A comb of origami curves in the moduli space M_3 with three dimensional closure

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Abstract

The first part of this paper is a survey on Teichmüller curves and Veech groups, with emphasis on the special case of origamis where much stronger tools for the investigation are available than in the general case.
In the second part we study a particular configuration of origami curves in genus 3: A "base" curve is intersected by infinitely many "transversal" curves. We determine their Veech groups and the closure of their locus in M_3, which turns out to be a three dimensional variety and the image of a certain Hurwitz space in M_3.

By an *origami* we mean a closed surface endowed with a translation structure that comes from glueing a certain number of squares. Variation of the translation structure leads to a 1-parameter family of Riemann surfaces, which in turn determines an algebraic curve in the moduli space of Riemann surfaces. For every genus g, there are countably many such origami curves in M_g. So far, not much is known about the configurations of origami curves in general.

In a previous paper [HeSc 05], we found an origami W of genus 3 for which the origami curve is intersected by infinitely many other origami curves. We call this configuration a "comb" although this name is misleading in several respects: first, it may (and does) happen that a transversal curve intersects the base curve in more than one point. And secondly, the transversal curves do not all lie in one plane, not even on a surface: in fact, one main result of this paper is that the closure of the set of transversal origamis is a three dimensional algebraic subvariety of the moduli space M_3 that we describe explicitly.

As a second main result we determine the Veech groups of the transversal origamis. We shall explain in Section 1.2 and 2.2 how to define the Veech group of a translation surface and that, for an origami O, it is a subgroup $\Gamma(O)$ of $\mathrm{SL}_2(\mathbb{Z})$ of finite

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index. Its importance for the origami curve $C(O)$ lies in the fact that $C(O)$ is birational to $\mathbb{H}/\Gamma(O)$ (where $\mathbb{H}$ is the upper half plane).

The first two sections of this paper give a survey of Teichmüller curves in general and origami curves as special cases. In particular we describe the Veech groups corresponding to Teichmüller disks and to origamis. In Section 3 we review the results of [HeSc 05] on the special origami curve W. The last two sections contain the new results of this paper. In Section 4 we determine the closure of the locus of the transversal origamis and relate it to the Hurwitz space of certain coverings of elliptic curves. In Section 5 we show that the Veech group of a transversal origami has index 3 in the stabilizer group of the corresponding configuration of critical points. These are n-torsion points on an elliptic curve for varying n. If n is odd, we explicitly describe the Veech group.

1 Teichmüller curves and Veech groups

In this section we explain what *Teichmüller curves* are. They are special algebraic subvarieties of the moduli space M_g of complex (regular) algebraic curves of genus g. They occur as images of one-dimensional analytic subvarieties of the Teichmüller space T_g, namely *Teichmüller disks*, which are natural with respect to the holomorphic structure on T_g as well as to the Teichmüller metric. Teichmüller curves are closely related to *Veech groups*, certain subgroups of $\mathrm{SL}_2(\mathbb{R})$. For a general introduction to Teichmüller spaces and the Teichmüller metric see e.g. [ImTa 92].

Definition 1. *Let $\iota : \mathbb{H} \hookrightarrow T_g$ be an embedding of the upper half plane $\mathbb{H}$ into Teichmüller space that is holomorphic and isometric (with respect to the Poincaré metric on $\mathbb{H}$ and the Teichmüller metric on T_g).*

a) The image Δ of such an embedding is called a Teichmüller disk.

b) If Δ projects to an algebraic curve C in the moduli space M_g, then this curve $C \subseteq M_g$ is called a Teichmüller curve.

First examples of Teichmüller curves were given by Veech in [Ve 89]. Since then, several authors have been working on this subject. The reader can find a detailed overview and further hints to literature e.g. in [HeSc 06]. In this section we give a short introduction without proofs in order to provide the reader with the general ideas and explain everything that one needs to understand the results in Sections 4 and 5. For details and proofs we refer the reader to [HeSc 06].

1.1 Construction of Teichmüller disks

Teichmüller disks are obtained in the following way: Let X be a Riemann surface and q a holomorphic quadratic differential. By integration q naturally defines a flat structure μ on $X\backslash\{\text{zeroes of } q\}$, i.e. an atlas such that all transition maps are of the form

$$z \mapsto \pm z + c \quad \text{with } c \in \mathbb{C}.$$

For each $A \in \mathrm{SL}_2(\mathbb{R})$, one gets a natural variation of the flat structure: Composing each chart with the affine map $z \mapsto Az$ defines a new flat structure μ_A. We use here and in the whole article the $\mathbb{R}$-linear identification of $\mathbb{C}$ with $\mathbb{R}^2$ that maps $\{1, i\}$ to the standard basis of $\mathbb{R}^2$.
Observe that the flat structure μ_A is in particular a holomorphic structure on the topological surface underlying X that will in general not be holomorphically equivalent to μ. Furthermore, the identity map defines a diffeomorphism $\mathrm{id} : X_\mu \to X_{\mu_A}$ between the Riemann surfaces $X_\mu := (X, \mu)$ and $X_{\mu_A} := (X, \mu_A)$ which is again in general not holomorphic. Altogether one obtains a map

$$\iota : \mathrm{SL}_2(\mathbb{R}) \to T_g, \quad A \mapsto [X_{\mu_A}, \mathrm{id}].$$

If A is in $\mathrm{SO}_2(\mathbb{R})$, then $z \mapsto A \cdot z$ is in fact biholomorphic, thus the map ι factors through the quotient by $\mathrm{SO}_2(\mathbb{R})$ and one obtains the map

$$\bar{\iota} : \mathbb{H} = \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) \to T_g \quad ;$$

one can show that $\bar{\iota}$ is a holomorphic and isometric embedding. Thus its image $\Delta = \Delta_\mu$ is a Teichmüller disk.

1.2 Veech groups

In order to study Teichmüller curves one needs to project the Teichmüller disk $\Delta = \Delta_\mu$ defined as above to the moduli space M_g by the natural projection $\mathrm{proj} : T_g \to M_g$. In general, the Zariski closure of the image will be large, i.e. higher-dimensional. Only occasionally it is one-dimensional and $\mathrm{proj}(\Delta)$ is a Teichmüller curve C. The Veech group introduced in this subsection makes it possible to decide whether this happens or not; and if it happens the Veech group "knows" how the Teichmüller curve looks like.

Definition 2. *Let X^* be a Riemann surface together with a flat structure μ.*

a) A diffeomorphism $f : X^* \to X^*$ is called affine*, if it is locally affine; i.e. in terms of the charts of the flat atlas μ it is of the form:*

$$z \mapsto Az + c \quad \textit{with } A \in \mathrm{GL}_2(\mathbb{R}) \textit{ and } c \in \mathbb{C}. \tag{1}$$

b) Observe that the matrix $A \in \mathrm{GL}_2(\mathbb{R})$ *is independent of the charts up to the sign. Let* $\mathrm{Aff}^+(X^*, \mu)$ *be the group of orientation preserving affine diffeomorphisms and*

$$D: \ \mathrm{Aff}^+(X^*, \mu) \to \mathrm{PSL}_2(\mathbb{R}), \quad f \mapsto [A],$$

with A *as in a) and* $[A]$ *its class in* $\mathrm{PSL}_2(\mathbb{R})$. *The* projective Veech group $\bar{\Gamma}(X^*, \mu)$ *is the image of* D *in* $\mathrm{PSL}_2(\mathbb{R})$. *Hence, it contains the classes of all matrices that occur in* (1) *for orientation preserving affine diffeomorphisms* f *on* (X^*, μ).

