Just a little bit of Linear Algebra

Let

$$\mathcal{C}^\bullet := \qquad \ldots \to V_4 \to V_3 \to V_2 \to V_1 \to V_0 \to 0$$

be a chain complex of k-vector spaces. Show that

$$\sum_{i=0}^{N} \dim V_i = \sum_{i=0}^{N} \dim H_i(\mathcal{C}^\bullet).$$

Hint: Recall that we have for a short exact sequence $0 \to A \to B \to C \to 0$ *of* k*-vector spaces that* $\dim(A) - \dim(B) + \dim(C) = 0$

What happens, if the V_i's are free $\mathbb{Z}$-modules?

Warm up on holomorphic functions

The following exercise is a warm up preparing Riemann surfaces recommended if you do not know holomorphic functions so far. The topic is not essential for the rest of the seminar but helps to understand a nice motivation for what we are doing which we will see in the next talk.

We consider in the following functions $f : \mathbb{C} \to \mathbb{C}$ from the complex plane to itself.

i) Look up the definition of *holomorphic*.

ii) For which points in $\mathbb{C}$ are the following functions well-defined and holomorphic?

$$z \mapsto \overline{z},\ z \mapsto |z|^2,\ z \mapsto Re(z)^2 + Im(z)^2 + (-Re(z)^2 + Im(z)^2)i,$$
$$z \mapsto \frac{az+b}{cz+d} \text{ with } a,b,c,d \in \mathbb{C}\text{Möbius transformation.}$$

iii) Construct a function which is holomorphic precisely on the diagonal

$$D = \{z \in \mathbb{C} | Re(z) = Im(z)\}.$$

Hint: Use the Cauchy-Riemann equations or the Wirtinger derivative.

Automorphisms of $\mathbb{D}$

One of the helpful tools one has for holomorphic functions is the *maximum modulus principle*. As applications for it one e.g. obtains the *Schwarz lemma*, explicit descriptions of the automorphism groups of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ and the *Riemann mapping theorem*. We see in the following how to obtain the automorphism groups $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$.

i) Look up the maximum modulus principle.

ii) Let us consider

$$\mathrm{Aut}(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{D} : \begin{array}{l} f \text{ is holomorphic and} \\ \text{has an inverse map } g \text{ which is also holomorphic.} \end{array} \right\}$$

Show that the map $z \mapsto e^{i\theta} \cdot z$ $(\theta \in \mathbb{R})$ is in $\mathrm{Aut}(\mathbb{D})$ and fixes 0.

iii) Consider the map

$$\mathbb{D} \to \mathbb{C}, \quad z \mapsto \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0. \\ f'(0), & \text{if } z = 0. \end{cases}$$

Convince yourself that the map is holomorphic.

iv) Restrict g to $\overline{\mathbb{D}}_r := \{z \in \mathbb{D} : |z| \leq r\}$. Show that

$$\forall z \in \overline{\mathbb{D}}_r : |g(z)| \leq \frac{1}{r}$$

and conclude that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$.
Hint: This is where the maximum modulus principle comes in. It also helps to observe that $\overline{\mathbb{D}}_r$ is compact.

v) Observe from the last step that $|f(z)| = |z|$.
Hint: f is invertible.

vi) Obtain finally:

$$\mathrm{Stab}_{\mathrm{Aut}(\mathbb{D})}(0) := \{f \in \mathrm{Aut}(\mathbb{D}) | f(0) = 0\} = \{z \mapsto e^{i\theta} \cdot z | \theta \in \mathbb{R}\}$$

vii) How can you conclude from this that $\mathrm{Aut}(\mathbb{D})$ and $\mathrm{Aut}(\mathbb{H})$ consist of Möbius transformations? Which Möbius transformations do you obtain?
Hint: The group of Möbius transformations which map $\mathbb{D}$ to $\mathbb{D}$ acts transitively on $\mathbb{D}$. Furthermore, there is a Möbius transformation which maps $\mathbb{D}$ to $\mathbb{H}$.