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p-adic Modular Forms

Problem Sheet 2

Problem 1 *The unit circle as a torus*
Let k be a field of characteristic $\mathrm{char}(k) \neq 2$ and let l be an extension of k that contains a square root i of (-1). Further let G be the affine k-group of 2×2-matrices of determinant one that satisfy the equation $AA^\top = 1$. These matrices are antisymmetric and one hence finds $\mathcal{O}(G) = k[a,b]/(a^2 + b^2 - 1)$. Finally, let $\mathbb{G}_m = \operatorname{Spec} l[x,y]/(xy - 1)$ be the multiplicative group over l.

Prove that the assignments $x \mapsto a + ib$ and $y \mapsto a - ib$ define an isomorphism $l \otimes_k G \cong \mathbb{G}_m$ of affine groups over l.

Problem 2 *Galois descent for vector spaces, part 1*
Let k be a field and let $l \mid k$ a Galois extension with Galois group $G = \mathrm{Gal}(l \mid k)$. We assume G to be equipped with its Krull topology, that is, a system of open neighbourhoods of some $\sigma \in G$ is given by $U_m = \{\tau \in G \mid \tau|_m = \sigma|_m\}$ where m ranges over finite extensions $k \subseteq m \subseteq l$.

(a) Let W be some k-vector space and let $V = l \otimes_k W$. Prove that the following conditions on some subspace $U \subseteq V$ are equivalent:

  (i) $GU = U$

  (ii) $GU \subseteq U$

  (iii) U is defined over k.

  In order to succeed, you might try to prove and use the following statements:

  (1) The orbit Gv for some $v \in V$ is finite and $\mathrm{tr}_G(v) = \sum_{w \in Gv} w$ is hence well-defined.

  (2) If $V \neq 0$ then there is some $v \in V$ such that $\mathrm{tr}_G(v) \neq 0$.

(b) Let A be some k-algebra. Prove that an ideal $\mathfrak{a} \subseteq l \otimes_k A$ is generated by elements of A if and only if $G\mathfrak{a} \subseteq \mathfrak{a}$.

(c) Consider the example $k = \mathbb{F}_p(T)$, $l = \mathbb{F}_p(\sqrt[p]{T})$ and $\mathfrak{a} = (X - \sqrt[p]{T}) \subseteq l[X]$. Use this example to deduce that the statement of part (b) is not necessarily valid if l is not Galois over k.

Problem 3 *Characters of tori*
Let $l \mid k$ be a Galois extension and $G = \mathrm{Gal}(l \mid k)$ be its Galois group. Further let T be a torus over k such that $T_l = l \otimes_k T$ splits.

(a) G acts on the characters $X^*(T_l)$ by $\chi^\sigma(t) = \sigma\chi(\sigma^{-1}t)$.

(b) A character $\chi \in X^*(T_l)$ is defined over k if and only if it is invariant under the action of G. Moreover, $X^*(T)$ is of finite index in $X^*(T_l)$.

Now let $\rho\colon T \to \mathrm{GL}(V)$ be a representation of T over k and let $\rho_l = l \otimes_k \rho\colon T_l \to \mathrm{GL}(V_l)$ be the corresponding representation over l.

(c) Let $\chi \in X^*(T_l)$ and let G_χ be its stabilizer in G. Prove that

$$V_{\chi,k} = \bigoplus_{\sigma \in G/G_\chi} V_{\chi^\sigma}$$

is defined over k and an irreducible representation of T.

(d) Show that

$$V = \bigoplus_{\chi} V_{\chi,k}\,,$$

where the direct sum is taken over representatives χ of the G-orbits in $X^*(T_l)$.

(e) The decomposition of V in part (d) is unique.

Hand in your solutions during the next problem session on Thursday, 7th November 2013