Revision and preparation towards "Homology and cohomolgy of groups"

We have already seen many of the things mentioned here in the last talk for homology of chain complexes. We will now observe that one can introduce cohomology very similarly, which is the analogon with differentials increasing the degree. These exercises serve as a chance to repeat and apply some of the technical framework of the last talk and to bring yourself in a perfect position to follow the next talks.

Exercise 0.1
Recall (or get to know) the definition of left and right modules over arbitary rings R and compare this definition to the one of a vector space over a field.[1]

Convention
From now on, we fix a ring R and all maps or morphisms are R-module homomorphisms for the contextual ring R.

Definition 0.2
A *chain (resp. cochain) complex* (A,d) of R-modules is a sequence $\{A_i\}_{i\in\mathbb{Z}}$ (resp. $\{A^i\}_{i\in\mathbb{Z}}$) of R-modules A_i together with R-module homomorphisms $d_i\colon A_i \to A_{i-1}$ (resp. $d^i\colon A^i \to A^{i+1}$), such that $d_i \circ d_{i+1} = 0$ (resp. $d^{i+1} \circ d^i = 0$).

$$\dots \xrightarrow{d_{i+2}} A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \xrightarrow{d_{i-1}} \dots$$

resp.

$$\dots \xrightarrow{d^{i-1}} A^{i-1} \xrightarrow{d^i} A^i \xrightarrow{d^{i+1}} A^{i+1} \xrightarrow{d^{i+2}} \dots$$

The *ith homology (resp. cohomology) module of* A is defined as $H_i(A) = ker(d_i)/im(d_{i+1})$ (resp. $H^i(A) = ker(d^{i+1})/im(d^i)$)[2].

Example 0.3
Fix a R-module A. Associating to each R-module B the $\mathbb{Z}$-module[3] $Hom_R(B,A)$[4] and to every R-module morphism $f\colon C \to D$ of R-modules C and D the $\mathbb{Z}$-module morphism

$$Hom_R(f,A)\colon Hom_R(D,A) \to Hom_R(C,A), g \mapsto g \circ f,$$

how can we get a cochain complex of $\mathbb{Z}$-modules starting from a chain complex of R-modules?[5]

Definition 0.4
A complex (A,d) is called exact, iff $ker(d_i) = im(d_{i+1})$ (resp. $ker(d^{i+1}) = im(d^i)$) for all i.

Example 0.5
Short exact sequences are just exact complexes of following shape.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Note that exactness at A is equivalent to f beeing injective and exactness at C to g beeing surjective.

The natural way to define morphisms between chain (resp. cochains) complexes and to make them form a category[6] is the following:[7]

Definition 0.6
Let (A,d), (B,e) be chain (resp. cochain) complexes. A *chain (resp. cochain) map* $f\colon A \to B$ is a family of morphisms $f = \{f_i\}_{i\in\mathbb{Z}}$ (resp. $f = \{f^i\}_{i\in\mathbb{Z}}$), such that

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{d_{i+2}} & A_{i+1} & \xrightarrow{d_{i+1}} & A_i & \xrightarrow{d_i} & A_{i-1} & \xrightarrow{d_{i-1}} & \dots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\dots & \xrightarrow{e_{i+2}} & B_{i+1} & \xrightarrow{e_{i+1}} & B_i & \xrightarrow{e_i} & B_{i-1} & \xrightarrow{e_{i-1}} & \dots
\end{array}$$

[1] http://en.wikipedia.org/wiki/Module_(mathematics) may help you.
[2] Why is this well-defined?
[3] Do you know another description of $\mathbb{Z}$-modules?
[4] With $Hom_R(C,D)$ for R-modules C and D we denote the set of all R-module homomorphisms from C to D. In which natural way gets $Hom_R(C,D)$ a $\mathbb{Z}$-module structure?
[5] $Hom_R(\_,A)$ is an example of an (additive) *functor* between module-categories. We will define them in the next talk and get to know some important examples.
[6] They form even an abelian one.
[7] You can just ignore that, if you haven't heard of the notion of a category before.

