

### ***A foreshadow of principal bundles***

Consider the the following map  $p$  from the unit circle  $S^1$  to itself:

$$p : X = S^1 \rightarrow B = S^1, z \mapsto z^2 \text{ where } S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$

Let  $G$  be the group  $(\mathbb{Z}/2\mathbb{Z}, +) = \langle \varphi \rangle$  endowed with the discrete topology. Show that there is an action of  $G$  on  $S^1$  from the right such that:

A)  $G$  acts effectively, i.e.:  $\forall x \in S^1, s \in G : (x \cdot s = x \Rightarrow s = \bar{0})$ .  
Here  $\bar{0}$  is the identity element in  $G$ .

B)  $X/G$  is isomorphic to  $B$ .

C) The map

$$\tau : X^* = \{(x, xs) \mid x \in X, s \in G\} \rightarrow G$$

with the property  $x \cdot \tau(x, y) = y$  is continuous.

What is the fibre of a point on  $S^1$ ?

Let now  $F$  be the unit interval  $[-1, 1]$ . Consider the action from the left of  $G$  by  $\varphi : z \mapsto -z$  for the non trivial element  $\varphi \in G$ . This action and the action of on  $X$  from the right from above give an action on the product  $X \times F$ :

$$G \ni s : (x, t) \mapsto (x \cdot s, s^{-1} \cdot t).$$

Does the map  $p_F : (X \times F)/G \rightarrow B, (x, t) \cdot G \mapsto p(x)$  also satisfy the properties A), B) and C) from above? The topological space  $(X \times F)/G$  is a topological space which you know very well. Which one? What is the fibre of a point on  $B$  via  $p_F$ ?

### ***Getting explicit about group homology***

Let  $G$  be a group. It holds

$$H_1(G; \mathbb{Z}) \cong \mathfrak{I}/\mathfrak{I}^2 \cong G/[G, G],$$

where  $\mathfrak{I}$  is the so called *augmentation ideal*, i.e. the kernel of the ring homomorphism

$$\varphi : \mathbb{Z}[G] \rightarrow \mathbb{Z}, f \mapsto \sum_{x \in \text{supp}(f)} f(x).$$

**Hint:**

First show  $H_1(G; \mathbb{Z}) \cong \mathfrak{I}/\mathfrak{I}^2$  by using the long exact sequence induced by the short exact sequence of  $\mathbb{Z}[G]$ -modules you get from  $\varphi$ . It is probably helpful to convince yourself that  $\{g - 1 \mid g \in G\}$  is a basis of  $\mathfrak{I}$  as a free  $\mathbb{Z}$ -module.

Again using this basis, you can show  $\mathfrak{I}/\mathfrak{I}^2 \cong G/[G, G]$  elementary.

### ***A useful property of the derived functor***

Prove that the zeroth derived functors are isomorphic to the original functor, i.e.  $R^0G(A) \cong G(A)$  and  $L_0F(A) \cong F(A)$  for right exact functors  $F$ , left exact functors  $G$  and all modules  $A$ .