Schottky space and Teichmüller disks

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If "group actions" means that an interesting group acts in an interesting way on an interesting space, the action of the mapping class group on Teichmüller space is one of the most prominent examples: In addition to the space and the group, also the orbit space, which is the moduli space of Riemann surfaces, is highly interesting. Perhaps the most important intermediate space between Teichmüller and moduli space is Schottky space.

To be a little more specific, fix a genus $g \geq 2$ and let X_0 be a fixed closed Riemann surface of genus g. As usual, we denote by M_g the moduli space of all isomorphy classes of closed Riemann surfaces of genus g, by T_g the Teichmüller space of all marked Riemann surfaces, i. e. equivalence classes of pairs (X, f), where X is Riemann surface of genus g and $f : X_0 \to X$ a diffeomorphism, and by Mod_g the mapping class group $\mathrm{Mod}_g = \mathrm{Diff}^+(X_0)/\mathrm{Diff}_0(X_0)$. We use the following well known facts:

- T_g is a contractible real manifold of dimension $6g - 6$
- Mod_g acts properly discontinuously on T_g
- $T_g/\mathrm{Mod}_g = M_g$
- Mod_g acts by isometries for the Teichmüller distance on T_g
- T_g carries a natural complex structure, and Mod_g acts by holomorphic automorphisms
- The stabilizer in Mod_g of a point $(X, f) \in T_g$ is isomorphic to the group $\mathrm{Aut}(X)$ of holomorphic automorphisms of X

The *Schottky space* S_g is the quotient of T_g by a subgroup $\mathrm{Mod}_g(\alpha)$ of Mod_g, (depending on a certain group homomorphism α which will be explained in Section 3). A basic result is

- $\mathrm{Mod}_g(\alpha)$ is torsion free (see Prop. 8)

This immediately implies

- S_g is a complex manifold of dimension $3g - 3$
- $T_g \to S_g$ is a universal covering

Moreover, $\mathrm{Mod}_g(\alpha)$ is an infinite subgroup of infinite index, but not a normal subgroup. As a consequence, the induced covering $S_g \to M_g$ is infinite, but not normal.

In fact the group of biholomorphic automorphisms of S_g is isomorphic to $\mathrm{Out}(F_g)$, the group of outer automorphisms of the free group F_g of rank g, see [E]. This group is also isomorphic to $H_g(\alpha)/\mathrm{Mod}_g(\alpha)$, where $H_g(\alpha)$ is the normalizer of $\mathrm{Mod}_g(\alpha)$ in Mod_g. For different α, the groups $\mathrm{Mod}_g(\alpha)$ are conjugated; the same holds for the groups $H_g(\alpha)$, which moreover are isomorphic to the mapping class group of a handlebody of genus g.

The covering $S_g \to M_g$ factors through $\tilde{S}_g = S_g/\mathrm{Out}(F_g)$, and the induced map $\tilde{S}_g \to M_g$ still has infinite fibres. Sometimes S_g is called the *marked* Schottky space and $\tilde{S}_g$ the *unmarked* Schottky space.

Of particular interest in this survey is the behaviour of Teichmüller disks under the covering map $T_g \to S_g$. A Teichmüller disk is an isometrically and holomorphically embedded complex unit disk (or upper half plane) in Teichmüller space. They naturally arise from quadratic differentials on Riemann surfaces and are closely related to flat structures on surfaces. In general, the stabilizer in Mod_g of a Teichmüller disk is trivial (or cyclic of order 2), but sometimes it is a lattice in $\mathrm{PSL}_2(\mathbb{R})$. In this case the image in M_g of the Teichmüller disk is a Riemann surface of finite type, i. e. an (affine) algebraic curve. These curves in moduli space are called *Teichmüller curves*; they have attracted a lot of attentionin the last 10 or 15 years. In Section 8 we show, following [DF], that the stabilizer in $\mathrm{Mod}_g(\alpha)$ of a Teichmüller disk that is induced by a translation surface is either trivial or infinite cyclic (in which case the image of the Teichmüller disk in S_g is a once punctured disk). Moreover, if the so called *Veech group* of the Teichmüller disk contains a parabolic element, we can always find α such that the stabilizer is infinite cyclic.

1 Schottky coverings

The original definition of a Schottky group is as follows [Sy]: Let $D_1, D'_1, \ldots, D_g, D'_g$ be $2g$ mutually disjoint closed disks in the complex plane $\mathbb{C}$ (or the Riemann sphere $\hat{\mathbb{C}}$) and choose Möbius transformations $\gamma_i \in \mathrm{PSL}_2(\mathbb{C})$ such that $\gamma_i(\partial D_i) = \partial D'_i$ and $\gamma_i(D_i) = \hat{\mathbb{C}} - \mathring{D}'_i$, $i = 1, \ldots, g$ $(g \geq 1)$. The subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$ that is generated by $\gamma_1, \ldots, \gamma_g$ is called a *Schottky group*; it has the following properties:

Proposition 1. **a)** Γ *is a free group freely generated by* $\gamma_1, \ldots, \gamma_g$.
b) *Let* $F = \hat{\mathbb{C}} - \bigcup_{i=1}^{g}(\mathring{D}_i \cup \mathring{D}'_i)$ *and* $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(F)$. *Then* Γ *acts properly discontinuously (and freely) on* Ω, *and* F *is a fundamental domain for this action.*
c) Ω/Γ *is a compact Riemann surface of genus* g.

Proof. **a)** This is the original situation of the well-known "ping-pong lemma": let $w = \gamma_{i_n}^{\varepsilon_n} \cdot \cdots \cdot \gamma_{i_1}^{\varepsilon_1}$ be a reduced word in $\gamma_1, \ldots, \gamma_g$ and γ the corresponding element in Γ. Then if $z \in F$ is any point we see by induction on n that $\gamma(z) \in \mathring{D}'_i$ if $\varepsilon_n = 1$ and $\gamma(z) \in \mathring{D}_i$ if $\varepsilon = -1$. In both cases $\gamma(z) \neq z$, thus $\gamma \neq \mathrm{id}$.
This argument also proves **b)**.

For **c)** observe that $X = \Omega/\Gamma$ is a Riemann surface by b) and that topologically it is obtained from F by identifying ∂D_i with $\partial D_i'$ for $i = 1, \ldots, g$. This clearly gives a sphere with g handles. $\square$

Note that the same proof holds if we replace the disks D_i and D_i' by simply connected closed domains that are bounded by Jordan curves. Of course, now we must require that there exists a Möbius transformation γ_i mapping ∂D_i to $\partial D_i'$ and $\mathring{D}_i$ to $\hat{\mathbb{C}} - D_i'$. We shall use this more general definition of a Schottky group throughout this paper.

