Exercises

Spectral Sequences

1 First Introduction

Definition: A *double complex* (or *bicomplex*) $C_{\cdot\cdot}$ is a family $\{C_{pq}\}$ of modules with maps

$$d^h : C_{pq} \to C_{p-1,q} \text{ and } d^v : C_{pq} \to C_{p,q-1}$$

such that: $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$.

Definition: The *total complexes* $Tot(C) = Tot^{\Pi}(C)$ and $Tot^{\oplus}(C)$ are defined as:

$$Tot^{\Pi}(C)_n = \prod_{p+q=n} C_{pq} \text{ and } Tot^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{pq}$$

The formula $d = d^h + d^v$ defines maps

$$d : Tot^{\Pi}(C)_n \to Tot^{\Pi}(C)_{n-1} \text{ and } d : Tot^{\oplus}(C)_n \to Tot^{\oplus}(C)_{n-1}$$

with $d \circ d = 0$ and turns $Tot^{\Pi}(C)$ and $Tot^{\oplus}(C)$ into chain complexes.

The following part is nearly literally taken from Charles A. Weibel's »An introduction to homological algebra« and can be read at the beginning of chapter 5 *Spectral Sequences*:

Now forget about the horizontal differentials and write $E^0_{pq} := C_{pq}$. The columns of the double complex now form chain complexes with differentials d^v. We assume that the modules are zero unless p and q both nonnegative.
If we write E^1_{pq} for the vertical homology $H_q(E^0_{p*})$ at the (p, q) spot, we may once again arrange the data in a lattice, this time using the horizontal differentials d^h.
Now we write E^2_{pq} for the horizontal homology $H_p(E^1_{*q})$ at the (p, q) spot.
In a sense made clearer by the following exercises, the elements of E^2_{pq} are a second-order approximation of the homology of $T_* = Tot(E_{**})$.

Exercise 1.1: Suppose that the double complex E consists solely of the two columns p and $p-1$. Fix n and set $q = n - p$, so that an element of $H_n(T)$ is represented by an element $(a, b) \in E_{p-1,q+1} \times E_{pq}$. Show that we have calculated the homology of $T = Tot(E)$ up to extension in the sense that there is a short exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \longrightarrow H_{p+q}(T) \longrightarrow E^2_{pq} \longrightarrow 0.$$

Exercise 1.2:

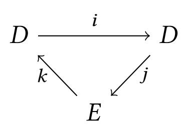
- Show that E^2_{pq} can be presented as the group of all pairs (a, b) in $E_{p-1,q+1} \times E_{pq}$ such that $0 = d^v b = d^v a + d^h b$, modulo the relation that these pairs are trivial: $(a, 0)$; $(d^h x, d^v x)$ for $x \in E_{p,q+1}$; and $(0, d^h c)$ for all $c \in E_{p+1,q}$ with $d^v c = 0$.
- If $d^h(a) = 0$, show that such a pair (a, b) determines an element of $H_{p+q}(T)$.
- Show that the formula $d(a, b) = (0, d^h(a))$ determines a well-defined map $d\colon E^2_{pq} \longrightarrow E^2_{p-2,q+1}$.

Exercise 1.3: Recall that we have assumed that E^0_{pq} vanishes unless both $p \geq 0$ and $q \geq 0$. By diagram chasing, show that $E^2_{00} = H_0(T)$ and that there is an exact sequence

$$H_2(T) \longrightarrow E^2_{20} \xrightarrow{\ d\ } E^2_{01} \longrightarrow H_1(T) \longrightarrow E^2_{10} \longrightarrow 0.$$

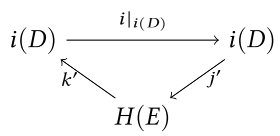
2 Exactness

Consider two modules D and E and morphisms such that the following diagram



is exact, i.e. kernel = image, at each vertex.

The map $d = jk$ fulfills $d^2 = jkjk = j(kj)k = j0k = 0$ and therefore we may form the homology module $H(E) = \ker(d)/\mathrm{im}(d)$. We get a second triangle



with morphisms $i' = i|_{i(D)}$, j' and k', which are given by $j'(i(x)) = [j(x)]$ and $k'([e]) = k(e)$ for $x \in D$ and $[e] \in H(E)$.

Exercise 2.1: Show that the maps i', j' and k' are well-defined and that this triangle, too, is exact at each vertex.