Remark:

1. For the examples of flat surfaces that we will study in this article, the flat structure is in fact a translation structure, i.e. all transition maps are translations. In this case the matrices in (1) are in fact independent of the charts. With the notation from above one has a map $\mathrm{Aff}^+(X^*, \mu) \to \mathrm{GL}_2(\mathbb{R})$, which we also denote by D. We call its image the *Veech group* $\Gamma(X^*, \mu)$. For $f \in \mathrm{Aff}^+(X^*, \mu)$, we call $D(f) \in \Gamma(X^*, \mu)$ the *derivative* of f.
Furthermore, if (X^*, μ) is obtained from a closed Riemann surface X together with a quadratic differential q as described in 1.1 with $X^* \subseteq X\backslash\{\text{zeroes of } q\}$, then q defines a translation structure if and only if q is the square of a holomorphic differential on X. In this case $\Gamma(X^*, \mu)$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$. We denote the group of orientation preserving affine diffeomorphisms of X^* also by $\mathrm{Aff}^+(X, \mu)$, the Veech group by $\Gamma(X, \mu)$ and the projective Veech group by $\bar{\Gamma}(X, \mu)$.
2. One may observe from the definitions that

$$\Gamma(X, \mu_A) = A \cdot \Gamma(X, \mu) \cdot A^{-1},$$

where the flat structure μ_A is defined as in Section 1.1. Hence, Veech groups for flat surfaces on the same Teichmüller disk are equal up to conjugation in $\mathrm{SL}_2(\mathbb{R})$.

Recall that the natural projection $T_g \to M_g$ is the quotient map by the mapping class group

$$\Gamma_g := \mathrm{Diff}^+(X)/\mathrm{Diff}^+_0(X),$$

where $\mathrm{Diff}^+(X)$ is the group of orientation preserving diffeomorphisms and $\mathrm{Diff}^+_0(X)$ consists of the diffeomorphisms homotopic to the identity.
Let $\Delta = \Delta_\mu$ be a Teichmüller disk as in Section 1.1. For our purposes, the following well known properties of the affine group and the Veech group Γ of (X, μ) are important:

- The affine diffeomorphisms are precisely the diffeomorphisms that stabilize Δ, i.e. map Δ to itself.

- The kernel of $D : \mathrm{Aff}^+(X,\mu) \to \Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ consists of those diffeomorphisms that act trivially on Δ. Consequently, one has:

$$\Gamma \;\cong\; \mathrm{Aff}^+(X,\mu)/\text{kernel of D} \;\cong\; \text{global stabilizer/pointwise stabilizer} \tag{2}$$

- By the identification $\iota : \mathbb{H} \to \Delta$, the action of Γ on Δ given by (2) becomes equal to the action of $\Gamma^* := R\Gamma R^{-1} \subseteq \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ as Fuchsian group with

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the map $\mathrm{proj} : \Delta \to \mathrm{proj}(\Delta) \subseteq T_g$ factors through $\mathbb{H}/\Gamma^*$.
Observe that $\mathbb{H}/\Gamma^*$ is an algebraic curve if and only if the quotient has finite volume, i.e. if Γ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. We put these results together in:

Theorem A[1] *Let Δ be a Teichmüller disk in the Teichmüller space T_g.*

- *The image* $\mathrm{proj}(\Delta)$ *is an algebraic curve C in M_g, if and only if the Veech group Γ is a lattice in* $\mathrm{SL}_2(\mathbb{R})$.
- *In this case, $\mathbb{H}/\Gamma^*$ is the normalization of the algebraic curve C.*

Hence, the Veech group detects, whether a Teichmüller disk leads to a Teichmüller curve and, if this happens, it determines the Teichmüller curve up to birationality.

2 Origamis

One way to obtain particular flat surfaces is given by the following construction: Take finitely many unit squares and glue their edges by translations, such that:

- Each left edge is glued to a right one and vice versa.
- Each upper edge is glued to a lower one and vice versa.
- The resulting surface X is connected.

X carries in a natural way a flat structure defined by the unit squares. Actually one has even more, namely a translation structure, since the glueings are done by translations.
Surfaces obtained this way are called *square tiled surfaces* or *origamis* as Pierre Lochak baptized them in [Lo 05] due to the playful character of their definition.
The baby example is to take only one square. There is only one possibility to glue its edges according to the rules and one obtains a torus. In fact this

[1]For a general introduction to algebraic curves and algebraic geometry we refer to [Ha 77].

example is universal in the following sense: Each other surface X obtained by the construction above allows a morphism to the torus by mapping each square on X to the unique square of the torus. This morphism is unramified except for the one point on the torus that comes from the vertices of the square. Conversely, a surface that allows such a covering can be obtained by glueing squares according to the rules above. This motivates the following definition for (oriented) origamis.

Definition 3. *An* origami *O is a covering $p : X \to E$ from a topological surface X to the torus E that is ramified at most over one point $\infty \in E$. Two origamis $O = (p : X \to E)$ and $O' = (p' : X' \to E)$ are said to be* isomorphic*, if there is some homeomorphism $f : X \to X'$ such that $p' \circ f = p$.*

One then obtains the flat structure μ on X equivalently as follows:

- Identify the torus E with $\mathbb{C}/\Lambda_I$, where Λ_I is the lattice in $\mathbb{C}$ generated by 1 and i.
- Take the natural translation structure on $E = \mathbb{C}/\Lambda_I$ coming from locally reversing the universal covering $\mathbb{C} \to E$.
- Lift it by the unramified covering $p : X^* = X - p^{-1}(\infty) \to E^* = E - \{\infty\}$ to the translation structure μ on X^*.

Obviously, isomorphic origamis define isomorphic translation surfaces. Note that for origamis the translation structure comes from the holomorphic quadratic differential $\omega^2 = p^*(\omega_E^2)$ on X, where ω_E is the (essentially unique) holomorphic differential on E.

2.1 Description of origamis

One advantage of flat surfaces coming from origamis is that they are described by only few combinatorial data: the *glueing data* that record which square is glued to which one (in horizontal resp. in vertical direction). The equivalent description in terms of Definition 3 is the monodromy

$$\pi_1(E^*) \to S_d,$$

where d is the degree of p and S_d is the symmetric group. Observe that the fundamental group $\pi_1(E^*)$ is the free group in two generators. The images in S_d of the two generators x, the horizontal closed curve on E^*, and y, the vertical closed geodesic, give precisely the glueing data.

For a third equivalent description pointed out in [Sc 04] observe that, by the theorem of the universal covering, the unramified covering $X^* \to E^*$ gives an embedding

$$U = \pi_1(X^*) \quad \hookrightarrow \quad \pi_1(E^*) = F_2 \tag{3}$$

of index d equal to the degree of p. Vice versa one obtains, again by the theorem of the universal covering, to each such subgroup U of F_2 a covering p as in Definition 3. Hence, origamis correspond to finite index subgroups of F_2.

2.2 Veech groups of origamis

A second advantage of origamis is that one has an alternative approach to their Veech groups, which we describe in this subsection. It makes them easier to handle than those of general flat surfaces.

Let O be an origami and (X, μ) the flat surface that it defines.

Definition 4. *The Veech group of the origami O is the Veech group of (X, μ)*

$$\Gamma(O) := \Gamma(X, \mu)$$

One may start with the baby example from above: the origami with only one square; in terms of Definition 3 it is the covering $O = (\mathrm{id} : E \to E)$. The flat surface then is $\mathbb{C}/\Lambda_I$. Its affine diffeomorphisms come from affine diffeomorphisms of $\mathbb{C}$ that descend to $\mathbb{C}/\Lambda_I$. Hence, they are precisely the affine diffeomorphisms preserving the lattice $\Lambda_I = \mathbb{Z} \oplus \mathbb{Z}i$. Thus $\Gamma(O) = \mathrm{SL}_2(\mathbb{Z})$.
For a general origami O the Veech group is always a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(O) \subseteq \mathrm{SL}_2(\mathbb{Z})$$

The converse also holds: By the Theorem of Gutkin and Judge in [GuJu 00] a Veech group is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ if and only if the translation surface comes from a once ramified covering of a torus.

