Polygonal billiards and homogeneous foliations

Ferrán Valdez

Centro de Ciencias Matemáticas
UNAM, Campus Morelia
www.matmor.unam.mx/~ferran/

December 22, 2011

Polygonal Billiards

Dynamical systems whose ingredients are:

Polygonal Billiards

Dynamical systems whose ingredients are: a table (Euclidean polygon P)

Polygonal Billiards

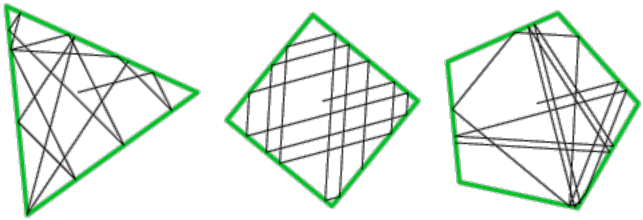
Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle)

Polygonal Billiards

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle) + reflection law (à la Descartes).

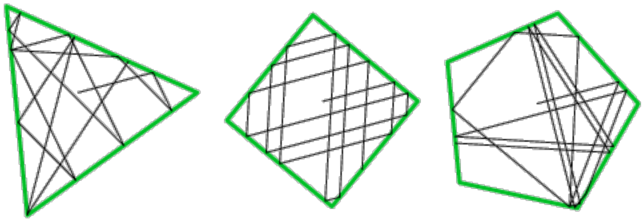
Polygonal Billiards

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle) + reflection law (à la Descartes).



Polygonal Billiards

Dynamical systems whose ingredients are: a table (Euclidean polygon P) + ball (point particle) + reflection law (à la Descartes).



(No friction, motion ends if we hit an infinitesimal pocket)

Polygonal billiards and translation surfaces

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a "translation" (or "flat") surface

$$S_P \to P$$

on which geodesics project to billiard trajectories.

Polygonal billiards and translation surfaces

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a “translation” (or “flat”) surface

$$S_P \to P$$

on which geodesics project to billiard trajectories.Studying the billiard is equivalent to studying the geodesic flow on S_P.

Polygonal billiards and translation surfaces

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a "translation" (or "flat") surface

$$S_P \to P$$

on which geodesics project to billiard trajectories.Studying the billiard is equivalent to studying the geodesic flow on S_P.
Main open questions: *is there a periodic trajectory in every triangular billiard?*

Polygonal billiards and translation surfaces

Idea (Fox-Kershner/Katok-Zemljakov): associate to each polygon P a "translation" (or "flat") surface

$$S_P \to P$$

on which geodesics project to billiard trajectories.Studying the billiard is equivalent to studying the geodesic flow on S_P.
Main open questions: *is there a periodic trajectory in every triangular billiard? is the geodesic flow on the generic S_P recurrent or dissipative?*

Homogeneous holomorphic foliations on $\mathbf{C}^2$

Homogeneous = invariant under the homothecy action of $\mathbf{C}^*$ on $\mathbf{C}^2$.

Homogeneous holomorphic foliations on $\mathbf{C}^2$

Homogeneous = invariant under the homothecy action of $\mathbf{C}^*$ on $\mathbf{C}^2$. We will consider foliations $\mathcal{F}_\lambda$ defined by the holomorphic 1-form ω_λ

$$\frac{\omega_\lambda}{z_1 z_2(z_2 - z_1)} = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \lambda_3 \frac{d(z_2 - z_1)}{z_2 - z_1},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi\lambda_i$ are the interior angles of a triangle P.