If $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ is a Schottky group and $\Omega = \Omega(\Gamma)$ its domain of discontinuity as in the proposition, we call the quotient map $\pi : \Omega \to \Omega/\Gamma =: X$ a *Schottky covering* of the Riemann surface X. A variant of the uniformization theorem states that this is possible for every Riemann surface:

Proposition 2. *Every compact Riemann surface X of genus $g \geq 1$ admits a Schottky covering with a Schottky group Γ of rank g.*

Idea of proof. Let $c_1, \ldots, c_g$ be a "cut system" on X, i. e. simple closed loops such that $F = X - \cup_{i=1}^{g} c_i$ is connected; equivalently $[c_1], \ldots, [c_g]$ are linearly independent in $H_1(X, \mathbb{C})$. Then F is biholomorphically equivalent to a plane region with $2g$ boundary components $C_1, C_1', \ldots, C_g, C_g'$, where C_i and C_i' are the two components corresponding to c_i. This implies that there is a Möbius transformation γ_i mapping C_i to C_i' and such that $F \cap \gamma_i(F) = C_i'$. The subgroup Γ of $\mathrm{PSL}_2(\mathbb{C})$ generated by $\gamma_1, \ldots, \gamma_g$ is a Schottky group and F is a fundamental domain for Γ. Then clearly $X = F/\Gamma = (\bigcup_{\gamma \in \Gamma} \gamma(F))/\Gamma$. The technical details skipped in this sketch can be found e. g. in [AS]. $\square$

The situation is particularly simple and explicit in genus 1: Here we can choose D_1 and D_1' to be disks around 0 and ∞, resp. Then F is an annulus of the form $\{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ for some positive real numbers $r_1 < r_2$, and $\gamma := \gamma_1$ is the transformation $\gamma(z) = \lambda \cdot z$ for some complex number λ satisfying $|\lambda| = \frac{r_2}{r_1}$. The group $\Gamma = <\gamma>$ is infinite cyclic, and its region of discontinuity is $\Omega = \bigcup_{n \in \mathbb{Z}} \gamma^n(F) = \hat{\mathbb{C}} - \{0, \infty\}$. Finally $E = \Omega/\Gamma$ is the complex torus obtained from F by identifying its two boundary components via γ.

Conversely, a cut system on a Riemann surface E of genus 1 consists of a single simple loop c. Realizing $E = \mathbb{C}/\Lambda$ as the quotient of the complex plane by a lattice Λ, we can represent c by one side of a fundamental parallelogram Π for Λ. Thus we may assume that c is the straight line from 0 to some $\omega_1 \in \mathbb{C} - \{0\}$ and that there is $\omega_2 \in \mathbb{C} - \{0\}$ such that $\Lambda = \mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2$. Cutting E along c yields a cylinder C which is also obtained by gluing the two sides of Π that are parallel to ω_2 by a translation by ω_1. The quotient of the complex plane by the translation $z \mapsto z + \omega_1$ is $\Omega = \hat{\mathbb{C}} - \{0, \infty\}$, and the quotient map is $z \mapsto q = \exp(\frac{z}{\omega_1})$ where $\exp(z) = e^{2\pi i z}$. The image of the parallelogram Π under this map is the annulus $A = \{z \in \mathbb{C} : |\lambda| \leq |z| \leq 1\}$ with $\lambda = \exp(\frac{\omega_2}{\omega_1})$, hence $|\lambda| = e^{-\mathrm{Im}(\frac{\omega_2}{\omega_1})}$ (which we may assume to be < 1 by choosing ω_2 suitably). The annulus A is conformally equivalent to the cylinder C, and the translation $z \mapsto z + \omega_2$ induces on Ω the Möbius transformation $\gamma(q) = \lambda \cdot q$ from above.

In higher genus an analogous idea leads to a less explicit, but still very conceptual way to find a Schottky covering of a given Riemann surface X: Let $a_1, b_1, \dots, a_g, b_g$ be a standard set of generators of the fundamental group $\pi_1(X)$, i. e. $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$ and the intersection form is given by $< a_i, a_j > = < b_i, b_j > = 0$ and $< a_i, b_j > = - < b_i, a_j > = \delta_{ij}$ for all i, j. Let $\tilde{X}$ be a universal covering of X and identify $\pi_1(X)$ with the group of deck transformations $\mathrm{Deck}(\tilde{X}/X)$. Since $g \geq 2$, by the uniformization theorem $\tilde{X}$ can be taken to be the upper half plane $\mathbb{H}$. Then $\mathrm{Deck}(\tilde{X}/X)$ is a Fuchsian group contained in $\mathrm{PSL}_2(\mathbb{R})$. Now let $N \leq \pi_1(X)$ be the normal subgroup generated by $a_1, \dots, a_g$. By the Galois theory of coverings, N corresponds to an intermediate covering $\Omega := \mathbb{H}/N \to X$, which is normal with Galois group $\pi_1(X)/N$; the latter group is isomorphic to the free group generated by $b_1, \dots, b_g$. It is not obvious, but true (see e. g. [AS]) that $\mathbb{H}/N$ is a plane domain. Once this is shown it follows immediately that every compact Riemann surface admits not only one, but many Schottky coverings, as there are plenty of standard sets of generators for $\pi_1(X)$.

2 Schottky space

Each element $\gamma \neq \mathrm{id}$ in a Schottky group Γ is hyperbolic, i. e. has two different fixed points, one of which is repelling, the other attracting; we denote them by $r(\gamma)$ and $a(\gamma)$ respectively. Moreover γ is conjugate to $z \mapsto \lambda \cdot z$ for a unique complex number λ with $|\lambda| > 1$; this λ is called the *multiplier* of γ. Conversely, a hyperbolic $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ is determined by its multiplier $\lambda(\gamma)$ and its two fixed points $(r(\gamma), a(\gamma))$ (as an ordered pair). Note that we do not distinguish between hyperbolic and loxodromic elements and only use the word "hyperbolic", no matter whether λ is real or not.

Definition 3. a) A *marked Schottky group* of rank g is a g-tupel $(\gamma_1, \dots, \gamma_g)$ of hyperbolic Möbius transformations in $\mathrm{PSL}_2(\mathbb{C})$ that freely generate a Schottky group.
b) Two marked Schottky groups $(\gamma_1, \dots, \gamma_g)$ and $(\gamma_1', \dots, \gamma_g')$ are called *equivalent* if there exists $\delta \in \mathrm{PSL}_2(\mathbb{C})$ such that $\delta\gamma_i\delta^{-1} = \gamma_i'$, $i = 1, \dots, g$.
c) The set S_g of equivalence classes of marked Schottky groups of rank g is called *Schottky space*.

If $g = 1$, a marked Schottky group consists of a single hyperbolic element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$. As mentioned above, γ is conjugate to $z \mapsto \lambda \cdot z$ for a unique $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Mapping λ to $1/\lambda$ we see that S_1 is the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ as a one-dimensional complex manifold.