One may now use the above description of origamis by finite index subgroups of F_2 to determine the Veech group. In [Sc 04], the following characterization was given:

Theorem B:

- *Let U be the finite index subgroup of F_2 defined in (3) for the origami O.*
- *Let $\hat{\beta} : \mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z}) = \mathrm{Out}(F_2)$ be the natural projection and* $\mathrm{Aut}^+(F_2) := \hat{\beta}^{-1}(\mathrm{SL}_2(\mathbb{Z}))$.
- *Let* $\mathrm{Stab}(U) := \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(U) = U\}$ *be the* stabilizing group of U

Then for the Veech group $\Gamma(O)$ holds: $\Gamma(O) = \hat{\beta}(\mathrm{Stab}(U))$.

Remark: The proof of Theorem B is based on the theorem of the universal covering. One uses a fixed universal covering $u : \mathbb{H} \to E_I^* = \mathbb{C}/\Lambda_I - \{\infty\}$. One then considers the group

$$\mathrm{Aff}(\mathbb{H}, u) := \{\tilde{f} \in \mathrm{Diff}^+(\mathbb{H}) |\ \tilde{f} \text{ descends via } u \text{ to an affine map on } E_I^*\}.$$

In fact, this is the group of affine diffeomorphisms of $\mathbb{H}$ with respect to the translation structure that comes as lift via u from the one on E_I^*. One then has the isomorphism:

$$* : \mathrm{Aff}(\mathbb{H}, u) \to \mathrm{Aut}^+(F_2), \quad \tilde{f} \mapsto (\gamma : w \mapsto \tilde{f} w \tilde{f}^{-1}).$$

Here we use that $F_2 = \mathrm{Gal}(\mathbb{H}/E_I^*)$ is the group of deck transformations. If one now has an origami $O = (p : X \to E)$, one may restrict to those affine diffeomorphisms on $\mathbb{H}$ that descend to X^* and obtains the following commutative diagram:

$$\begin{array}{ccc}
\{\tilde{f} \in \mathrm{Aff}(\mathbb{H}, u) | \tilde{f} \text{ descends to } X^*\} & \xrightarrow{\ \star\ } & \mathrm{Stab}(U) \\
\downarrow D & & \downarrow \hat{\beta} \\
\Gamma(O) & \xrightarrow{\ \mathrm{id}\ } & \hat{\beta}(\mathrm{Stab}(U)) \quad \subseteq \mathrm{SL}_2(\mathbb{Z})
\end{array}$$

The map $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ can be given explicitly as follows: Let $\sharp_x(w)$ be the number of occurrences of x in w (with x^{-1} counted as -1) and similarly for $\sharp_y(w)$. Then

$$\hat{\beta}(\gamma) = \begin{pmatrix} \sharp_x(\gamma(x)) & \sharp_x(\gamma(y)) \\ \sharp_y(\gamma(x)) & \sharp_y(\gamma(y)) \end{pmatrix}.$$

2.3 Teichmüller curves of origamis

A further advantage of flat surfaces coming from origamis is that they always define Teichmüller curves: As we have mentioned in the last subsection, their Veech groups are finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ and thus lattices in $\mathrm{SL}_2(\mathbb{R})$. Hence the Teichmüller disk Δ defined by such a surface always projects to a Teichmüller curve in the moduli space (see Section 1.2).

Observe that composing each chart of the translation surface $\mathbb{C}/\Lambda_I$ with the affine map $z \mapsto Az$ with $A \in \mathrm{SL}_2(\mathbb{R})$ is equivalent to taking the translation surface $\mathbb{C}/\Lambda_A$, where Λ_A is the lattice

$$\Lambda_A =< \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} > \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here again we identify $\mathbb{C}$ with $\mathbb{R}^2$.
Therefore one may describe the Teichmüller disk Δ defined by an origami $O = (p : X \to E)$ as follows:

- For each $A \in \mathrm{SL}_2(\mathbb{R})$ let η_A be the translation structure on E obtained by identifying E with $\mathbb{C}/\Lambda_A$.
- Let μ_A be the lift of η_A to $X^* = X \backslash p^{-1}(\infty)$ and call $X_A = (X, \mu_A)$.

Then the Teichmüller disk Δ is given as

$$\Delta = \{[X_A, \mathrm{id}] |\ A \in \mathrm{SL}_2(\mathbb{R})\},$$

where, as before, id is topologically the identity on X, whereas the holomorphic structure on X is changed.

We denote the Teichmüller curve defined by an origami $O = (p : X \to E)$ by $C(O)$ and call it an *origami curve*; recall that it is the image in the moduli space of the Teichmüller disk Δ. It is possible to decide whether two origamis induce the same origami curve in terms of the subgroup $U \subset F_2$ associated to an origami in Section 2.1:

Proposition 5. *Let $O = (p : X \to E)$ and $O' = (p' : X' \to E)$ be two origamis and U (resp. U') the corresponding subgroups of F_2. Then*
a) *O is isomorphic to O' if and only if U is conjugate to U' in F_2.*
b) *$C(O) = C(O')$ if and only if there is a $\gamma \in \mathrm{Aut}^+(F_2)$ such that $\gamma(U) = U'$.*

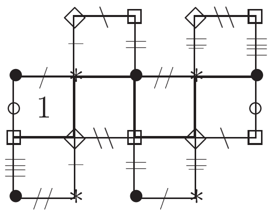
Proof. a) By definition, O is isomorphic to O' if and only if there is a diffeomorphism $f : X \to X'$ satisfying $p = p' \circ f$. Then also the unramified coverings $X^* \to E^*$ and $(X')^* \to E^*$ are isomorphic, and hence $\pi_1(X^*)$ is conjugate to $\pi_1(X')^*$ in $\pi_1(E^*)$.
b) $C(O)$ is equal to $C(O')$ if and only if there is a diffeomorphism $f : X \to X'$ such that $p^*(\omega_E)$ is equal to $f^*((p')^*(\omega_E))$ up to a multiplicative constant in $\mathbb{C}$, where ω_E, as before, is the holomorphic differential on $E = E_I$. Equivalent conditions are that f descends to an affine diffeomorphism of E or that f lifts to an element $\tilde{f} \in \mathrm{Aff}(\mathbb{H}, u)$ in the notation of the remark following Theorem B. Under the isomorphism $*$ in the same remark, such an f corresponds to an automorphism of F_2 that maps U to U'. □

3 The quaternion origami

In this section we illustrate the general concepts that we introduced in the first two sections by a specific example of an origami curve in genus 3. In addition to the general features shared by all origami curves, our origami W has several remarkable properties which make it particularly interesting. We restrict ourselves to a short description and refer to [HeSc 05] for more details and, in particular, for proofs.

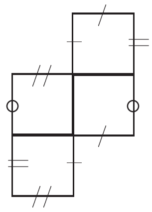
3.1 W as an origami

Using the combinatorial definition of origamis, W can be described by 8 squares that are glued as indicated (edges are glued if they have the same label).

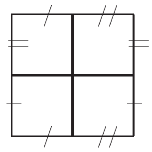


Note that every vertex is glued to precisely one of the vertices of the square labeled "1". Euler's formula gives $2 - 2g = 8 - 16 + 4$, thus the genus of W is 3.

The total angle at every vertex is 4π, so they are all ramification points of order 2 for the covering $p : W \to E$ of degree 8 to the torus E. Recall that p is obtained by mapping each of the eight squares to E. The map p can be decomposed as follows: Observe that "translation by 2 to the right" is an automorphism of W, and let $q : W \to \overline{W}$ be the quotient map for this automorphism. Then $\overline{W}$ is the origami



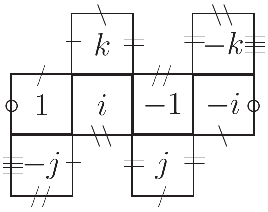
which is equivalent to



Thus $\overline{W}$ is the torus E. As in Section 2, we identify E with $\mathbb{C}/\Lambda_I$ in such a way that the vertex of the square becomes the origin of the group structure. In this identification, the 4 vertices of W are mapped by q to the points of order 2 on E, and p is obtained by postcomposing q with the multiplication by 2 on E. As usual (see e. g. [Si 86]) we call, for $n \geq 2$, the points of order dividing n in the group $\mathbb{C}/\Lambda_I$ the *n-torsion points* of E. They form a group $E[n]$ of order n^2, isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$; in other words, $E[n]$ is the kernel of the multiplication $[n]$ by n on E.