Homogeneous holomorphic foliations on $\mathbf{C}^2$

Homogeneous = invariant under the homothecy action of $\mathbf{C}^*$ on $\mathbf{C}^2$. We will consider foliations $\mathcal{F}_\lambda$ defined by the holomorphic 1-form ω_λ

$$\frac{\omega_\lambda}{z_1 z_2 (z_2 - z_1)} = \lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \lambda_3 \frac{d(z_2 - z_1)}{z_2 - z_1},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi\lambda_i$ are the interior angles of a triangle P. In the dual picture, $\mathcal{F}_\lambda$ is given by the integral curves of the vector field X_λ

$$(\lambda_2 z_1 (z_2 - z_1) + \lambda_3 z_1 z_2) \frac{\partial}{\partial z_1} + (\lambda_1 z_2 (z_1 - z_2) + \lambda_3 z_1 z_2) \frac{\partial}{\partial z_2}$$

Homogeneous holomorphic foliations on $\mathbf{C}^2$

Using E. Paul's work we can assure that the fibers of

$$F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2} (z_2 - z_1)^{\lambda_3}$$

over $\mathbb{C}^*$ are the leaves of $\mathcal{F}_\lambda$ in $\mathbb{C}^2 \setminus \{z_1 z_2 (z_2 - z_1) = 0\}$. In other words, F_λ is a *first integral* for $\mathcal{F}_\lambda$.

Homogeneous holomorphic foliations on $\mathbf{C}^2$

Using E. Paul's work we can assure that the fibers of

$$F_\lambda(z_1, z_2) = z_1^{\lambda_1} z_2^{\lambda_2}(z_2 - z_1)^{\lambda_3}$$

over $\mathbb{C}^*$ are the leaves of $\mathcal{F}_\lambda$ in $\mathbb{C}^2 \setminus \{z_1 z_2(z_2 - z_1) = 0\}$. In other words, F_λ is a *first integral* for $\mathcal{F}_\lambda$. Hence, there are 3 types of *leaves* : those in $F_\lambda^{-1}(0)$ (the tangent cone) and $F_\lambda^{-1}(t)$ with $t \in \mathbf{C}^*$ (generic leaf, denoted L).

Flat surface structure of leaves

Consider the 1-form η on $\mathbb{C}^2$ satisfying $\eta(X_\lambda) = 1$.

Flat surface structure of leaves

Consider the 1-form η on $\mathbb{C}^2$ satisfying $\eta(X_\lambda) = 1$. Then, for every $L \in \mathcal{F}_\lambda$, $\eta_{|L}$ is zero free. We can then define new coordinates in L by the map:

$$z(p) = \int_{p_0}^{p} \eta_{|L}$$

Flat surface structure of leaves

Consider the 1-form η on $\mathbb{C}^2$ satisfying $\eta(X_\lambda) = 1$. Then, for every $L \in \mathcal{F}_\lambda$, $\eta_{|L}$ is zero free. We can then define new coordinates in L by the map:

$$z(p) = \int_{p_0}^{p} \eta_{|L}$$

Remark that, if we change base points in some small patch, then our new coord. change by a translation:

$$c := \int_{p_0}^{p} \eta_{|L} - \int_{p_1}^{p} \eta_{|L} = \int_{p_0}^{p_1} \eta_{|L}$$

Flat surface structure of leaves

Consider the 1-form η on $\mathbb{C}^2$ satisfying $\eta(X_\lambda) = 1$. Then, for every $L \in \mathcal{F}_\lambda$, $\eta_{|L}$ is zero free. We can then define new coordinates in L by the map:

$$z(p) = \int_{p_0}^{p} \eta_{|L}$$

Remark that, if we change base points in some small patch, then our new coord. change by a translation:

$$c := \int_{p_0}^{p} \eta_{|L} - \int_{p_1}^{p} \eta_{|L} = \int_{p_0}^{p_1} \eta_{|L}$$

In other words, the leaves of $\mathcal{F}_\lambda$ have a "natural" translation surface structure. Notation $(L, \eta_|)$

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1.

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1. The restriction of $\mathcal{F}_0$ to a generic leaf $L \in \mathcal{F}$ *is a foliation*.

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1. The restriction of $\mathcal{F}_0$ to a generic leaf $L \in \mathcal{F}$ *is a foliation*. Locally, the map $z : L \to (L, \eta_|)$ sends $\mathcal{F}_0$ to the set of integral curves of $\frac{\partial}{\partial t}$.

