**A foreshadow of principal bundles**

Consider the following map \( p \) from the unit circle \( S^1 \) to itself:

\[
p : X = S^1 \rightarrow B = S^1, \ z \mapsto z^2 \text{ where } S^1 = \{ z \in \mathbb{C} | |z| = 1 \}
\]

Let \( G \) be the group \((\mathbb{Z}/2\mathbb{Z}, +) = < \varphi >\) endowed with the discrete topology.

Show that there is an action of \( G \) on \( S^1 \) from the right such that:

A) \( G \) acts effectively, i.e.: \( \forall x \in S^1, s \in G : (x \cdot s = x \Rightarrow s = \bar{0}) \).

Here \( \bar{0} \) is the identity element in \( G \).

B) \( X/G \) is isomorphic to \( B \).

C) The map

\[
\tau : X^* = \{(x, xs) | x \in X, s \in G \} \rightarrow G
\]

with the property \( x \cdot \tau(x, y) = y \) is continuous.

What is the fibre of a point on \( S^1 \)?

Let now \( F \) be the unit interval \([-1, 1]\). Consider the action from the left of \( G \) by \( \varphi : z \mapsto -z \) for the non trivial element \( \varphi \in G \). This action and the action of on \( X \) from the right from above give an action on the product \( X \times F \):

\[
G \ni s : (x, t) \mapsto (x \cdot s, s^{-1} \cdot t).
\]

Does the map \( p_F : (X \times F)/G \rightarrow B, (x, t) \cdot G \mapsto p(x) \) also satisfy the properties A), B) and C) from above? The topological space \( (X \times F)/G \) is a topological space which you know very well. Which one? What is the fibre of a point on \( B \) via \( p_F \)?

**Getting explicit about group homology**

Let \( G \) be a group. It holds

\[
H_1(G; \mathbb{Z}) \cong \mathcal{I}/\mathcal{I}^2 \cong G/[G, G],
\]

where \( \mathcal{I} \) is the so called augmentation ideal, i.e. the kernel of the ring homomorphism

\[
\varphi : \mathbb{Z}[G] \rightarrow \mathbb{Z}, f \mapsto \sum_{x \in \text{supp}(f)} f(x).
\]

**Hint:**

First show \( H_1(G; \mathbb{Z}) \cong \mathcal{I}/\mathcal{I}^2 \) by using the long exact sequence induced by the short exact sequence of \( \mathbb{Z}[G] \)-modules you get from \( \varphi \). It is probably helpful to convince yourself that \( \{ g - 1 | g \in G \} \) is a basis of \( \mathcal{I} \) as a free \( \mathbb{Z} \)-module.

Again using this basis, you can show \( \mathcal{I}/\mathcal{I}^2 \cong G/[G, G] \) elementary.

**A useful property of the derived functor**

Prove that the zeroth derived functors are isomorphic to the original functor, i.e. \( R^0G(A) \cong G(A) \) and \( L_0F(A) \cong F(A) \) for right exact functors \( F \), left exact functors \( G \) and all modules \( A \).