Let now $g \geq 2$. With a marked Schottky group $(\gamma_1, \dots, \gamma_g)$ we can associate the $3g$-tupel $(r(\gamma_1), a(\gamma_1), \lambda(\gamma_1), \dots, r(\gamma_g), a(\gamma_g), \lambda(\gamma_g)) \in \hat{\mathbb{C}}^{3g}$. As noted above it uniquely determines the marked Schottky group. Since the fixed points of the γ_i are pairwise distinct, we find a unique $\delta \in \mathrm{PSL}_2(\mathbb{C})$ such that $r(\delta\gamma_1\delta^{-1}) = 0$, $a(\delta\gamma_1\delta^{-1}) = \infty$ and $r(\delta\gamma_2\delta^{-1}) = 1$. Thus each equivalence class of marked Schottky groups contains a unique elelment of the form $(0, \infty, \lambda_1, 1, a_2, \lambda_2, \dots, r_g, a_g, \lambda_g)$. In this way an element of S_g determines a $(3g-3)$-tupel $(\lambda_1, a_2, \lambda_2, \dots, r_g, a_g, \lambda_g) \in \mathbb{C}^{3g-3}$. We have proved most of

Proposition 4. *For $g \geq 2$, S_g can be embedded as an open subset into $\mathbb{C}^{3g-3}$.*

Proof. It remains to show that the points in $\mathbb{C}^{3g-3}$ parametrizing equivalence classes of marked Schottky groups form an open subset. For this we may assume that a given point in S_g is repesented by a basis as in Proposition 1, i. e. we have mutually disjoint closed subsets D_i and D'_i of $\hat{\mathbb{C}}$ that are mapped by the γ_i in the specified way. Hence it suffices to show that the D_i and D'_i depend continuously on the parameters r_i, a_i and λ_i. If they are disks this can be seen directly, e. g. from the formulas for Ford's isometric circles (cf. [M] I.C. and II.H.). The general case is a continuous deformation of this situation. □

There is a natural action of the automorphism group $\mathrm{Aut}(F_g)$ of the free group of rank g on the set of marked Schottky groups. To describe it we slightly reformulate the definition of marked Schottky groups:

Definition 5. Let F_g be the free group of rank g on generators $x_1, \ldots, x_g$.
a) A *marked Schottky group* of rank g is an injective group homomorphism $\sigma : F_g \to \mathrm{PSL}_2(\mathbb{C})$ such that $\Gamma := \sigma(F_g)$ is a Schottky group of rank g.
Clearly the $\gamma_i = \sigma(x_i)$ satisfy the original definition, and vice versa.
b) σ and σ' are equivalent if there is $\delta \in \mathrm{PSL}_2(\mathbb{C})$ such that $\sigma' = c_\delta \circ \sigma$, where c_δ is conjugation with δ.

We shall identify S_g with the set of equivalence classes of homomorphisms σ as in part a) of the definition. Note that any such σ determines a Schottky covering of a Riemann surface of genus g: Let Ω_σ be the region of discontinuity of the Schottky group $\Gamma = \sigma(F_g)$; then $X_\sigma = \Omega_\sigma/\Gamma$ is a compact Riemann surface and $\pi : \Omega_\sigma \to X_\sigma$ is a Schottky covering. Usually we shall retain the Riemann surface in the notation of points in Schottky space, i. e. a point in S_g will be denoted by (X, σ) with a compact Riemann surface X of genus g and a homomorphism $\sigma : F_g \to \mathrm{PSL}_2(\mathbb{C})$ as in Def. 5 a) such that $X_\sigma \cong X$.

The automorphism group $\mathrm{Aut}(F_g)$ of F_g now acts on S_g by composition: For $\varphi \in \mathrm{Aut}(F_g)$ and $(X, \sigma) \in S_g$ we define

$$\varphi \cdot (X, \sigma) = (X, \sigma \circ \varphi^{-1})$$

Note first that $(\sigma \circ \varphi^{-1})(F_g) = \sigma(F_g)$, thus $\Omega_{\sigma\circ\varphi^{-1}} = \Omega_\sigma$ and $X_{\sigma\circ\varphi^{-1}} = X_\sigma$.
Note further that the action of $\mathrm{Aut}(F_g)$ (from the right) commutes with the conjugation by elements of $\mathrm{PSL}_2(\mathbb{C})$, which is an action from the left. Hence the action of $\mathrm{Aut}(F_g)$ on marked Schottky groups respects the equivalence classes and thus really defines an action on S_g.
Finally note that an inner automorphism of F_g maps each equivalence class of marked Schottky groups to itself, i. e. it acts trivially on S_g and we obtain an action of the outer automorphism group

$$\mathrm{Out}(F_g) = \mathrm{Aut}(F_g)/\mathrm{Inn}(F_g)$$

of F_g on S_g.

Proposition 6. $\mathrm{Out}(F_g)$ *acts on* S_g *by biholomorphic automorphisms.*

Proof. $\mathrm{Aut}(F_g)$ can be generated by the permutations and inversions of $x_1, \ldots, x_g$ and the automorphism $x_1 \mapsto x_2x_1, x_i \mapsto x_i$ for $i \geq 2$. For these automorphisms the action on the parameters λ_i, a_i and r_i can be calculated explicitly. This is carried out in [E] Sect. 8.
We shall give an indirect proof of the proposition in Section 4, where we show that there is a family of curves over S_g. This implies that the forgetful map $(X, \sigma) \mapsto X$ from S_g to the moduli space of Riemann surfaces is holomorphic. This map factors through the quotient space $\tilde{S}_g := S_g/\mathrm{Out}(F_g)$, and therefore the quotient map and thus also the action of $\mathrm{Out}(F_g)$ is holomorphic. □

In [E], C. Earle proves the stronger result that $\mathrm{Out}(F_g)$ is the full group of biholomorphic self-mappings of S_g; we shall not need this result in the sequel.

Note that the action of $\mathrm{Out}(F_g)$ identifies all possible markings of a given Schottky group. Thus the points of $\tilde{S}_g$ can be seen as the conjugacy classes of Schottky groups of rank g in $\mathrm{PSL}_2(\mathbb{C})$. This is the reason that some authors prefer to call $\tilde{S}_g$ the Schottky space und use "marked Schottky space" for S_g.

The action of $\mathrm{Out}(F_g)$ on S_g is properly discontinuous but not free. Therfore $\tilde{S}_g$ is still a complex space of dimension $3g - 3$, but no longer a complex manifold.

3 Schottky and Teichmüller space

In this section we study natural maps from Teichmüller space to Schottky space that decompose the forgetful map from Teichmüller to moduli space. What "natural" means in this context will be explained in the next section.

As is well known, the Teichmüller space T_g classifies marked Riemann surfaces of genus g. There are several equivalent ways of describing markings: Choose a fixed Riemann surface X_0 of genus g as reference surface and denote by π_g its fundamental group. Usually, a *marking* of a Riemann surface X of genus g is defined as an orientation preserving diffeomorphism $f : X_0 \to X$, and two marked Riemann surfaces (X, f) and (Y, g) are considered equivalent if $g \circ f^{-1} : X \to Y$ is homotopic to a biholomorphic map.

Any such marking induces an isomorphism $f_* : \pi_g \to \pi_1(X)$ of the fundamental groups. Conversely the Dehn-Nielsen theorem implies that every group isomorphism $\tau : \pi_g \to \pi_1(X)$ is induced by some diffeomorphism $f : X_0 \to X$, i. e. $\tau = f_*$. Thus we may as well define a marking on X as an isomorphism $\tau : \pi_g \to \pi_1(X)$. In these terms, (X, τ) and (Y, τ') are equivalent if there is a biholomorphic map $h : X \to Y$ such that $(\tau')^{-1} \circ h_* \circ \tau$ is an inner automorphism of π_g.

