Introduction to the Langlands Program

Winter 2013

We will approach the subject in two stages. The main stage consists of the seminar weekend in december, and before we will have a collection of introductory talks in Karlsruhe, which may be helpful in the understanding of the workshop.

During the workshop weekend we won't be able to go into the classical theories of Hecke characters and modular forms. However the speakers are invited to refer to the latter two as important and motivating examples. Therefore it would be helpful to get acquainted with the basics of the classical picture, for which we refer to [11, Chapter 1], [8, Chapter 1], [3, Chapter 3], [27, Chapters 1-3].

We will focus on automorphic representations of $\mathrm{GL}(n)$ over number fields. The situation in the geometric Langlands program (characteristic $p > 0$) is often simpler, as the representation theory at the archimedean places doesn't interfere. Consequently the seminar implicitly also lays the foundations for this case.

In any case our goal is to get a feeling for the fundamental objects of the Langlands program and thus bridge the gap between classical textbooks and the vast literature on the subject. Our references are by far not complete, but they are a useful starting point.

In the first workshop lectures (V1)-(V8) we define the underlying objects and sketch the corresponding representation theory. For $\mathrm{GL}(n)$ the picture here is complete and does not depend on conjectures. In the lectures (V9)-(V11) we apply what we have learned to dig deeper into the Jacquet-Langlands Correspondence for $\mathrm{GL}(2)$, one of the first proven instances of Landlands Functoriality, the local and global Langlands Correspondence for $\mathrm{GL}(n)$ (only the local one is known), as well as Landlands Functoriality (known nowadays in a few cases).

As our time is limited we won't be able to go into base change (Arthur-Clozel), potential modularity, existence of Galois representations, the Arthur-Selberg trace formula, etc.

Lectures

The following preliminary lectures are possible:

(E1) Representation theory of (pro)finite groups and (Artinian) L-functions for Galois representations, Artin formalism.

(E2) Class field theory and consequences for L-functions.

(E3) The classification of complex semi-simple Lie algebras.

(E4) The irreducible finite-dimensional representations of complex semi-simple Lie algebras.

(E5) Reductive algebraic groups, their classification and rational representations.

(E6) Adeles, ideles, lattices and adelization of algebraic groups.

(E7) Modular forms for $\mathrm{GL}(2)/\mathbb{Q}$ and their L-functions.

On the workshop weekend, lecture V1 is planned for friday evening, lectures V2-V6 for saturday, and V7-V11 for sunday.

(V1) **Representation theory of compact Lie groups, real reductive groups.** Lie algebra, maximal tori, roots, finite-dimensional (resp. unitary) representations of compact groups, and some structure theory for real reductive Lie grupps: existence and uniqueness of maximal compact subgroups, complexification of linear reductive groups, Iwasawa decomposition, symmetric spaces; Ref: Knapp's book [20], and Helgason's [17].

(V2) **Reductive algebraic groups, adelization.** Reductive algebraic groups, parabolic subgroups, Levi decomposition, adelization, arithmetic subgroups, strong approximation and consequences; Ref: Springer's lecture in Corvallis [9, Reductive groups], Murnaghan's chapter in [2], Borel's book [4], Borel-Tits' article [5, 6], and Tits' Corvallis lecture [9, Reductive groups over local fields] (optional), and furthermore [8, Chapter 3, Section 3.3], [11, Chapter 3], Kudla's lectures in [3, 6. Tate's Thesis and Section 1 in 7. From Modular Forms to Automorphic Representations] for $\mathrm{GL}(2)$.

(V3) **Automorphic forms on $\mathrm{GL}(n)$.** K-finite, smooth and L^2-forms on $\mathrm{GL}(n)$, essentially as in Cogdell's lecture 2 in [10], and Borel-Jacquet's [9, Automorphic forms and automorphic representations] (here we focus only on forms, not yet on representations); further references are [8, Chapter 3, Section 3.3], [11, Chapter 3], and Kudla's 2nd lecture in [3, Sections 1 and 2 in 7. From Modular Forms to Automorphic Representations] (the latter only discuss $\mathrm{GL}(2)$).

