to find one critical point that is a torsion point.
Explicit calculation shows that for each $n \geq 3$ and each n-torsion point P on E_{-1}, there is $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$ such that P is a critical point of κ_λ. □

Corollary 5.9. *For each torsion point $P \in E_{-1}$ (of order $n \geq 3$) there is an origami D_P of degree $2n$ and genus 3 such that $C(D_P)$ intersects $C(W)$.*

It is a nice challenge to describe the origami D_P in terms of squares. By its construction, D_P is a double covering of a "trivial" $n \times n$-origami. more precisely, D_P consists of two copies of the trivial $n \times n$-origami, glued in such a way that we have 4 ramification points that form an orbit under rotation by 90.

Example 5.10. For $n = 3$, there are two different possibilities for the ramification points:



Here the leaf is changed at the highlighted edges.

In [8] we also determined explicitly the Veech group of D_P: it is a congruence group of level $2n$ if n is odd, and of level n if n is even.
Finally, in [9] we studied the configuration of all the origami curves $C(D_P)$ in M_3:

Theorem 5.11. *The closure in M_3 of the set of all origami curves $C(D_P)$, where as above, P is an n-torsion point on E_{-1} and D_P is the associated origami, is a three-dimensional algebraic subvariety H.*
More precisely, let $\tilde{H}$ be the Hurwitz space of coverings $X \to E$ of degree 2, where X has genus 3 and E is an elliptic curve, which are ramified over four points that are, for a suitable choice of the origin on E, of the form P, $-P$, Q, $-Q$. Then $\tilde{H}$ has four irreducible components, three of which consist of hyperelliptic Riemann surfaces X.
Then H is the image in M_3 of the non-hyperelliptic component of $\tilde{H}$.

References

[1] O. Bauer, Das 108er Origami, Diploma thesis, Karlsruhe 2005.

[2] O. Bauer, Familien von Jacobischen von Origamikurven, *PhD thesis, Universität Karlsruhe*, 2009.

[3] C. Earle and F. Gardiner, Teichmüller disks and Veech's $\mathcal{F}$-structures, *Contemp. Math.* 201 (1997), 165–189.

[4] J. Ellenberg and D. McReynolds, Every curve is a Teichmüller curve, arXiv 0909.1831.

[5] E. Gutkin and C. Judge, Affine mappings of translation surfaces, *Duke Math. J.* 103 (2000), 191–212.

[6] J. Harris and I. Morrison, Moduli of Curves, *Grad. Texts in Math.* 187, Springer 1998.

[7] F. Herrlich, Teichmüller curves defined by characteristic origamis, *Contemp. Math.* 397 (2006), 133–144.

[8] F. Herrlich and G. Schmithüsen, An extraordinary origami curve, *Math. Nachrichten* 281 No.2 (2008), 219–237.

[9] F. Herrlich and G. Schmithüsen, A comb of origami curves in the moduli space M_3 with three dimensional closure, *Geom. Dedicata* 124 (2007), 69–94.

[10] F. Herrlich and G. Schmithüsen, The boundary of Teichmüller disks in Teichmüller and in Schottky space, in A. Papadopoulos (ed.) *Handbook of Teichmüller theory, Volume I*, Chapter 6. European Mathematical Society (2006).

[11] F. Herrlich and G. Schmithüsen, Dessins d'enfants and origami curves, in A. Papadopoulos (ed.) *Handbook of Teichmüller theory, Volume II*, Chapter 18. European Mathematical Society (2009).

[12] P. Hubert and S. Lelièvre, Prime arithmetic Teichmüller disks in $\mathcal{H}_2$, *Israel J. Math.* 151 (2006), 281–321.

[13] P. Lochak, On arithmetic curves in the moduli spaces of curves, *J. Inst. Math. Jussieu* 4(3) (2005), 443–508.

[14] G. Schmithüsen, An algorithm for finding the Veech group of an origami, *Experimental mathematics* 13 (2004), 459–472.

[15] G. Schmithüsen, Veech Groups of Origamis. *PhD thesis, Universität Karlsruhe*, 2005.

[16] G. Schmithüsen, Origamis with noncongruence Veech groups, *Proc. 34th Symp. on Transformation Groups*, Wing Co., Wakayama 2007, 31–55.

[17] G. Schmithüsen, Generalized imprimitive Veech groups in Outer space, Manuscript 2010.

[18] J. Smillie and B. Weiss Finiteness results for flat surfaces: A survey and problem list, in G. Forni (ed.) *Partially hyperbolic dynamics, laminations, and Teichmüller flow*, Proc. of a conf., Fields Institute 2006.

[19] W.A. Veech, Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Inventiones Mathematicae*, 97(3) (1989), 553–583.