3.2 The automorphism group of W

W is a Galois origami, i. e. the covering $p : W \to E$ is a normal covering. The Galois group is the quaternion group Q of order 8. This can be seen e. g. by labeling the squares of W as follows:



As usual, the elements of Q are denoted $\pm 1, \pm i, \pm j$ and $\pm k$, with relations $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$.

The full automorphism group G of W consists of the affine diffeomorphisms with derivative I and $-I$. The first ones are precisely the elements of Q. G is an extension of Q of degree 2; more precisely, G is a group of order 16 that contains (besides the elements of Q) 6 involutions which act on W by rotating the squares by an angle of π; each of these rotations has four fixed points which are midpoints either of squares or of edges. The last two elements of G, c and c^{-1}, are the rotation by π around the four vertices, and its inverse; note that c has order 4 since the total angle at a vertex is 4π. It is a useful observation that c^2 equals the translation by 2 squares (in horizontal or vertical direction: they are equal!); thus c^2 is the element -1 in Q. Moreover, c generates the center of G.

3.3 The equation of $C(W)$

In this section we determine the origami curve $C(W)$ by explicitly giving the algebraic equation of the covering Riemann surfaces. Since c has 4 fixed points, it follows from the Riemann-Hurwitz formula that $W/<c>$ has genus 0. Thus $W \to W/<c>$ is a cyclic covering of degree 4 of the projective line, branched over 4 points which we can normalize to be 0, 1, ∞ and some parameter $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$. Therefore, as an algebraic curve, W is given by an equation of the form

$$y^4 = x^{\varepsilon_0}(x-1)^{\varepsilon_1}(x-\lambda)^{\varepsilon_\lambda}. \tag{4}$$

The exponents ε_0, ε_1 and ε_λ are determined by the monodromy (which can be read off from the origami) and turn out to be 1. The complex structure on W is determined by λ up to replacing λ by one of $\frac{1}{\lambda}$, $1-\lambda$, $1-\frac{1}{\lambda}$, $\frac{\lambda}{\lambda-1}$ and $\frac{1}{1-\lambda}$. The covering $q : W \to E$ from Section 3.1 is given by $(x, y) \mapsto (x, y^2)$, and E is the elliptic curve with equation $y^2 = x(x-1)(x-\lambda)$.

In the last equation, λ is the *Legendre parameter* of the elliptic curve E. It is related to the lattice description $E = \mathbb{C}/\Lambda_A$, $A \in \mathrm{SL}_2(\mathbb{R})$, in Section 2.3 by the classical modular function λ on the upper half plane, which is the universal covering $\mathbb{H} \to \mathbb{P}^1 - \{0, 1, \infty\}$ (see e.g. [FiLi 88, Chap. VI.3]): For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, the complex torus $\mathbb{C}/\Lambda_A$ is biholomorphic to the elliptic curve E_{λ_A} with affine equation $y^2 = x(x-1)(x-\lambda_A)$, where $\lambda_A = \lambda(\frac{d \cdot i + b}{c \cdot i + a})$. For example, the unit lattice Λ_I corresponds to $\lambda_I = -1$ (or to $\lambda = 2$ or $\lambda = \frac{1}{2}$, if we choose a different basis for Λ_I).

The holomorphic differential $\omega = p^*(\omega_E)$ that according to the introduction of Section 2 determines the translation structure on W, can also be written down explicitly: it is

$$\omega = p^*(\omega_E) = q^*[2]^*(\frac{dx}{2y}) = q^*(\frac{dx}{y}) = \frac{dx}{y^2}. \tag{5}$$

An equivalent form of the equation (4) is

$$y^4 = x^4 + 2ax^2 + 1 \tag{6}$$

where now a is in $\mathbb{P}^1 - \{1, -1, \infty\}$. The two equations are transformed into each other by a projective change of coordinates, see e. g. [Gu 01]. The relation between λ and a is given by

$$a = \frac{\lambda + 1}{\lambda - 1}, \qquad \lambda = \frac{a+1}{a-1}. \tag{7}$$

The result of Section 3.2 states that, for every admissible value of a (or λ), the projective curve given by (6) (resp. by (4)) has an automorphism group containing G. There are precisely two curves in the family that have a larger automorphism group: one is the Fermat curve $y^4 = x^4 + 1$, which has 96 automorphisms. The other exceptional curve occurs for $\lambda = \frac{1}{2} \pm \frac{i}{2}\sqrt{3}$; it has 48 automorphisms. See [KuKo 79] for a detailed discussion of these automorphism groups.

3.4 The Veech group of W

We have seen in Section 2.2 that the Veech group of any origami is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. For the origami W we have:

Proposition 6. $\Gamma(W) = \mathrm{SL}_2(\mathbb{Z})$.

This can be checked by showing that the two generators of $\mathrm{SL}_2(\mathbb{Z})$ – e. g. the matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ – are in fact affine automorphisms of W. A more conceptual argument is the following:

Let $U = \pi_1(W^*)$ be the subgroup of F_2 that corresponds to the origami W as in Section 2.1. By Section 3.2, U is the kernel of a surjective homomorphism $F_2 \to Q$. It is not hard to see that all surjective homomorphisms $F_2 \to Q$ have the same kernel. Therefore U is invariant under all automorphisms of F_2, which by Theorem B implies that $\Gamma(W) = \mathrm{SL}_2(\mathbb{Z})$.

As a consequence, $\mathbb{H}/\Gamma(W)$ is isomorphic to the affine line. It turns out that, in this case, the map $\mathbb{H}/\Gamma(W) \to C(W)$ to the origami curve in M_3 (which by Theorem A is birational in general) is an isomorphism. Thus we have

Proposition 7. *The origami curve $C(W)$ is an affine line, embedded into the moduli space M_3.*

3.5 Origami curves crossing W

The origami W has several other remarkable properties. One of them is that the Jacobian of each of the Riemann surfaces W_λ, $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$, splits (up to isogeny) into a product of three elliptic curves, see [HeSc 05, Prop. 1.6]. More precisely, $\mathrm{Jac}(W_\lambda)$ is isogenous to $E_\lambda \times E_{-1} \times E_{-1}$, where E_λ is the elliptic curve with equation $y^2 = x(x-1)(x-\lambda)$ and E_{-1} (given by the equation $y^2 = x^3 - x$) is independent of λ; see [BiLa 04, Chap. XI] for background information on Jacobian varieties of Riemann surfaces. The fact that the Jacobian of the family of curves defined by W has a constant part of codimension 1 shows that $C(W)$ is a Shimura curve, see [Mö 05]. In that paper, Möller also shows that among all Teichmüller curves that are induced by the square of a holomorphic differential, $C(W)$ is the only one that is at the same time a Shimura curve.