(V4) **Automorphic representations on $\mathrm{GL}(n)$.** Automorphic representations, Hecke algebra, as in Cogdell's lecture 3 in [10], as well as Borel-Jacquet's and Flath's Corvallis lectures [9, Automorphic forms and automorphic representations, Decomposition of representations into tensor products], also [8, Chapter 3, Section 3.3]; further references are [11, Chapter 3 and Chapter 5], and Kudla's 2nd lecture in [3, Section 2 in 7. From Modular Forms to Automorphic Representations] (the latter restrict all to $\mathrm{GL}(2)$).

(V5) **Langlands Classification: Archimedean case.** Langlands' classification of irreducible admissible representations of $\mathrm{GL}(n,k)$, $k \cong \mathbb{R}$ and $k \cong \mathbb{C}$. Ref: Langlands' original [22], Wallach's and Knapp's Corvallis lectures [9, Representations of reductive Lie groups, Representations of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{C})$], Murty's Chapter 4 in [10], for $\mathrm{GL}(2)$ also [11, Chapters 2 and 4], [8, Chapter 2], and Kudla's 2nd lecture in [3, Section 3 in 7. From Modular Forms to Automorphic Representations]. For those who want to dig deeper we refer to [1, 7, 13, 14, 15].

(V6) **Langlands Classification: Non-archimedean case.** The classification of irreducible admissible representations of $\mathrm{GL}(n,k)$ for k non-archimedean. Ref: Cartie's Corvallis lecture [9, Representations of $\mathfrak{p}$-adic groups], Murty's Chapter 4 in [10], for $\mathrm{GL}(2)$ also [11, Chapter 4], [8, Chapter 4], Kudla's 2nd lecture in [3, Section 3 in 7. From Modular Forms to Automorphic Representations].

(V7) **Langlands Classification: Global.** Langlands' (first) Corvallis lecture [9, On the notion of an automorphic representation], for Langlands' essential general theory of Eisenstein series, see [23]. Also Murty's chapters 4 and 5 in [10].

(V8) **Fourier transform and multiplicity one.** Via non-abelian Fourier transform, which leads to Whittaker models, we conclude that the space of cusp forms for $\mathrm{GL}(n)$ is multiplicity free [26, 25]. Further references: Cogdell's lecture 4 in [10], Piatetski-Shapiro's Corvallis lecture [9, Multiplicity one theorems], Cogdell's first lecture in [3, Section 1 in 9. Analytic Theory of L-Functions for GL_n], and for $\mathrm{GL}(2)$ the original in [19], and as usual [11] and [8, Chapter 3, Section 3.5].

(V9) **The Jacquet-Langlands Correspondence.** Jacquet-Langlands' original [19, Chapter III], [8], [11], Gelbart's lectures [12], and for $\mathrm{GL}(n)$ [16, Chapter VI, Section VI.1]. Foundations of quaternion algebras may be found in [28].

(V10) **The local and global Langlands Correspondence.** The local non-archimedean Langlands correspondence is treated in Harris-Taylor's [16] (the introduction is very readable!) and in Henniart's original work [18]; Murty's Chapter 10 in [10] sketches the local correspondence and Cogdell discusses in his 2nd lecture in [3, 10. Langlands Conjectures for GL_n] the local and global (incl. the geometric) setup, furthermore Borel's Corvallis lecture [9, Automorphic L-functions, Section 12] is a good reference (at the time the local Langlands correspondence was still an open problem). Additionally [8, Chapter 4, Section 4.9] is worth reading. Historically of course also [19, Chapter I] for $\mathrm{GL}(2)$.

(V11) **Langlands' Functoriality Conjectures.** Langlands' [24], Cogdell's 3rd lecture in [3, 11. Dual Groups and Langlands Functoriality], Cogdell's lectures 11, 12 and 13 in [10]. An informal discussion may be found in [8, Chapter 3, Section 3.9].

Literatur

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Weblinks

Some literature is available online:
Casselman's online collection of Langlands' work:
`sunsite.ubc.ca/DigitalMathArchive/Langlands/`
and Langlands' page at the IAS:
`publications.ias.edu/rpl/`
The Corvallis proceedings [9] were completely available online on the AMS website in old days.
Gelbart's lectures [12] are online:
`arxiv.org/abs/math/9505206`
As is [2] officially:
`www.claymath.org/library/proceedings/cmip04.pdf`