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1. The restriction of $\mathcal{F}_0$ to a generic leaf $L \in \mathcal{F}$ *is a foliation*. Locally, the map $z : L \to (L, \eta_|)$ sends $\mathcal{F}_0$ to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1. The restriction of $\mathcal{F}_0$ to a generic leaf $L \in \mathcal{F}$ *is a foliation*. Locally, the map $z : L \to (L, \eta_|)$ sends $\mathcal{F}_0$ to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

1. There exits a translation surface isomorphism $\phi : (L, \eta_|) \longrightarrow S_P$.

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1. The restriction of $\mathcal{F}_0$ to a generic leaf $L \in \mathcal{F}$ *is a foliation*. Locally, the map $z : L \to (L, \eta_|)$ sends $\mathcal{F}_0$ to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

1. There exits a translation surface isomorphism $\phi : (L, \eta_|) \longrightarrow S_P$. In particular, the foliation $(L, \mathcal{F}_0)$ is conjugated by ϕ to a foliation on S_P formed by geodesics parallel to some direction θ.

Dictionary

Write $X_\lambda = X_1 + iX_2$ and define $\mathcal{F}_0$ to be the (real) foliation on $\mathbf{C}^2$ defined by the integral curves of X_1. The restriction of $\mathcal{F}_0$ to a generic leaf $L \in \mathcal{F}$ *is a foliation*. Locally, the map $z : L \to (L, \eta_|)$ sends $\mathcal{F}_0$ to the set of integral curves of $\frac{\partial}{\partial t}$.

Theorem.

Let $L \in \mathcal{F}_\lambda$ be a fixed generic leaf. Then,

1. There exits a translation surface isomorphism $\phi : (L, \eta_|) \longrightarrow S_P$. In particular, the foliation $(L, \mathcal{F}_0)$ is conjugated by ϕ to a foliation on S_P formed by geodesics parallel to some direction θ.
2. If we change L for $\rho e^{i\alpha} L$, then $\mathcal{F}_0$ restricted to $\rho e^{i\alpha} L$ is conjugated to the foliation on S_P formed by geodesics parallel to $\theta + \alpha$.

Sketch of proof.

Aim: construct $\phi : (L, \eta_|) \longrightarrow S_P$.

Sketch of proof.

Aim: construct $\phi : (L, \eta_|) \longrightarrow S_P$.

Locally, X_λ restricted to L is of the form

$$t^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t, \qquad t \in \mathbf{H}$$

Sketch of proof.

Aim: construct $\phi : (L, \eta_|) \longrightarrow S_P$.

Locally, X_λ restricted to L is of the form

$$t^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t, \qquad t \in \mathbf{H}$$

Therefore

$$z(t) := \int^t \xi^{\lambda_2-1}(\xi-1)^{\lambda_3-1}d\xi$$

defines the translation surface structure of L (& locally rectifies X_1).

Sketch of proof.

Aim: construct $\phi : (L, \eta_|) \longrightarrow S_P$.

Locally, X_λ restricted to L is of the form

$$t^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t, \qquad t \in \mathbf{H}$$

Therefore

$$z(t) := \int^t \xi^{\lambda_2 - 1}(\xi - 1)^{\lambda_3 - 1} d\xi$$

defines the translation surface structure of L (& locally rectifies X_1). Remark that f is a Schwarz-Christoffel transformation.

Sketch of proof.

Aim: construct $\phi : (L, \eta_|) \longrightarrow S_P$.