For the present note, the following result on $C(W)$, proved in [HeSc 05, Thm. 3.1], is the most interesting:

Theorem C: $C(W)$ *intersects infinitely many other origami curves.*

Since all origami curves are defined over number fields, there can be at most countably many origami curves that intersect a given one. In particular, their image in the moduli space M_3 of Riemann surfaces of genus 3 cannot be closed. We shall study the closure of the set of origami curves intersecting $C(W)$ in the next section. Here we briefly recall how these origamis arise:

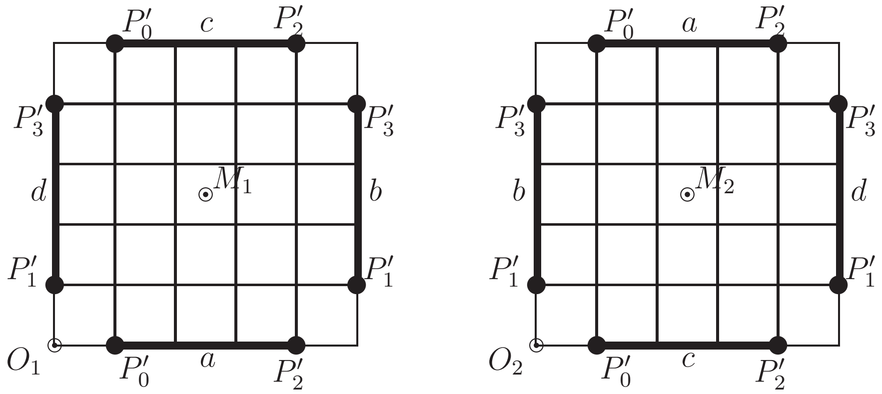
We have seen in Section 3.2 that, for all $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$, $\mathrm{Aut}(W_\lambda)$ contains 7 involutions. All of them have 4 fixed points on W_λ, so the quotient curve has genus 1 in all cases. One involution is the translation by 2 squares which induces the map q to E_λ, see Section 3.1. For all the other involutions the central automorphism $c \in \mathrm{Aut}(W_\lambda)$ descends to an automorphism $\bar{c}$ of order 4 on the quotient. Since $\bar{c}$ has two fixed points, which we call O and M, the quotient must be the elliptic curve E_{-1}, and $\bar{c}$ is a rotation around O and M of angle $\frac{\pi}{2}$.

Let us denote by σ one of these six involutions and by $\kappa = \kappa_\lambda : W_\lambda \to E_{-1}$ the quotient map. The ramification points of κ are the four fixed points of σ. Their images $P_0(\lambda), \ldots, P_3(\lambda)$ form an orbit under $\bar{c}$. Choose the fixed point O of $\bar{c}$ as the origin on E_{-1}. Then if $P_0(\lambda)$ is an n-torsion point for some n, the same holds for $P_1(\lambda)$, $P_2(\lambda)$ and $P_3(\lambda)$. Hence in this case, $[n] \circ \kappa$ is ramified only over the origin and thus defines an origami. We showed in [HeSc 05] by explicit calculations that, for any $n \geq 3$ and any n-torsion point P on E_{-1}, there is a $\lambda \in \mathbb{P}^1 - \{0, 1, \infty\}$ such that $P_0(\lambda) = P$.

4 The Hurwitz space H

4.1 The origamis D_P

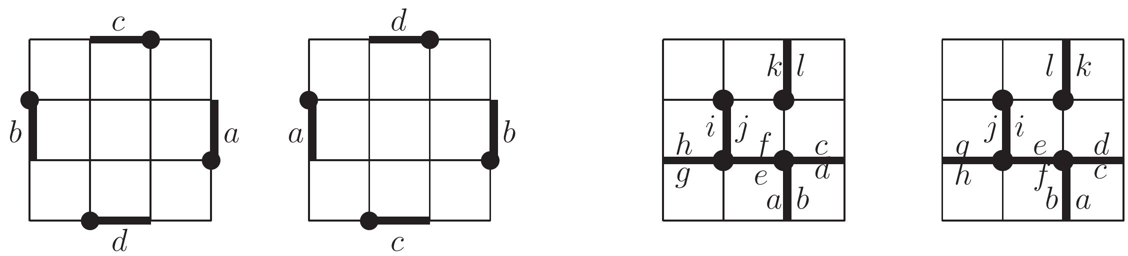
We have seen that, for each $n \geq 3$, every n-torsion point P on the elliptic curve $E_{-1} : y^2 = x^3 - x$ induces an origami D_P of degree $2n^2$ whose origami curve $C(D_P)$ intersects the origami curve $C(W)$. Combinatorially, D_P consists of two large squares made of n^2 small squares each; the point $P = P_0$ corresponds to a (primitive) vertex of one of the small squares. P_1, P_2 and P_3 are obtained from P_0 by rotation by an angle of $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$, resp., around the center of the large square. The two large squares are glued in such a way that the canonical map (of degree 2) from the resulting surface X_P to the torus corresponding to the large square is ramified exactly over P_0, P_1, P_2 and P_3. For the precise description of the glueing we refer to [HeSc 05]; here we confine ourselves to an example with $n = 5$:



Highlighted edges: those with same labels are glued.
Other edges: opposite edges are glued.

Here, P_0', P_1', P_2' and P_3' are the preimages of the ramification points P_0, P_1, P_2 and P_3, M_1 and M_2 the preimages of M and O_1, O_2 the preimages of O, respectively. As in Section 3.5, O and M denote the fixed points of the automorphism $\bar{c}$ of order 4 on the torus E.

For $n = 3$ there are only two different possibilities for the orbit P_0, P_1, P_2, P_3:



Highlighted edges: those with same labels are glued.
Other edges: opposite edges are glued.

In this case the two corresponding origamis are related by an affine diffeomorphism which is given explicitly in [HeSc 05, Example 3.4]; therefore their origami curves are equal by Proposition 5 b).

Let $d_P : X \to E$ denote the covering of degree $2n^2$ that defines the origami D_P. Recall from Section 2.3 that the points on $C(D_P)$ are given by the Riemann surfaces X_A for $A \in \mathrm{SL}_2(\mathbb{R})$. Here X_I corresponds to the intersection point of $C(D_P)$ with $C(W)$ described in Section 3.5. The affine diffeomorphism c on X_I has derivative $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and therefore is not an element of the automorphism group of D_P (cf. Section 3.2). Explicitly, c defines an affine diffeomorphism c_A on X_A with derivative $A \cdot S \cdot A^{-1}$ which, in general, is not holomorphic.

On the other hand, the square $\tau = c^2$ of c has derivative $-I$ which is central in $\mathrm{SL}_2(\mathbb{R})$; therefore τ defines an affine and holomorphic automorphism τ_A of order 2 and derivative $-I$ on each X_A. By inspection of the possible positions of P, see [HeSc 05, Section 3.2], one finds that, in any case, τ fixes the inverse images of O and M under d_P (and no other points of X).

4.2 Coverings with given ramification data

The origamis D_P introduced in the previous section all are defined by coverings with the same ramification behaviour. It is classically known that coverings with a prescribed kind of ramification are classified by an algebraic variety, called a *Hurwitz space*.

More precisely Hurwitz spaces classify coverings $p : X \to Y$ of compact Riemann surfaces with the following data fixed: the degree of p, the genus of Y, the number of critical points and the ramification orders of their preimages. It is also possible to specify a certain geometric configuration of the critical points and/or the monodromy of the covering. Note that once Y and the critical points of p together with their ramification orders are fixed, there are only finitely many possibilities for the monodromy homomorphism $\mu : \pi_1(Y - \{\text{critical points}\}) \to S_n$, where n is the degree of p. Thus restricting the monodromy typically leads to an irreducible component of the Hurwitz space defined by the other data.

In our case, we have coverings of degree 2 of an elliptic curve, that are ramified over 4 points P, P', Q, Q'. Moreover, for a suitable choice of a base point on the elliptic curve, we have $P' = -P$ and $Q' = -Q$. Note that we can always achieve $P' = -P$ by choosing an appropriate base point; but then "$Q' = -Q$" is a proper condition. We cannot formulate in an algebraic way the condition that P_0, P_1, P_2, P_3 form an orbit under the affine diffeomorphism $\bar{c}_A$ induced by $\bar{c}$, since we saw above that $\bar{c}_A$ is in general not an algebraic automorphism on the elliptic curve E_A.

We denote by $\tilde{H}$ the Hurwitz space of isomorphism classes of such coverings.

Thus every element of $\tilde{H}$ is represented by a pair (X, p) where X is a compact Riemann surface and $p : X \to E$ is a covering of degree 2 to an elliptic curve E with a base point O such that the four critical points P, Q, P', Q' of p satisfy $P' = -P$ and $Q' = -Q$. Two such coverings $p : X \to E$ and $p' : X' \to E'$ are considered equivalent (or isomorphic) if there are isomorphisms $\varphi : X \to X'$ and $\bar{\varphi} : E \to E'$ such that $p' \circ \varphi = \bar{\varphi} \circ p$.

