Seminar: Stable Cohomology of the Mapping Class Group

The Arc Complex – Part Two

Warm-up Exercises

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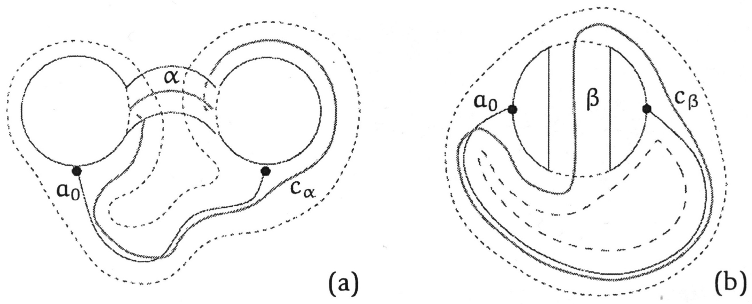
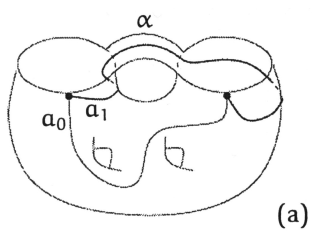
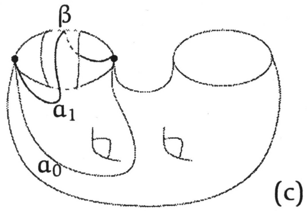
1 Fun with Dehn Twists

Recall. Let S be a surface. In the last talk we learned how to glue in strips with the maps α and β (see below for an image). We denote by S_α and S_β the surface S together with the strip glued in via α and β, respectively.

Problem. Consider the loops c_α and c_β on the surface S_α and S_β as shown in the following pictures.







Let a_0 and a_1 be the arcs defined in the image above and denote with t_c the Dehn twist along a loop c. Show that the arc a_1 is mapped to a_0 by the Dehn twist t_{c_α} respectively by t_{c_β}.

2 A Whiff of Relative Homology

Recall. Let G be a group and $\mathrm{B}G$ its classifying space. There is a canonical isomorphism

$$\mathrm{H}_\bullet(G;\mathbb{Z}) \cong \mathrm{H}_\bullet(\mathrm{B}G;\mathbb{Z}).$$

For the rest of this sheet we will omit the $\mathbb{Z}$, i.e. write $\mathrm{H}(G)$ instead of $\mathrm{H}(G;\mathbb{Z})$.

Problem. (i) Let H be a subgroup of G. Show that there is a canonical embedding $\mathrm{B}H \hookrightarrow \mathrm{B}G$ of the classifying spaces.

(ii) Determine the chain complexes that we use to compute the homology of the classifying spaces $\mathrm{B}G$ and $\mathrm{B}H$.

Recall. For topological spaces X and A with $A \subseteq X$ we have

$$\mathrm{H}_n(X, A) = \mathrm{H}_n(\mathcal{C}_\Delta(X)/\mathcal{C}_\Delta(A)).$$

Definition. Let $H \leq G$ be groups. We define the relative group homology, i.e. the homology of G relative to H, as

$$\mathrm{H}_n(G, H) \coloneqq \mathrm{H}_n(\mathrm{B}G, \mathrm{B}H).$$

Problem. Let $G \leq H$ be groups. We write G-**Mod** for the category of G-Modules. Use the standard resolutions of $\mathbb{Z}$ in G-**Mod** and H-**Mod** to construct an explicit chain complex whose homology is the relative homology $\mathrm{H}(G, H)$.

Problem. Let C_k be the cyclic group of order $k \in \mathbb{N}$. Compute $\mathrm{H}_\bullet(C_{k^2}, C_k)$.

Hint. For the cyclic group C_k we have

$$\mathrm{H}_n(C_k) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}/k\mathbb{Z} & \text{for } n \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Hint. Let $A \subseteq X$ be a subspace. The snake lemma yields a long exact sequence

$$\cdots \longrightarrow \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(X, A) \longrightarrow \mathrm{H}_{n-1}(A) \longrightarrow \cdots$$