Another consequence of the Dehn-Nielsen theorem is that the mapping class group $\mathrm{Mod}_g^{\pm}$ is isomorphic to the outer automorphism group $\mathrm{Out}(\pi_g)$ of π_g. Under this isomorphism, the subgroup Mod_g of isotopy classes of orientation preserving homeomorphisms corresponds to a subgroup $\mathrm{Out}^+(\pi_g)$ of index 2. This group acts on Teichmüller space and identifies all possible markings of a given Riemann surface:

$$T_g/\mathrm{Mod}_g = M_g.$$

It is well known that Teichmüller space carries a natural structure as real manifold of dimension $6g-6$ and that this manifold is contractible. Mod_g acts properly discontinuously on T_g. Moreover, T_g also has the structure of a complex manifold, and the action of Mod_g is holomorphic. The induced analytic structure on M_g is that of a complex algebraic variety.

To define maps from Teichmüller to Schottky space we use the construction of a Schottky covering at the end of Section 1: Fix standard generators $a_1, b_1, \ldots, a_g, b_g$ of π_g; then for any marked Riemann surface $(X, \tau) \in T_g$, $\tau(a_1), \tau(b_1), \ldots, \tau(a_g), \tau(b_g)$ are standard generators for $\pi_1(X)$, and $\tau(b_1), \ldots, \tau(b_g)$ is a cut system for X. In Section 1 we saw how this cut system determines a Schottky covering $\Omega \to \Omega/\Gamma \cong X$ and an isomorphism between the factor group $\pi_1(X)/N$ and the Schottky group $\Gamma \in \mathrm{PSL}_2(\mathbb{C})$, where $N = N_\tau$ is the normal subgroup generated by $\tau(a_1), \ldots, \tau(a_g)$. Thus τ induces an isomorphism $\sigma : F_g \to \pi_1(X)/N_\tau$ where the F_g is the free group π_g/N; here $N = N_{\mathrm{id}}$ is the normal subgroup generated by $a_1, \ldots, a_g$.

Denote by $\alpha : \pi_g \to F_g$ the quotient map, i. e. the homomorphism defined by $\alpha(a_i) = 1$, $\alpha(b_i) = \gamma_i$, $i = 1, \ldots, g$. Then we can summarize the preceding discussion in the following commutative diagram:

$$\begin{array}{ccccccc} \pi_g & \xrightarrow{\tau} & \pi_1(X) & \xrightarrow{\sim} & \mathrm{Deck}(\tilde{X}/X) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{R}) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_\tau} & & \downarrow & & \\ F_g & \xrightarrow[\sigma]{} & \pi_1(X)/N_\tau & \xrightarrow{\sim} & \Gamma & \hookrightarrow & \mathrm{PSL}_2(\mathbb{C}) \end{array}$$

The diagram gives us a map

$$s_\alpha : T_g \to S_g, \quad (X, \tau) \mapsto (X, \sigma)$$

since the equivalence relations are compatible: conjugation by c in π_g corresponds to conjugation by $\alpha(c)$ in F_g.

Definition 7. A homomorphism $\alpha : \pi_g \to F_g$ is called *symplectic* if there are standard generators $a_1, b_1, \ldots, a_g, b_g$ of π_g such that $\alpha(a_i) = 1$ and $\alpha(b_i) = \gamma_i$, $i = 1, \ldots, g$.

Given a symplectic homomorphism α, two points (X, τ) and (X, τ') have the same image in S_g under s_α if and only if $\alpha \circ \tau = \alpha \circ \tau'$ up to an inner auotmorphism (on either side). In other words $s_\alpha(X, \tau) = s_\alpha(X, \tau')$ if and only if the automorphism $\varphi = \tau^{-1} \circ \tau'$ of π_g satisfies $\alpha \circ \varphi \equiv \alpha$ mod inner automorphisms. This proves the first part of

Proposition 8. *For any symplectic homomorphism* $\alpha : \pi_g \to F_g$ *we have:*
a) s_α *is the quotient map for the subgroup*

$$\mathrm{Mod}_g(\alpha) := \{\varphi \in \mathrm{Mod}_g = \mathrm{Out}^+(\pi_g) : \alpha \circ \varphi = \alpha \text{ up to inner automorphisms}\}$$

of Mod_g.
b) $s_\alpha : T_g \to S_g$ *is a universal covering.*

Proof. Part b) can be seen as a special case of [M71], Cor. 7 which is a more general statement about deformation spaces of Kleinian groups. A more direct proof using mainly topological arguments is given in [Hj], Lemma 5.11. A group theoretic proof is sketched in [DF], Satz 3.10: Since T_g is simply connected and Mod_g acts properly discontinuously, it suffices to show that $\mathrm{Mod}_g(\alpha)$ is torsion free. An elementary argument shows that a torsion element $\varphi \in \mathrm{Mod}_g(\alpha)$ would act trivially on the homology ([DF], Lemma 3.1). Thus φ would be an element of the Torelli group, but this group is known to be torsion free. □

Example. Let us illustrate the preceding in genus 1: Here X_0 is a torus, π_1 is the free abelian group on two generators a and b, α is the homomorphism $\pi_1 \to \mathbb{Z}$ that maps a to 0 and b to 1. $\mathrm{Mod}_1 = \mathrm{Aut}^+(\mathbb{Z}^2)$ is equal to $\mathrm{SL}_2(\mathbb{Z})$ and

$$\mathrm{Mod}_1(\alpha) = \{M \in \mathrm{SL}_2(\mathbb{Z}) : \alpha \circ M = \alpha\}.$$

Note that the condition is really $\alpha \circ M = \alpha$ since $\mathbb{Z}^2$ is abelian. It implies that $M \cdot a$ is a multiple of a; as M is invertible it can only be a or $-a$. For b the condition implies $M \cdot b = b + k \cdot a$ for some $k \in \mathbb{Z}$. Finally, since $\det(M) = 1$, $M \cdot a$ cannot be $-a$, thus

$$\mathrm{Mod}_1(\alpha) = \{\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z}\}$$

is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the parabolic matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Note that $\mathrm{Mod}_g(\alpha)$ can also be considered as a subgroup of the *handlebody group*: Let $\alpha : \pi_g \to F_g$ be the symplectic homomorphism corresponding to standard generators $a_1, b_1, \dots, a_g, b_g$ of π_g, and let H_g be the handlebody with boundary $\partial H_g = X_0$ such that the a_i are nullhomotopic in H_g and the b_i freely generate the fundamental group $\pi_1(H_g)$. Denoting the mapping class group of H_g by $\mathrm{Mod}(H_g)$ we can identify $\mathrm{Mod}_g(\alpha)$ with the normal subgroup of $\mathrm{Mod}(H_g)$ of self-homeomorphisms that induce the identity on the fundamental group $\pi_1(H_g)$. Moreover we have:

Proposition 9.

$$\mathrm{Mod}(H_g)/\mathrm{Mod}_g(\alpha) \cong \mathrm{Out}(F_g).$$

Restriction to the boundary $\partial H_g = X_0$ gives a homomorphism $\mathrm{Mod}(H_g) \to \mathrm{Mod}_g$ which is injective. Thus $\mathrm{Mod}(H_g)$ can be considered as a subgroup of Mod_g, contained in the normalizer of $\mathrm{Mod}_g(\alpha)$.