By the Riemann-Hurwitz formula, for every $(X, p) \in \tilde{H}$, the genus of X is 3. Therefore there is a natural morphism π (of algebraic varieties) from $\tilde{H}$ to the moduli space M_3 of isomorphism classes of Riemann surfaces of genus 3.

Proposition 8. *Let (X, p) be a point in $\tilde{H}$. Then* $\mathrm{Aut}(X)$ *contains a subgroup isomorphic to the Klein four group V_4.*

Proof. Every covering of degree 2 is normal, therefore there is an automorphism σ of X that interchanges the two inverse images under p; the ramification points $\tilde{P}$, $\tilde{P}'$, $\tilde{Q}$, $\tilde{Q}'$, that are mapped to P, P', Q, and Q' by p, are the fixed points of σ.
Next we want to show that the automorphism $[-1]$ on the elliptic curve E lifts to an automorphism τ on X of order 2 that commutes with σ. Note that $[-1]$ also is an automorphism of $E^{**} := E - \{P, P', Q, Q'\}$ and thus induces an automorphism $\tilde{\tau}$ of $\pi_1(E^{**})$. It follows from surface topology that $[-1]$ lifts to X if and only if $\tilde{\tau}$ preserves the subgroup U of $\pi_1(E^{**})$ that corresponds to the unramified covering $p : X - \{\tilde{P}, \tilde{P}', \tilde{Q}, \tilde{Q}'\} \to E^{**}$.
We take a fixed point Z_0 of $[-1]$ as base point for $\pi_1(E^{**})$. Let l_x and l_y be simple geodesic loops around Z_0 that are mapped onto their inverses by $[-1]$ and such that their homotopy classes x and y generate $\pi_1(E)$. Together with the loops l_P, $l_{P'}$, l_Q and $l_{Q'}$ around the critical points, we obtain a set of generators for $\pi_1(E^{**})$. Using them, the automorphism $\tilde{\tau}$ of order 2 is given by

$$\tilde{\tau}(x) = x^{-1}, \quad \tilde{\tau}(y) = y^{-1}, \quad \tilde{\tau}(l_P) = l_{P'}, \quad \tilde{\tau}(l_Q) = l_{Q'} \tag{8}$$

The subgroup U is the kernel of the monodromy homomorphism $\mu = \mu_P : \pi_1(E^{**}) \to S_2$. Since p is ramified over P, P', Q and Q', we have $\mu(l_P) = \mu(l_{P'}) = \mu(l_Q) = \mu(l_{Q'}) = (1\ 2)$. Moreover $\mu(x) = \mu(x^{-1})$ and $\mu(y) = \mu(y^{-1})$, thus the kernel of μ clearly is preserved by $\tilde{\tau}$. Hence $\tilde{\tau}$ induces an automorphism τ of X of order 2 which by construction acts on the fibers of p and thus commutes with σ. □

Note that a covering $p : X \to E$ of degree 2 is completely determined by its deck transformation σ. Therfore we can describe the Hurwitz space $\tilde{H}$ also as the set of equivalence classes of pairs (X, σ), where X is a Riemann surface of genus 3 and $\sigma \in \mathrm{Aut}(X)$ an involution with exactly 4 fixed points whose images on $E = X/ < \sigma >$ are symmetric w. r. t. some point on E. According to the proof of Proposition 8, the last condition is equivalent to the existence of a second involution τ which commutes with σ.

The equivalence relation can be rewritten as

$$(X, \sigma) \cong (X', \sigma') \Leftrightarrow \exists \text{ isomorphism } \varphi : X \to X' \text{ s. t. } \sigma' = \varphi\sigma\varphi^{-1} \tag{9}$$

4.3 Origami curves in *H*

If we are given an elliptic curve E and four points P, $-P$, Q, $-Q$ on E, there are four different coverings of E of degree 2 that are ramified exactly over these points: this is due to the fact, pointed out at the end of the proof of Proposition 8, that the monodromy homomorphism $\mu : \pi_1(E^{**}) \to S_2$ is fixed by these data except for the choice of $\mu(x)$ and $\mu(y)$. The covering surface X is hyperelliptic if and only if it admits an involution with 8 fixed points. A (general) point $(X, \sigma) \in \tilde{H}$ has 3 involutions: σ, τ and $\sigma\tau$ in the notation of Proposition 8. By definition, σ has 4 fixed points. Possible fixed points of τ and $\sigma\tau$ are the inverse images of the four fixed points of $[-1]$ on E. If τ fixes exactly 4 of them, the other four are fixed by $\sigma\tau$; if τ has 8 fixed points, $\sigma\tau$ has none, and vice versa. In particular, at least one of τ and $\sigma\tau$ has fixed points on X; thus we may assume that τ has a fixed point.

It is easy to see that for precisely one choice of the monodromy map, τ has 8 fixed points. For the other three choices of μ, τ and $\sigma\tau$ have 4 fixed points each.

Definition 9. *Let H be the subspace of $\tilde{H}$ that consists of the pairs (X, σ) for which the involution τ has exactly 4 fixed points.*

For $(X, \sigma) \in H$, the quotients of X by the involutions σ, τ and $\sigma\tau$ all have genus 1. We shall see in Corollary 14 that for generic $(X, \sigma) \in H$, $\mathrm{Aut}(X) = \{\mathrm{id}, \sigma, \tau, \sigma\tau\}$; in particular, $\mathrm{Aut}(X)$ does not contain an involution with quotient curve isomorphic to $\mathbb{P}^1$, i. e. X is not hyperelliptic. Thus we could alternatively characterize H as the closure in $\tilde{H}$ of the set of those (X, σ) for which X is not hyperelliptic.

Remark. For every n-torsion point P on E_{-1}, the origami D_P defines an algebraic curve C_P in H. In the same way, W defines an algebraic curve C_W in H.

Namely, $\pi : \tilde{H} \to M_3$ is a finite morphism of varieties, and the origami curves $C(D_P)$ resp. $C(W)$ are algebraic curves in M_3 which lie in the image $\pi(\tilde{H})$. This was shown in Section 4.2 for D_P and in Section 3.5 for W. The inverse image $C_P := \pi^{-1}(C(D_P))$ resp. $C_W = \pi^{-1}(C(W))$ is therefore an algebraic curve in $\tilde{H}$. It remains to show that they lie in H. But for C_P, this follows from the observation in Section 4.1 that the automorphism τ has 4 fixed points on every Riemann surface X_A corresponding to a point $(X_A, \sigma) \in C_P$. On W, τ is the automorphism c^2 (or $-1 \in Q$) described in Section 3.2 which fixes the four vertices of the squares.

Theorem 1. *The union of the curves C_P, P a torsion point on E_{-1}, is dense in H.*

Proof. We shall prove the statement for the complex topology on the Hurwitz space H. Then it holds a fortiori also for the Zariski topology on the algebraic variety H.

Let (X, σ) be a point in H with non-hyperelliptic X. This point is determined by an elliptic curve E (with a base point O), two pairs $(P, -P)$ and $(Q, -Q)$ of opposite points on E, and a choice of one of three possible monodromy maps. Clearly we can approximate P and Q by torsion points, more precisely by pairs P_n, Q_n of n-torsion points for the same n. For a suitable choice of the monodromy map, the twofold covering of E ramified over P_n, $-P_n$, Q_n and $-Q_n$ then defines a point $(X_n, \sigma_n) \in H$ which is close to (X, σ). Note that the composition of the covering $X_n \to E$ of degree 2 with multiplication by n on E is an origami O, hence (X_n, σ_n) lies on the inverse image $C = \pi^{-1}(C(O))$ of the origami curve $C(O)$ in M_3. In the sequel, we also call $C \subset H$ an origami curve.

We now have to show that there is an n-torsion point P on E_{-1} such that $(X_n, \sigma_n) \in C_P$. We shall in fact prove the stronger statement $C = C_P$, but only for the case that $n = p$ is a prime (which clearly suffices to approximate the original point (X, σ)).

