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Modular Forms: some prerequisites

Here comes a sketch of the prerequisites from complex analysis which will be made use of during my course on modular forms in winter 2006/07.

Analyticity: Let $D \subseteq \mathbb{C}$ be a domain, i.e. open and connected. For every holomorphic function $f : D \longrightarrow \mathbb{C}$ its Taylor-series at any point $p \in D$ converges to f locally uniformly on some neighborhood of p. The radius of convergence of the Taylor-series is at least as large as the distance from p to $\mathbb{C} \smallsetminus D$.

Around any isolated singularity of f, it can be expanded as a Laurent-series.

Openness: If f is a non-constant holomorphic function on D, then it is open, i.e. for every open $U \subseteq D$, $f(U)$ is open in $\mathbb{C}$.

Identity principle: Two holomorphic functions f and g on D coincide if and only if they coincide on some subset of D which has an accumulation point inside D.

Liouville's Theorem: A bounded, holomorphic function on $D = \mathbb{C}$ is constant.

Residue Theorem: If D is an elementary domain (i.e. simply connected) and $S \subset D$ is a discrete subset, and if $f : D \smallsetminus S \longrightarrow \mathbb{C}$ is holomorphic, then for every closed curve $\gamma : [0,1] \longrightarrow D \smallsetminus S$ we get

$$\int_\gamma f(\zeta)\mathrm{d}\zeta = 2\pi\mathrm{i} \sum_{s \in S} \mathrm{Res}(f,s) \cdot \chi(\gamma, s),$$

where $\mathrm{Res}(f,s)$ is the residue of f at s, i.e. the coefficient at $(z-s)^{-1}$ in the Laurent-series of f at s, and $\chi(\gamma,s)$ is the winding number of γ around s, counting how often γ goes round s. Formally,

$$\chi(\gamma, s) := \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{1}{\zeta - s}\mathrm{d}\zeta.$$

Argument principle: It is worth to notice that the residue of f'/f at $z \in D$ is the order of f at z, if f does not have an essential singularity at z. If all singularities of f at S are poles (i.e. inessential, i.e. f is meromorphic on D) and if the curve γ above avoids the zeros of f, then (if D is elementary)

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\mathrm{d}\zeta = \sum_{z \in D} \mathrm{ord}(f,z)\chi(\gamma, z).$$

If f is holomorphic on D, we get

$$\int_\gamma \frac{f'(\zeta)}{f(\zeta)}\mathrm{d}\zeta = \chi(f \circ \gamma, 0).$$

This is the number of zeros of f in the open set bounded by the image of γ, counted with multiplicities coming from the parametrization of γ.

Hurwitz' Theorem: If D is any domain and $(f_n)_{n \in \mathbb{N}}$ is a sequence of holomorphic functions on D without zeros which converge locally uniformly to some function f, then either f has no zeros as well or f is the zero function on D.