4 Schottky space as a moduli space

In this section we sketch the construction of a family of Riemann surfaces $\mathcal{C}_g$ over the Schottky space S_g. Each individual Riemann surface in this family comes along with a Schottky covering, and they all fit together to a "Schottky structure". It turns out that $\mathcal{C}_g \to S_g$ is the universal family of Riemann surfaces with Schottky structure.

The family $\mathcal{C}_g$ is constructed as follows: Let $s = (X, \sigma)$ be a point in S_g. We have seen in Section 2 that σ uniquely defines a Schottky group Γ_s and generators $\gamma_1^{(s)}, \dots, \gamma_g^{(s)}$ of

Γ_s that satisfy the normalization conditions for the fixed points. Let Ω_s be the region of discontinuity of Γ_s; it depends holomorphically on s in the sense that

$$\Omega_g = \{(s,z) \in S_g \times \hat{\mathbb{C}} : z \in \Omega_s\}$$

is a complex manifold.

The free group F_g acts on Ω_g via the action of Γ_s on Ω_s: Let $\gamma \in F_g$, $(s,z) \in \Omega_g$ with $s = (X,\sigma)$; for simplicity of notation we denote the isomorphism $F_g \to \Gamma_s$ also by σ. Then

$$\gamma(s,z) = (s, \sigma(\gamma)(z)).$$

The action of F_g is holomorphic and free since in each fibre Ω_s over a point $s \in S_g$, F_g acts via Γ_s. Hence the quotient space $\mathcal{C}_g = \Omega_g/F_g$ is again a complex manifold with a projection $p : \mathcal{C}_g \to S_g$, and the fibre of $\mathcal{C}_g$ over $s = (X,\sigma) \in S_g$ is the Riemann surface $\Omega_s/\Gamma_s = X$. We have shown

Proposition 10. *$p : \mathcal{C}_g \to S_g$ is a family of compact Riemann surfaces, and $p^{-1}(s) \cong X$ for every $s = (X,\sigma) \in S_g$.*

That p is a "family" means, besides the statement about the fibres, that p is a proper holomorphic map.

Corollary 11. *The forgetful map $\mu : S_g \to M_g$, $(X,\sigma) \mapsto X$, is holomorphic.*

Proof. This follows from the property of M_g of being a coarse moduli space for compact Riemann surfaces of genus g. □

In [GH] the notion of a *Schottky structure* on a family of Riemann surfaces was introduced, and it was shown that $p : \mathcal{C}_g \to S_g$ is a universal family of Riemann surfaces with Schottky structure.

A similar construction as above is possible for the Teichmüller space to obtain a family $\tilde{\mathcal{C}}_g \to T_g$ of Riemann surfaces over T_g, which again is universal for the appropriate notion of Teichmüller structure. These families are compatible with the action of $\mathrm{Mod}_g(\alpha)$ and the quotient map to M_g, see [HS] Section 5.3.

With some extra efforts the preceding construction can be extended to suitable boundaries of T_g, S_g and M_g, namely the "augmented" Teichmüller space, the "extended" Schottky space, and the Deligne-Mumford compactification $\overline{M}_g$ of M_g. Details can be found in [HS] Section 5.

5 Teichmüller disks

Teichmüller disks have been studied a lot over the last 15 or more years. There are several introductions from different points of view in the literature, see e. g. [EG], [HS], [Mö]. Therefore we confine ourselves to a short statement of definitions and main properties.

Fix a genus $g \geq 1$ and endow T_g with its natural complex structure (see Section 4) and with the Teichmüller metric. Recall that for marked Riemann surfaces (X, f) and (Y, g) (with homeomorphisms $f : X_0 \to X$, $g : X_0 \to Y$) the *Teichmüller distance* is the logarithm of the smallest complex dilatation of a quasiconformal homeomorphism in the isotopy class of $g \circ f^{-1}$. Furthermore endow the complex upper half plane $\mathbb{H}$ with its usual complex structure and with the hyperbolic metric.

Definition 12. Let $\iota : \mathbb{H} \to T_g$ be a holomorphic and isometric embedding. Then ι is called a *Teichmüller embedding* and $\Delta := \iota(\mathbb{H}) \subseteq T_g$ is called a *Teichmüller disk* in T_g.

There is a well known construction of Teichmüller disks which in fact turns out to be universal in the sense that every Teichmüller disk can be obtained this way:

Let (X, τ) be an arbitrary point in T_g (with an isomorphism $\tau : \pi_1(X_0) \to \pi_1(X)$), and let $q \in H^0(X, \Omega_X^{\otimes 2}) \setminus \{0\}$ be a nonzero holomorphic quadratic differential on X. On $X^* := X \setminus \{\text{zeroes of } q\}$ we obtain charts as follows: Cover X^* by simply connected open subsets U_i and in each U_i choose a point P_i. Then

$$\mu_i : U_i \to \mathbb{C},\ P \mapsto \int_{P_i}^{P} \sqrt{q}$$

is a well defined homeomorphism onto some open $V_i \subseteq \mathbb{C}$ (with the convention that the integral is taken along a path in U_i). The transition maps $\mu_j \circ \mu_i^{-1} : V_i \to V_j$ are of the form

$$z \mapsto \pm z + c_{ij}$$

for some constants $c_{ij} \in \mathbb{C}$. The surface X^* (and also X) together with the atlas defined by the μ_i is called a *flat surface*.

If $q = \omega^2$ is the square of a holomorphic differential, there is a consistent choice of the square root $\sqrt{q}$ (namely ω) and thus all transition maps are of the form

$$z \mapsto z + c_{ij}.$$

A surface with such an atlas is called a *translation surface*.

For $A \in \mathrm{SL}_2(\mathbb{R})$ we obtain new chart maps on the open sets U_i by composing μ_i with the $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ (here we identify $\mathbb{C}$ with $\mathbb{R}^2$). The $A \circ \mu_i$ define a new flat surface X_A, or more precisely a new flat structure on the same surface X. Since the topological surface has not changed, we can consider the marking τ also as an isomorphism $\pi_1(X_0) \to \pi_1(X_A)$ and thus obtain a new point (X_A, τ) in Teichmüller space. Note that $X_A = X$ as Riemann surfaces if and only if A is $\mathbb{C}$-linear, i.e. orthogonal.

Proposition 13. **a)** *Given $(x, \tau) \in T_g$ and $q \in H^0(X, \Omega_X^{\otimes 2}) \setminus \{0\}$ as above, the map*

$$\iota_{X,q} : \mathbb{H} = \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) \to T_g,\ \mathrm{SO}_2(\mathbb{R}) \cdot A \mapsto (X_A, \tau)$$

is a Teichmüller embedding.
Denote by $\Delta(X, q) := \iota_{X,q}(\mathbb{H})$ the corresponding Teichmüller disk and by $C(X, q)$ the image of $\Delta(X, q)$ in the moduli space M_g.
b) *Every Teichmüller embedding $\iota : \mathbb{H} \to T_g$ can be obtained by the above construction.*

Proof. A proof of part a) can be found e. g. in [H], Thm. 3.4.
For part b) note that a Teichmüller embedding $\iota : \mathbb{H} \to T_g$ is uniquely determined by $\iota(\mathrm{i})$ and the complex tangent vector to $\Delta = \iota(\mathbb{H}) \subseteq T_g$ in $\iota(\mathrm{i})$. As is well known, the tangent space to T_g in a point (X, τ) can in a natural way be identified with the space $H^0(X, \Omega_X^{\otimes 2})$ of holomorphic quadratic differentials on X. □

From the point of view of $\mathrm{SL}_2(\mathbb{R})$-actions on translation surfaces it is very natural to study the above construction of Teichmüller disks not in Teichmüller space but rather in the bundle ΩT_g over T_g, cf. [McM], Sect. 3. ΩT_g is the complement of the zero section in the rank g vector bundle of holomorphic 1-forms over T_g and thus consists of the triples (X, τ, ω) where (X, τ) is a point in T_g and ω a nonzero holomorphic differential on X.