Locally, X_λ restricted to L is of the form

$$t^{1-\lambda_2}(t-1)^{1-\lambda_3}\partial/\partial t, \qquad t \in \mathbf{H}$$

Therefore

$$z(t) := \int^t \xi^{\lambda_2 - 1}(\xi - 1)^{\lambda_3 - 1} d\xi$$

defines the translation surface structure of L (& locally rectifies X_1). Remark that f is a Schwarz-Christoffel transformation. *"Hard" part*: show that z extends to define ϕ. □

Remarks

The triangular billiard has a periodic orbit iff the vector field

Remarks

The triangular billiard has a periodic orbit iff the vector field

$$\begin{aligned} 2X_1 &= [\lambda_2(x_1^2 - y_1^2) - (\lambda_2+\lambda_3)(x_1x_2 - y_1y_2)]\partial/\partial x_1 \quad + \\ &\quad [2\lambda_2 x_1 y_1 - (\lambda_2+\lambda_3)(x_1y_2 + x_2y_1)]\partial/\partial y_1 \quad + \\ &\quad [\lambda_1(x_2^2 - y_2^2) - (\lambda_1+\lambda_3)(x_1x_2 - y_1y_2)]\partial/\partial x_2 \quad + \\ &\quad [2\lambda_1 x_2 y_2 - (\lambda_1+\lambda_3)(x_1y_2 + x_2y_1)]\partial/\partial y_2 \end{aligned}$$

has a periodic orbit.

Foliations in $\mathbf{RP}(3)$

The phase space of the billiard is three dimensional!

Foliations in **RP**(3)

The phase space of the billiard is three dimensional! Since everything is homogeneous look at the induced foliations in **RP**(3) (or in S^3).

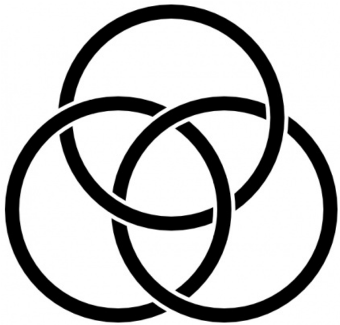
Foliations in **RP**(3)

The phase space of the billiard is three dimensional! Since everything is homogeneous look at the induced foliations in **RP**(3) (or in S^3).
Let $\mathcal{G}_\lambda$ be the foliation defined in homogeneous coordinates by:

$$\alpha := i_R i_{X_1} i_{X_2}(dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2)$$

The singular locus of $\mathcal{G}_\lambda$ is $\pi(z_1 z_2(z_2 - z_1) = 0)$

Foliations in $\mathbf{RP}(3)$



The Borromean rings "=" $Sing\mathcal{G}_\lambda$.

Foliations in $\mathbf{RP}(3)$

Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi_| : L \to \pi(L)$$

is a $2:1$ map.

Foliations in $\mathbf{RP}(3)$

Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi_| : L \to \pi(L)$$

is a $2:1$ map. Else, it is a bijection. Hence we can talk about the generic leaf of $\mathcal{G}_\lambda$.

Foliations in $\mathbf{RP}(3)$

Let $L \in \mathcal{F}_\lambda$ be fixed. If there exist $n, m \in \mathbb{Z}$ such that $n\lambda_i + m\lambda_j - \frac{1}{2} \in \mathbb{Z}$, then

$$\pi_| : L \to \pi(L)$$

is a $2:1$ map. Else, it is a bijection. Hence we can talk about the generic leaf of $\mathcal{G}_\lambda$.

Local picture near $Sing\mathcal{G}_\lambda$: (up to analytic change of coordinates) $\mathcal{G}_\lambda$ is given by

$$XdY - YdX$$

(Open Book / Kupka phenomenon)

Foliations in **RP**(3)

Global picture near $Sing\mathcal{G}_\lambda$. Let

$$\pi : \widetilde{\mathbf{RP}(3)} \to \mathbf{RP}(3)$$

be the blow-up along $Sing(\mathcal{G}_\lambda)$ and $\widetilde{\mathcal{G}_\lambda}$ the foliation defined by $\pi^*\alpha$.

Foliations in $\mathbf{RP}(3)$

Global picture near $Sing\mathcal{G}_\lambda$. Let

$$\pi : \widetilde{\mathbf{RP}(3)} \rightarrow \mathbf{RP}(3)$$

be the blow-up along $Sing(\mathcal{G}_\lambda)$ and $\widetilde{\mathcal{G}_\lambda}$ the foliation defined by $\pi^*\alpha$.Each component of the Borromean rings defines a "torus" T_i.