Recall from Proposition 5 that the origami curves C and C_P are equal if and only if the corresponding subgroups U and U' of F_2 are mapped onto each other by an automorphism of F_2. If such an automorphism exists, it is induced by an affine diffeomorphism $\bar{f} : E \to E_{-1}$ which transforms the configuration $\{P_p, Q_p, -P_p, -Q_p\}$ of p-torsion points on E to an orbit $\{P, \bar{c}(P), -P, -\bar{c}(P)\}$ under $\bar{c}$ on E_{-1} (for some P), and which moreover respects the monodromy. We shall see in the proof of Proposition 18 that, for a given pair of p-torsion points on E, all three non-hyperelliptic choices of the monodromy map lead to points on the same origami curve; so we do not have to care about the monodromy.

To find such an affine diffeomorphism $\bar{f}$, consider a point (X', σ') on the origami curve C through (X_p, σ_p) which lies over the elliptic curve E_{-1}. To (X', σ') there corresponds an affine transformation $A \in \mathrm{SL}_2(\mathbb{R})$ that maps E into E_{-1}, and P_p and Q_p to p-torsion points on E_{-1}. We identify the group $E_{-1}[p]$ of p-torsion points on E_{-1} with $(\mathbb{Z}/p\mathbb{Z})^2$ and keep the notation P_p and Q_p for the points in $(\mathbb{Z}/p\mathbb{Z})^2$ induced by the original p-torsion points on E. Thus in order to show that the origami curves C (defined by O) and C_P (defined by $P \in E_{-1}[p]$) are equal it suffices to find some $B \in \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and some $P \in (\mathbb{Z}/p\mathbb{Z})^2$ such that

$$\{B(P_p), B(Q_p)\} = \{P, \bar{c}(P)\}. \tag{10}$$

Since $\bar{c}$ is the rotation by $\frac{\pi}{2}$, it acts on $E_{-1}[p]$ through the matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now we identify $\mathbb{Z}/p\mathbb{Z}$ with the field $\mathbb{F}_p$ and use linear algebra over $\mathbb{F}_p$: we consider

P_p and Q_p as vectors $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} c \\ d \end{pmatrix}$ in $\mathbb{F}_p^2$; we may assume that $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \neq 0$, because we can approximate (X, σ) by p-torsion points that are not collinear. Choose $M \in \mathrm{SL}_2(\mathbb{F}_p)$ with $M(P_p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let $\begin{pmatrix} u \\ v \end{pmatrix} = M(Q_p)$. Since $v = ad - bc$ is invertible in $\mathbb{F}_p$, we find a matrix $S'' \in \mathrm{SL}_2(\mathbb{F}_p)$ with first column $\begin{pmatrix} u \\ v \end{pmatrix}$ and $\mathrm{tr}(S'') = 0$. For the conjugate $S' := M^{-1}S''M$ we then obtain $S'(P_p) = Q_p$. We show in Lemma 10 below that S' is conjugate to S or to S^{-1} in $\mathrm{SL}_2(\mathbb{F}_p)$, say $S' = B^{-1}SB$. Then

$$SB(P_p) = BS'(P_p) = B(Q_p),$$

i. e. (10) is satisfied for $P = B(P_p)$, and the proposition is proved. □

Lemma 10. *Let p be a prime and $T \in \mathrm{SL}_2(\mathbb{F}_p)$ with $\mathrm{tr}(T) = 0$. Then T is conjugate in $\mathrm{SL}_2(\mathbb{F}_p)$ to S or to S^{-1}, where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.*

Proof. First observe that $T^2 = -I$ and the characteristic polynomial of T is $x^2 + 1$. We distinguish two cases:

1. If $p \equiv -1 \bmod 4$, T has no eigenvalue in $\mathbb{F}_p$. Let v_1 be any nonzero vector in $\mathbb{F}_p^2$ and $v_2 = Tv_1$. With respect to the basis v_1, v_2, T is represented by the matrix S. Hence T and S are conjugate by an invertible matrix $M \in \mathrm{GL}_2(\mathbb{F}_p)$. Replacing v_1, v_2 by λv_1 and λv_2 for some $\lambda \in \mathbb{F}_p^\times$ multiplies $\det(M)$ by λ^2. Thus if the determinant of the original matrix M was a square in $\mathbb{F}_p^\times$, we can find a suitable λ such that the base change matrix has determinant 1. Otherwise $-\det(M)$ is a square (since $p \equiv -1 \bmod 4$ and hence -1 is not a square in $\mathbb{F}_p$). We now replace v_2 by $-v_2$ and obtain a base change matrix with determinant $-\det(M)$. T is conjugate to S^{-1} in this case.

2. If $p \equiv 1 \bmod 4$, T has two eigenvalues α and $-\alpha$ in $\mathbb{F}_p$ (satisfying $\alpha^2 = -1$). Thus T is conjugate to $\tilde{S} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$ by a matrix M whose rows are eigenvectors of T. Multiplying the first row of M by $\det(M)^{-1}$ then yields a base change matrix of determinant 1. In particular, S itself is conjugate to $\tilde{S}$ in $\mathrm{SL}_2(\mathbb{F}_p)$, thus all other matrices $T \in \mathrm{SL}_2(\mathbb{F}_p)$ with $\mathrm{tr}(T) = 0$ are conjugate to S. □

4.4 Affine coordinates for *H*

Proposition 11. *For any point $(X, \sigma) \in H$, X can be represented by a plane quartic with equation*

$$x^4 + y^4 + z^4 + 2ax^2y^2 + 2bx^2z^2 + 2cy^2z^2 = 0 \tag{11}$$

for some complex numbers a, b and c.

Proof. Since X is not hyperelliptic, the canonical map on X gives an embedding into the projective plane as a smooth quartic. Moreover, every automorphism of X is induced by a projective automorphism of $\mathbb{P}^2$. Thus we can choose coordinates on $\mathbb{P}^2$ such that σ acts by $(x:y:z) \mapsto (-x:y:z)$ and τ by $(x:y:z) \mapsto (x:-y:z)$. Then $\sigma\tau$ acts by $(x:y:z) \mapsto (-x:-y:z) = (x:y:-z)$. A quartic that is invariant under σ and τ must therefore be a polynomial in x^2, y^2 and z^2. We can still multiply x, y and z by suitable constants and thus obtain that the coefficients of x^4, y^4 and z^4 are 1, which gives us an equation of the form (11). □

The equation (11) describes a family of plane projective curves of degree 4 over the affine parameter space $\mathbb{A}^3(\mathbb{C}) = \mathbb{C}^3$. The nonsingular curves in this family are determined by

Proposition 12. a) *The plane curve C_{abc} with equation* (11) *is singular if and only if $a = \pm 1$ or $b = \pm 1$ or $c = \pm 1$ or $a^2 + b^2 + c^2 - 2abc - 1 = 0$.*
b) *Let $V \subset \mathbb{A}^3(\mathbb{C})$ be the zero set of the polynomial $(a^2-1)(b^2-1)(c^2-1)(a^2 + b^2 + c^2 - 2abc - 1)$, $U = \mathbb{A}^3(\mathbb{C}) - V$ and*

$$\mathcal{C} = \{((x:y:z),(a,b,c)) \in \mathbb{P}^2(\mathbb{C}) \times U : (x:y:z) \in C_{abc}\}.$$

Then the projection $\mathrm{proj}_2 : \mathcal{C} \to U$ to the second factor is a family of nonsingular projective curves of genus 3.

Proof. The first part is easily checked using Jacobi's criterion $\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0$ for a singular point P on the plane projective curve with equation $f = 0$. The second statement is a consequence of the first. □

It follows from the property of M_3 as a coarse moduli space, that the map $m : U \to M_3$ that sends a point (a,b,c) to the isomorphism class of C_{abc}, is an algebraic morphism. To better understand this map we first determine the automorphisms of U and of $\mathcal{C}$.