In the local charts on X defined by ω we can decompose ω into real and imaginary part: $\omega = \operatorname{Re}\omega + \mathrm{i}\operatorname{Im}\omega$. For a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ we can thus define the real differential

$$\omega_A := (a\operatorname{Re}\omega + b\operatorname{Im}\omega) + \mathrm{i}\,(c\operatorname{Re}\omega + d\operatorname{Im}\omega).$$

Clearly ω_A is holomorphic for the complex structure on X defined by the translation structure μ_A, i. e. on the Riemann surface X_A. We thus obtain an action of $\mathrm{SL}_2(\mathbb{R})$ on ΩT_g by

$$A \cdot (X, \tau, \omega) = (X_A, \tau, \omega_A).$$

This action obviously commutes with the action of the mapping class group (which only changes the marking τ), hence it descends to an action of $\mathrm{SL}_2(\mathbb{R})$ on the bundle ΩM_g over the moduli space M_g. Since the action of $\mathrm{SL}_2(\mathbb{R})$ preserves the norm of the differential it is natural to restrict it to the subspaces $\Omega^1 T_g$ and $\Omega^1 M_g$ of differentials of norm 1. The study of the orbits of this action is an active field of research, see e. g. [Mö]. Of particular interest are closed orbits in ΩM_g, as we shall see in the next section.

6 Veech groups

Let (X, μ) be a translation surface. A homeomorphism $f : X \to X$ is called *affine* if on any open set $U \subseteq U_i$ with $f(U) \subseteq U_j$, in the local charts μ_i and μ_j f can be expressed by

$$\mu_j(f(x)) = A_{ij} \cdot \mu_i(x) + b_{ij}$$

with a matrix $A_{ij} \in \mathrm{GL}_2(\mathbb{R})$ and a constant $b_{ij} \in \mathbb{R}^2$. Since the transition maps are translations, the matrices A_{ij} are equal for all i and j.

If (X, μ) is only a flat surface the matrix part of an affine homeomorphism may change its sign from one chart to another; hence the matrix is only well defined in $\mathrm{PGL}_2(\mathbb{R})$.

Definition 14. Let (X, μ) be a translation surface.

a) $\mathrm{Aff}^+(X, \mu)$ denotes the set of all orientation preserving homeomorphisms $f : X \to X$ that are affine with respect to μ. Moreover we require that the matrix part A_f of f

has determinant 1; note that this condition is always satisfied if X has finite area.
b) $\Gamma(X,\mu) := \{A_f : f \in \mathrm{Aff}^+(X,\mu)\} \subseteq \mathrm{SL}_2(\mathbb{R})$ is called the *Veech group* of (X,μ).

For a flat surface we obtain in the same way a Veech group in $\mathrm{PSL}_2(\mathbb{R})$.

Note that the map $f \mapsto A_f$ is a group homomorphism der: $\mathrm{Aff}^+(X,\mu) \to \mathrm{SL}_2(\mathbb{R})$, called the *derivative map*. Its kernel consists of the *translations*.

Now let (X,μ) be the translation surface associated to a holomorphic differential ω on a Riemann surface X. Then we denote the Veech group by $\Gamma(X,\omega)$ and the Teichmüller disk obtained by the construction in Section 5 by $\Delta(X,\omega)$. Similarly we use the notations $\Delta(X,q)$ and $\Gamma(X,q)$ for the Teichmüller disk and the Veech group obtained from a holomorphic quadratic differential q on X.

Observe that every affine homeomorphism $f \in \mathrm{Aff}^+(X,\omega)$ can be considered as an element φ_f of the mapping class group Mod_g (or $\mathrm{Mod}_{g,n}$ where $n = |X^* \setminus X|$). Clearly φ_f stabilizes $\Delta(X,\omega)$. More precisely we have the following result, see [EG]:

Proposition 15. *For the Teichmüller disk associated with a holomorphic 1-form ω on a Riemann surface X we have*
a) $\mathrm{Aff}^+(X,\omega) = \mathrm{Stab}_{\mathrm{Mod}_g}(\Delta(X,\omega))$.
b) *Under the identification in a), the translations in* $\mathrm{Aff}^+(X,\omega)$ *correspond to the pointwise stabilizer of* $\Delta(X,\omega)$ *in* Mod_g.
c) $\Gamma(X,\omega) \cong \mathrm{Stab}(\Delta(X,\omega))/\mathrm{Stab}^{pw}(\Delta(X,\omega))$.

The question which subgroups of $\mathrm{SL}_2(\mathbb{R})$ arise as Veech groups of Teichmüller disks (or equivalently precompact translation surfaces) is still largely open. The most general restrictions were already found by Veech himself in his seminal paper [V]:

Proposition 16. *Let $\Gamma = \Gamma(X,q)$ be the Veech group of a Teichmüller disk $\Delta(X,q)$. Then*

a) Γ *is a discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *resp.* $\mathrm{PSL}_2(\mathbb{R})$.

b) Γ *is not cocompact, i. e. the Riemann surface* $\mathbb{H}/\Gamma$ *is not compact.*

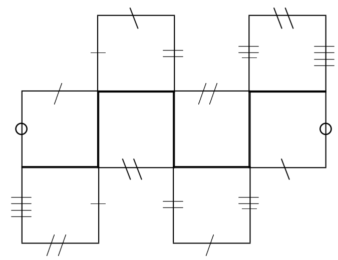
For a general Teichmüller disk the Veech group is either trivial or cyclic of order 2. At the other extreme there are examples of Teichmüller disks $\Delta(X,q)$ whose Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$, i. e. $\mathbb{H}/\Gamma(X,q)$ is a Riemann surface of finite area, hence compact minus finitely many points. Since by Proposition 15 the map from $\Delta(X,q)$ to the moduli space M_g factors through $\mathbb{H}/\Gamma(X,q)$, the image $C(X,q)$ in M_g is a Riemann surface of finite type, or equivalently an algebraic curve (necessarily affine by Prop. 16 b)). There can be only little difference between $\mathbb{H}/\Gamma(X,q)$ and $C(X,q)$:

Proposition 17. *For any Teichmüller disk $\Delta(X,q)$ the induced map $\mathbb{H}/\Gamma(X,q) \to C(X,q)$ is birational.*

Proof. This follows from the fact that Mod_g acts properly discontinuously on T_g. □

If $\Gamma(X,q)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ (or $\mathrm{PSL}_2(\mathbb{R})$) the algebraic curve $C(X,q)$ in M_g is called a *Teichmüller curve*. In this case the image of $\Delta(X,q)$ in M_g is closed. A theorem of Smillie assures that the converse also holds: $C(X,q)$ is closed in M_g if and only if $\Gamma(X,q)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{PSL}_2(\mathbb{R})$).