Foliations in $\mathbf{RP}(3)$

Global picture near *Sing*$\mathcal{G}_\lambda$. Let

$$\pi : \widetilde{\mathbf{RP}(3)} \to \mathbf{RP}(3)$$

be the blow-up along $Sing(\mathcal{G}_\lambda)$ and $\widetilde{\mathcal{G}_\lambda}$ the foliation defined by $\pi^*\alpha$.Each component of the Borromean rings defines a "torus" T_i.Let (x, y) be local coordinates for T_i, then the trace of a leaf $L \in \widetilde{\mathcal{G}_\lambda}$ in T_i is of the form:

$$\lambda_i x + (1 - \lambda_i)y = k, \quad k \in \mathbb{R}$$

In other words, $\widetilde{\mathcal{G}_\lambda}_{|T_i}$ is defined by "parallel lines".

Foliations in $\mathbf{RP}(3)$

Let $\mathcal{G}_0$ be the foliation on $\mathbf{RP}(3)$ defined by X_1.

Foliations in $\mathbf{RP}(3)$

Let $\mathcal{G}_0$ be the foliation on $\mathbf{RP}(3)$ defined by X_1.Remark that $Sing(\mathcal{G}_0) \subset Sing(\mathcal{G})$.

Foliations in $\mathbf{RP}(3)$

Let $\mathcal{G}_0$ be the foliation on $\mathbf{RP}(3)$ defined by X_1.Remark that $Sing(\mathcal{G}_0) \subset Sing(\mathcal{G})$. Since X_1 restricted to the tangent cone is conjugated to $Re(z^2\frac{\partial}{\partial z})$, $Sing(\mathcal{G}_0)$ is given by:

$$p_1 = [1 : 0 : 0 : 0] \quad p_2 = [0 : 0 : 1 : 0] \quad p_3 = [1 : 0 : 1 : 0]$$

Foliations in **RP**(3)

Let $\mathcal{G}_0$ be the foliation on **RP**(3) defined by X_1.Remark that $Sing(\mathcal{G}_0) \subset Sing(\mathcal{G})$. Since X_1 restricted to the tangent cone is conjugated to $Re(z^2 \frac{\partial}{\partial z})$, $Sing(\mathcal{G}_0)$ is given by:

$$p_1 = [1:0:0:0] \quad p_2 = [0:0:1:0] \quad p_3 = [1:0:1:0]$$

Local picture near $Sing(\mathcal{G}_0)$. The foliation is conjugated to the foliation defined by

$$\lambda_j(x+x^3)\frac{\partial}{\partial x} + [-y + x(\ldots)]\frac{\partial}{\partial y} + [-z + x(\ldots)]\frac{\partial}{\partial z}$$

Foliations in $\mathbf{RP}(3)$

Let $\mathcal{G}_0$ be the foliation on $\mathbf{RP}(3)$ defined by X_1.Remark that $Sing(\mathcal{G}_0) \subset Sing(\mathcal{G})$. Since X_1 restricted to the tangent cone is conjugated to $Re(z^2\frac{\partial}{\partial z})$, $Sing(\mathcal{G}_0)$ is given by:

$$p_1 = [1:0:0:0] \quad p_2 = [0:0:1:0] \quad p_3 = [1:0:1:0]$$

Local picture near $Sing(\mathcal{G}_0)$. The foliation is conjugated to the foliation defined by

$$\lambda_j(x+x^3)\frac{\partial}{\partial x} + [-y + x(\dots)]\frac{\partial}{\partial y} + [-z + x(\dots)]\frac{\partial}{\partial z}$$

Remark the invariant (local) manifold $\{x=0\}$!

¡Muchas gracias a todos y feliz Navidad!

vielen Dank und schöne Weihnachten!