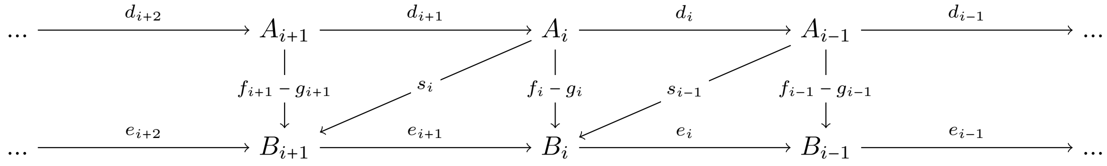
resp.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d^{i-1}} & A^{i-1} & \xrightarrow{d^{i}} & A^{i} & \xrightarrow{d^{i+1}} & A^{i+1} & \xrightarrow{d^{i+2}} & \cdots \\ & & \downarrow f^{i-1} & & \downarrow f^{i} & & \downarrow f^{i+1} & & \\ \cdots & \xrightarrow{e^{i-1}} & B^{i-1} & \xrightarrow{e^{i}} & B^{i} & \xrightarrow{e^{i+1}} & B^{i+1} & \xrightarrow{e^{i+2}} & \cdots \end{array}$$

commutes.

Exercise 0.7
Prove that every chain (resp. cochain) map $f: A \to B$ induces a morphism $f_{i,*}: H_i(A) \to H_i(B)$ (resp. $f^{i,*}: H^i(A) \to H^i(B)$) on each homology (resp. cohomology) module.
If you know the notion of a *functor*, you can check that each H_n (resp. H^n) forms one from the category of chain (resp. cochain) complexes to the category of R-modules.

Definition 0.8
Let $f, g: A \to B$ be two chain (resp. cochain) maps. One says *f and g are chain homotopic*, iff $f - g$ is *nullhomotopic*, i.e. iff there is a family of morphisms $\{s_i: A_i \to B_{i+1}\}_{i \in \mathbb{Z}}$ (resp. $\{s^i: A^i \to B^{i-1}\}_{i \in \mathbb{Z}}$) such that $f_n - g_n = s_{n-1} \circ d_n + e_{n+1} \circ s_n$ (resp. $f_n - g_n = s_{n+1} \circ d_n + e_{n-1} \circ s_n$).



Be aware that the diagram above does not necessarily commute in this situation. It's just an overview of the bunch of arrows in the definition.

Exercise 0.9
Convince yourself that two homotopic cochain maps induce the same morphisms on the cohomology modules.[8]

Definition 0.10
Let $f: A \to B$ and $g: B \to C$ be chain (resp. cochain) maps. One calls

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

a *short exact sequence of chain (resp. cochain) complexes*, iff

$$0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$$

resp.

$$0 \longrightarrow A^i \xrightarrow{f^i} B^i \xrightarrow{g^i} C^i \longrightarrow 0$$

is a short exact sequence of R-modules for all i.

Remark 0.11
I have to admit: The following exercise maybe isn't a little homework task to rush through, because initially one might have to prove some technical lemmas. Choose between or combine the options to give it a try on your own, to just believe it or to look at a proof in an introductory textbook about homological algebra.

Exercise 0.12
Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of cochain complexes. Construct natural maps $\partial^i: H^i(C) \to H^{i+1}(A)$[9], such that the sequence

$$\cdots \xrightarrow{g^{i-1,*}} H^{i-1}(C) \xrightarrow{\partial^{i-1}} H^i(A) \xrightarrow{f^{i,*}} H^i(B) \xrightarrow{g^{i,*}} H^i(C) \xrightarrow{\partial^{i}} H^{i+1}(A) \xrightarrow{f^{i-1,*}} \cdots$$

is exact.[10]

[8] We saw this for chain maps in the last talk. You can either try to deduce it from that or just compute it for cochain maps seperatley.
[9] They are called *connecting homomorphisms*.
[10] Again, we saw this already for chain complexes in the last talk.