An interesting class of examples for Teichmüller curves can be obtained from origamis: An *origami* (or *square-tiled surface*) is a translation surface that can be obtained by gluing finitely many plane unit squares by translations. Thus the left edge of each square is glued to the right edge of one of the squares, its upper edge is glued to a lower edge, and the resulting compact surface has to be connected. Here is an example, where edges with the same label are glued:



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An origami comes along with a natural map $p : X \to E$ to the torus E which is obtained by gluing the opposite edges of a single unit square: p maps each square of X to the unique square of E and is unramified outside the vertices of the squares. In particular, p is ramified (if at all) only over the point $\infty \in E$ which results from gluing the four vertices of the square. Conversely every finite covering $p : X \to E$ which is ramified over at most one point defines an origami by pulling back the translation structure on E via p.

The Teichmüller disk $\Delta(O)$ corresponding to an origami $O = (p : X \to E)$ is obtained via the construction in Section 5 by considering X with its translation atlas as a Riemann surface and using the holomorphic 1-form $p^*\omega_E$, where ω_E is a nonzero holomorphic differential on E (which is unique up to multiplication by a nonzero complex number).

The Veech group of the "basic" origami E is easily seen to be $\mathrm{SL}_2(\mathbb{Z})$. In general, the Veech group $\Gamma(O)$ of an origami O turns out to be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. By a result of Gutkin and Judge [GJ] the converse also holds. Thus we have:

Proposition 18. *The Veech group $\Gamma(\Delta)$ of a Teichmüller disk $\Delta \subset T_g$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ if and only if $\Delta = \Delta(O)$ for an origami O.*

There is an efficient algorithm to determine the Veech group of an origami [Sch1]. For many subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index it is known that they arise as the Veech group of an origami: most of the congruence groups [Sch2], and all subgroups of the principal congruence group $\Gamma(2)$ that contain $-I$ [EM]. Nevertheless there remain many subgroups of $\mathrm{SL}_2(\mathbb{Z})$ for which it is open whether they are Veech groups or not.

7 Horizontal cut systems

In this section we show that on a flat surface with a horizontal cylinder decomposition there exist cut systems that consist entirely of “horizontal” curves. This will be crucial for the analysis of the image of Teichmüller disks in Schottky space that will be carried out in the next section. The concept of horizontal cut systems and the proof of their existence are due to Diego De Filippi in his PhD thesis [DF].

Let (X, q) be a flat surface defined by a holomorphic quadratic differential on a Riemann surface X, and let $\mu_i : U_i \to \mathbb{R}^2$ be the chart maps defined by q as in Section 5. A maximal (real) curve in X that is mapped by the μ_i to a straight line segment in $\mathbb{R}^2$, is called a *trajectory* of q. The *direction* of a trajectory is the direction vector $v \in \mathbb{R}^2$ of the line segment. By postcomposing the μ_i with a rotation (which does not change the complex structure), we can always achieve that a given direction v becomes horizontal. A direction $v \in \mathbb{R}^2$ is called a *Strebel direction* if all trajectories in direction v are closed or hit a zero of q.

If v is a Strebel direction for q, X is decomposed into finitely many *cylinders* in direction v; more precisely, the complement of the “critical” trajectories, i. e. those containing a zero of q, is a finite union of annuli that are swept out by closed trajectories in direction v. This follows from work of Strebel and Masur, cf. [HS], Sect. 4.1.

We shall always assume that the horizontal direction is a Strebel direction on the flat surface X, in other words that X is decomposed into horizontal cylinders $C_1, \ldots, C_p$. Such a decomposition is e. g. guaranteed if the Veech group of X contains a parabolic element $A \in \mathrm{PSL}_2(\mathbb{R})$: Veech showed in [V] Prop. 2.4 that in this case an eigenvector $v \in \mathbb{R}^2$ of A determines a Strebel direction on X. This result is a converse to the following observation:

Remark 19. *Let (X, q) be a flat surface on which the horizontal direction is Strebel. Assume moreover that the lengths of the horizontal cylinders are commensurable. Then there exists an affine homeomorphism φ in $\mathrm{Aff}^+(X, q)$ which on each cylinder C_i acts as horizontal shearing by the l.c.m. of the cylinder lengths. The derivative of φ is a parabolic element A in the Veech group $\Gamma(X, q)$.*

In each horizontal cylinder C_i of X we can choose an interior horizontal trajectory m_i as *core line*. Then the homeomorphism φ in the previous remark acts on C_i as a power of the Dehn twist along m_i. Following [DF] we call a path in X^* *horizontal* if it does not intersect any of the core lines.

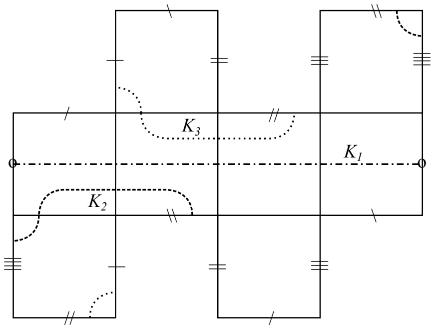
Definition 20. Let X be a flat surface of genus $g \geq 1$ with a horizontal cylinder decomposition. Disjoint simple closed curves $c_1, \ldots, c_g$ are called a *horizontal cut system* if

(i) all c_i are horizontal

(ii) $X - \cup_{i=1}^{g} c_i$ is connected.

Proposition 21. *On each flat surface with a horizontal cylinder decomposition, there exist horizontal cut systems.*

Proof. An explicit construction, starting from a maximal non-separating set of horizontal trajectories, is given in [DF] Ch. 6. We illustrate the result with an example, which shows a horizontal cut system on the origami we have already seen in Section 6:



□

8 Teichmüller disks in Schottky space

The image of a Teichmüller disk $\Delta \subset T_g$ in the Schottky space S_g first of all depends on the choice of the covering map s_α, i. e. on the choice of a symplectic homomorphism $\alpha : \pi_g \to F_g$. For given α, the image $s_\alpha(\Delta)$ is isomorphic to the quotient of Δ by the intersection $\Gamma(\Delta, \alpha)$ of $\mathrm{Mod}_g(\alpha)$ with the stabilizer $\mathrm{Stab}(\Delta)$ of Δ in Mod_g. Recall that if Δ is a Teichmüller disk corresponding to the flat surface (X, q), $\mathrm{Stab}(\Delta)$ is isomorphic to $\mathrm{Aff}^+(X, q)$ and $\mathrm{Stab}(\Delta)/\mathrm{Stab}^{pw}(\Delta)$ is isomorphic to the Veech group $\Gamma(X, q)$. Hence if $\Gamma(\Delta, \alpha)$ is trivial, $s_\alpha(\Delta)$ is again biholomorphic to a disk. Thus the interesting question is whether there can be nontrivial elements in the intersection $\mathrm{Mod}_g(\alpha) \cap \mathrm{Stab}(\Delta)$. A satisfactory answer is given by the first main result of [DF]:

