Connectivity of arc complexes - Part I

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In this seminar we consider the simplicial complex made of isotopy classes of arcs in a surface S, with their end points in the boundary of the surface.

Let Δ denote a finite set of points in the boundary of S. An arc in S with its ends in Δ is said to be trivial if it separates the surface into two parts with one of them being a disk containing no other points in Δ than its end points. For example the arc in the following figure is trivial:

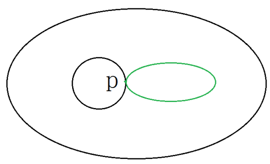

Figure 1: The black circles are boundary components of the annulus; the green arc is trivial

Let $A(S, \Delta)$ denote the simplicial complex whose vertices are the isotopy classes of non-trivial arcs in S with their end points in Δ; a q-simplex in $A(S, \Delta)$ is a collection of $q+1$ distinct isotopy classes of such arcs which mutually do not intersect except at their end points.

Now we have the following challenge problems:

1. **Simplicial complex of a disk**

   Suppose the surface is a disk.

   (a) Please check that for $A(S, \Delta)$ to be non-empty, Δ has to contain at least four points in the boundary of the disk.

(b) What happens if Δ contains 5 or 6 points? Which topological spaces are the two corresponding complexes? Can you guess which topological spaces the complexes for a disk are in general?

2. An example of contractible arc complexes

In this part we are going to learn a theorem which states that in general the arc complexes are contractible, except for some special cases (for example, S is a disk). Can you convince yourself of the contractibility for the arc complex corresponding to the surface shown in the following figure and $\Delta = \{p_1, p_2, p_3\}$?

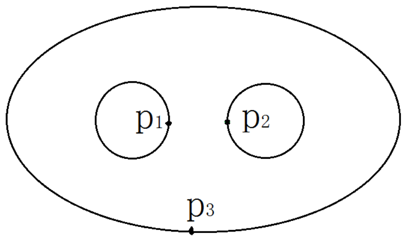

Figure 2: Δ contains one point in each boundary component of the pants.