Problem 1  The unit circle as a torus
Let $k$ be a field of characteristic $\text{char}(k) \neq 2$ and let $l$ be an extension of $k$ that contains a square root $i$ of $(-1)$. Further let $G$ be the affine $k$-group of $2 \times 2$-matrices of determinant one that satisfy the equation $AA^\top = 1$. These matrices are antisymmetric and one hence finds $\mathcal{O}(G) = k[a,b]/(a^2 + b^2 - 1)$. Finally, let $\mathbb{G}_m = \text{Spec } l[x,y]/(xy - 1)$ be the multiplicative group over $l$.

Prove that the assignments $x \mapsto a + ib$ and $y \mapsto a - ib$ define an isomorphism $l \otimes_k G \cong \mathbb{G}_m$ of affine groups over $l$.

Problem 2  Galois descent for vector spaces, part 1
Let $k$ be a field and let $l | k$ a Galois extension with Galois group $G = \text{Gal}(l | k)$. We assume $G$ to be equipped with its Krull topology, that is, a system of open neighbourhoods of some $\sigma \in G$ is given by $U_m = \{\tau \in G | \tau|_m = \sigma|_m\}$ where $m$ ranges over finite extensions $k \subseteq m \subseteq l$.

(a) Let $W$ be some $k$-vector space and let $V = l \otimes_k W$. Prove that the following conditions on some subspace $U \subseteq V$ are equivalent:

(i) $GU = U$
(ii) $GU \subseteq U$
(iii) $U$ is defined over $k$.

In order to succeed, you might try to prove and use the following statements:

(1) The orbit $Gv$ for some $v \in V$ is finite and $\text{tr}_G(v) = \sum_{w \in Gv} w$ is hence well-defined.

(2) If $V \neq 0$ then there is some $v \in V$ such that $\text{tr}_G(v) \neq 0$.

(b) Let $A$ be some $k$-algebra. Prove that an ideal $\mathfrak{a} \subseteq l \otimes_k A$ is generated by elements of $A$ if and only if $Ga \subseteq \mathfrak{a}$.

(c) Consider the example $k = \mathbb{F}_p(T)$, $l = \mathbb{F}_p(\sqrt[3]{T})$ and $\mathfrak{a} = (X - \sqrt[3]{T}) \subseteq l[X]$. Use this example to deduce that the statement of part (b) is not necessarily valid if $l$ is not Galois over $k$. 
Problem 3  Characters of tori
Let $l \mid k$ be a Galois extension and $G = \text{Gal}(l \mid k)$ be its Galois group. Further let $T$ be a torus over $k$ such that $T_l = l \otimes_k T$ splits.

(a) $G$ acts on the characters $X^*(T_l)$ by $\chi^\sigma(t) = \sigma \chi(\sigma^{-1}t)$.

(b) A character $\chi \in X^*(T_l)$ is defined over $k$ if and only if it is invariant under the action of $G$. Moreover, $X^*(T)$ is of finite index in $X^*(T_l)$.

Now let $\rho : T \to \text{GL}(V)$ be a representation of $T$ over $k$ and let $\rho_l = l \otimes_k \rho : T_l \to \text{GL}(V_l)$ be the corresponding representation over $l$.

(c) Let $\chi \in X^*(T_l)$ and let $G_\chi$ be its stabilizer in $G$. Prove that
\[ V_{\chi,k} = \bigoplus_{\sigma \in G/G_\chi} V_{\chi^\sigma} \]
is defined over $k$ and an irreducible representation of $T$.

(d) Show that
\[ V = \bigoplus_{\chi} V_{\chi,k}, \]
where the direct sum is taken over representatives $\chi$ of the $G$-orbits in $X^*(T_l)$.

(e) The decomposition of $V$ in part (d) is unique.