Theorem 1. *Let (X, q) be a flat surface which admits a decomposition into horizontal cylinders of commensurable lengths. Then there exists a symplectic homomorphism $\alpha : \pi_g \to F_g$ such that $\mathrm{Mod}_g(\alpha) \cap \mathrm{Stab}(\Delta(X, q))$ contains an element of infinite order.*

Proof. The assumption on the cylinder decomposition implies that the horizontal direction is a Strebel direction and therefore by Remark 19 that there is an affine automorphism $\varphi \in \mathrm{Aff}^+(X, q)$ that preserves the horizontal cylinders, and on each of them acts as a power of the Dehn twist along the core line.
By Prop. 21 there exists a horizontal cut system $c_1, \ldots, c_g$ on (X, q). Fix a base point $x_0 \in X$ for the fundamental group and denote by α_i the class of c_i in $\pi_1(X, x_0)$. Complete $\alpha_1, \ldots, \alpha_g$ to a standard set of generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ of $\pi_1(X, x_0)$ by choosing appropriate paths in X. Then considering φ as an element of $\mathrm{Out}(\pi_g)$ we find $\varphi(\alpha_i) = \alpha_i$ and $\varphi(\beta_i) = \beta_i$ up to multiplication by a product of the α_j and conjugation. Fix a marking $\tau : \pi_g \to \pi_1(X, x_0)$ of X and let $\Delta(X, q)$ be the Teichmüller disk through (X, τ) defined by q. Let $a_i := \tau^{-1}(\alpha_i)$, $b_i := \tau^{-1}(\beta_i)$, $i = 1, \ldots, g$, and $\alpha : \pi_g \to F_g$ the symplectic homomorphism defined by $\alpha(a_i) = 1$, $\alpha(b_i) = \gamma_i$, $i = 1, \ldots, g$. As we have seen, the class of φ in Mod_g is an element of both $\mathrm{Mod}_g(\alpha)$ and $\mathrm{Stab}(\Delta(X, q))$. □

Now that we have seen that $\mathrm{Mod}_g(\alpha) \cap \mathrm{Stab}(\Delta(X,q))$ can contain parabolic elements, the next natural question to ask is how big this intersection can be. For translation surfaces the answer is given by the second main theorem of [DF]:

Theorem 2. *Let (X,ω) be a translation surface of genus $g \geq 2$. Then for any symplectic homomorphism $\alpha : \pi_g \to F_g$, $\mathrm{Mod}_g(\alpha) \cap \mathrm{Stab}(\Delta(X,\omega))$ is cyclic.*

In fact the statement can be made a little more precise: the intersection is either trivial or infinite cyclic generated by a parabolic element.

A corresponding statement for flat surfaces that are not translation surfaces is not known. The main argument in the proof of Theorem 2 does not hold in this case, but nevertheless there is no example so far of a noncyclic intersection $\mathrm{Mod}_g(\alpha) \cap \mathrm{Stab}(\Delta(X,q))$.

Proof of Theorem 2. First note that $\Gamma(\Delta,\alpha) := \mathrm{Mod}_g(\alpha) \cap \mathrm{Stab}(\Delta(X,\omega))$ cannot contain an element of finite order since $\mathrm{Mod}_g(\alpha)$ is torsion free. Next suppose that $\Gamma(\Delta,\alpha)$ contains two parabolic elements with different fixed points on $\partial\mathbb{H}$: their commutator would be hyperbolic. Thus to prove the theorem it suffices to show that $\Gamma(\Delta,\alpha)$ cannot contain a hyperbolic element. This is done in two steps:

Proposition 22 ([DF] Satz 5.7)**.** *Let (X,ω) be a translation surface of genus $g \geq 2$ and $\varphi \in \mathrm{Aff}^+(X,\omega)$. Denote by $A = \mathrm{der}(\varphi) \in \mathrm{SL}_2(\mathbb{R})$ the matrix part of φ and by $M_\varphi \in \mathrm{GL}_{2g}(\mathbb{C})$ the automorphism of $H^1(X,\mathbb{C})$ induced by φ.*
Then the eigenvalues of A are also eigenvalues of M_φ.

Proof of Proposition 22. Identify $H^1(X,\mathbb{C})$ with $H^1_{DR}(X) \otimes \mathbb{C}$ and consider the 2-dimensional subspace U generated by $\mathrm{Re}\,\omega$ and $\mathrm{Im}\,\omega$. By the definition of an affine automorphism, U is φ-invariant, and by Section 5, φ acts on U via the matrix A. □

In the second step one shows that M_φ can only have the eigenvalue 1. Then A has the eigenvalue 1 with multiplicity 2 and hence is parabolic. This finishes the proof of Theorem 2. □

In fact, the second step in the above proof follows from the following explicit result:

Proposition 23 ([DF] Lemma 3.1)**.** *Let $\alpha : \pi_g \to F_g$ be a symplectic homomorphism, $\varphi \in \mathrm{Mod}_g(\alpha)$ and $M_\varphi \in \mathrm{GL}_{2g}(\mathbb{Z})$ the matrix describing the automorphism of $\pi_g^{\mathrm{ab}} \cong \mathbb{Z}^2$ induced by φ with respect to the basis $a_1, b_1, \ldots, a_g, b_g$ of π_g^{ab}. Then*

$$M_\varphi = \begin{pmatrix} I_g & B \\ 0 & I_g \end{pmatrix}$$

with the $g \times g$ unit matrix I_g and some $B \in \mathbb{Z}^{g \times g}$.

Proof. Consider $\varphi \in \mathrm{Mod}_g \cong \mathrm{Out}(\pi_g)$ as an (outer) automorphism of π_g. Then the (i,j)-th entry in the first $g \times g$ block of M_φ is the number of occurrences of a_i in $\varphi(a_j)$ written as word in $a_1, b_1, \ldots, a_g, b_g$ and counted with sign, and similarly for the other three blocks. Then the result follows from the defining properties of φ being an element of $\mathrm{Mod}_g(\alpha)$. □

Finally we want to describe the image of a Teichmüller disk $\Delta = \Delta(X, \omega)$ (for a translation surface (X, ω) of genus $g \geq 2$) in Schottky space S_g: We pick a symplectic homomorphism $\alpha : \pi_g \to F_g$, and we know that $s_\alpha(\Delta)$ is isomorphic to $\Delta/\Gamma(\Delta, \alpha)$. From Theorem 2 we know that $\Gamma(\Delta, \alpha)$ is either trivial or infinite cyclic generated by a parabolic element. Since every parabolic element in $\mathrm{SL}_2(\mathbb{R})$ is conjugated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the quotient in the second case is isomorphic to $\mathbb{H}/ < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >$. This is the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$, the quotient map being the exponential map $z \mapsto e^{2\pi i z}$. Thus we can summarize our observations in

Corollary 24. *Let (X, ω) be a translation surface of genus $g \geq 2$ and $\Delta = \Delta(X, \omega)$ a corresponding Teichmüller disk in T_g. For any symplectic homomorphism $\alpha : \pi_g \to F_g$, the image $s_\alpha(\Delta)$ of Δ in S_g is either a disk or a once punctured disk. The latter happens if and only if the Veech group $\Gamma(X, \omega)$ contains a parabolic element, and α is suitably chosen.*

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