Proposition 13. a) *For every $(a,b,c) \in U$, C_{abc} admits the automorphisms*

$$\alpha : (x:y:z) \mapsto (-x:y:z) \quad \textit{and} \quad \beta : (x:y:z) \mapsto (x:-y:z);$$

they generate a Klein four group G_0 in the group

$$\mathrm{Aut}(\mathcal{C}) = \{\varphi : \mathcal{C} \to \mathcal{C} : \varphi \text{ automorphism}, \ \mathrm{proj}_2 \circ \varphi = \ \mathrm{proj}_2\}$$

of relative automorphisms of the family $\mathcal{C}$.
b) *The group L of linear automorphisms of U is generated by the permutations of a, b and c, and by the map $s : (a,b,c) \mapsto (-a,-b,c)$; L is isomorphic to S_4.*

c) *There is a short exact sequence*

$$1 \to G_0 \to \mathrm{Aut}(\mathcal{C}) \to L \to 1. \tag{12}$$

d) $\mathrm{Aut}(\mathcal{C})$ *has 96 elements and is isomorphic to the automorphism group of the Fermat curve* $C_{000} : x^4 + y^4 + z^4 = 0$.

Proof. a) is obvious.
b) Clearly the group generated by s and the permutations of a, b and c is contained in $\mathrm{Aut}(U)$, and it is easy to check that it is isomorphic to S_4.
On the other hand, any linear automorphism f of U must permute the six planes $a = \pm 1$, $b = \pm 1$ and $c = \pm 1$. The nonlinear component of V also must be preserved, which gives a restriction on the distribution of signs: the product of the three signs must be $+1$. Thus f is in the group generated by S_3 and s.
c) Observe that every element of L can be lifted to an automorphism of $\mathcal{C}$: For a permutation of a, b and c, apply the same permutation to z, y and x (in this order!); s can be lifted to $\tilde{s} : ((x : y : z), (a, b, c)) \mapsto ((ix : y : z), (-a, -b, c))$. The subgroup G of $\mathrm{Aut}(\mathcal{C})$ generated by G_0 and these lifts clearly fits into the exact sequence (12). We shall see later that $G = \mathrm{Aut}(\mathcal{C})$ (see the remark following Proposition 16).
d) G has $|G_0| \cdot |L| = 96$ elements, and all of them map C_{000} onto itself. Recall that C_{000} is the Fermat curve of degree 4 and has precisely 96 automorphisms as we have mentioned in Section 3.3. □

Corollary 14. *The kernel of the homomorphism* $\rho : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(U)$ *is* G_0. *In particular, for general* $(a, b, c) \in U$, $\mathrm{Aut}(C_{abc}) = G_0$.

Proof. As observed in the proof of Proposition 11, any automorphism of a curve in $\mathcal{C}$ is induced by an automorphism of $\mathbb{P}^2(\mathbb{C})$. Thus any $\varphi \in \ker(\rho)$ is a linear change of the homogeneous coordinates x, y, z that preserves the terms ax^2y^2, bx^2z^2 and cy^2z^2 for all $(a, b, c) \in U$. This is only possible if $\varphi \in G_0$. Alternatively, the result can be checked by comparison with the list of automorphism groups in genus 3 in [KuKo 79]. □

4.5 Maps to moduli space

In this subsection we study the relations between the spaces U, H and M_3. To do so, we shall factor the morphism $m : U \to M_3$, $(a, b, c) \mapsto [C_{abc}]$, in two ways.

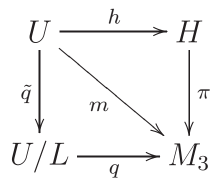
We saw in the proof of Proposition 13 that the subgroup L of $\mathrm{Aut}(U)$ is contained in the image of the natural homomorphism $\rho : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(U)$. Therefore m factors through U/L.

On the other hand we know that , for every $(a, b, c) \in U$, C_{abc} admits the automorphism $\alpha : (x : y : z) \mapsto (-x : y : z)$. We find:

Proposition 15. $(a, b, c) \mapsto (C_{abc}, \alpha)$ *is a surjective morphism* $h : U \to H$.

Proof. The fixed points of α in $\mathbb{P}^2(\mathbb{C})$ are the point $(1 : 0 : 0)$ (which is not on C_{abc} for any $(a, b, c) \in U$) and all points having $x = 0$. On C_{abc} there are precisely 4 points $(0 : y_i : 1)$ of this form, where the y_i are the solutions of $y^4 + 2cy^2 + 1 = 0$ (note that they are all different because $c^2 \neq 1$). Thus $E_{abc} := C_{abc}/ < \alpha >$ is a curve of genus 1 by the Riemann-Hurwitz formula. To see that (C_{abc}, α) defines a point in H, it remains to show that the critical points of the covering $p : C_{abc} \to E_{abc}$ are symmetric w. r. t. an involution $[-1]$. For this consider the automorphism $\beta : (x : y : z) \mapsto (x : -y : z)$ of C_{abc} which, like α, is an element of G_0: β also has 4 fixed points on C_{abc}, different from those of α, and it maps every fixed point of α to a fixed point of α. Since α and β commute, β descends to an involution on E_{abc} with 4 fixed points that acts as $[-1]$ on the critical points of p.
This defines the morphism h; the surjectivity of h was proved in Proposition 11. □

So far we have found a commutative diagram



The final goal in this section is to show

Proposition 16. q *is birational.*

Remark. From this it follows in particular that $L = \rho(\mathrm{Aut}(\mathcal{C}))$ (because m also factors through $U/\rho(\mathrm{Aut}(\mathcal{C}))$. Together with Corollary 14 this shows that $\mathrm{Aut}(\mathcal{C})$ fits in the middle of the exact sequence (12), and thereby finishes the proof of Proposition 13.

Proof of Proposition 16. Since U is irreducible, the same holds for H, U/L and their image $m(U) \subset M_3$. Moreover all morphisms in the diagram are finite, so we can compare their degrees.
Obviously the degree of $\tilde{q}$ is $|L| = 24$.
Next we observe that $\deg(\pi) = 3$: Any covering $C \to E$ of degree 2 is induced by an automorphism of order 2 on C. Since for a general point $(C, p) \in H$, C has precisely 3 such involutions, the degree of π is at most 3. But if some $(C, \alpha) \in H$ would be isomorphic to (C, β) for different automorphisms α and β, these would by (9) have to be conjugate in $\mathrm{Aut}(C)$; this is impossible, again because $\mathrm{Aut}(C) = G_0$ for generic C by Corollary 14.
Finally we claim that h is of degree 8. To see this we have to determine, for generic $(a, b, c) \in U$, all (a', b', c') such that there is an isomorphism $\varphi : C_{abc} \to$

$C_{a'b'c'}$ satisfying $\varphi\alpha\varphi^{-1} = \alpha$ (see (9)). As before, φ has to be induced by an automorphism of $\mathbb{P}^2(\mathbb{C})$ and, since we want (a, b, c) to be generic, even by an automorphism of $\mathcal{C}$.
As a projective transformation, φ commutes with α if and only if it is given by a matrix of the form

$$D_\varphi = \begin{pmatrix} v & 0 & 0 \\ 0 & r & s \\ 0 & t & u \end{pmatrix} \in \mathrm{PGL}_3(\mathbb{C}). \tag{13}$$

The condition that φ maps every curve C_{abc} to a curve of the same type implies by a straightforward (but unpleasant) calculation that

$$v^4 = 1 \quad \text{and either} \qquad t = s = 0, \quad r^4 = u^4 = 1$$
$$\text{or} \qquad u = r = 0, \quad t^4 = s^4 = 1$$

This give $4 \cdot 4 \cdot 4 \cdot 2$ matrices; multiplying each entry by a power of i gives the same element in $\mathrm{PGL}_3(\mathbb{C})$. Thus we have found a subgroup of order 32 of $\mathrm{Aut}(\mathcal{C})$ that generically determines h. It contains the kernel G_0, therefore its image L_H in L is a subgroup of order 8. □

Note: q is not an isomorphism, nor is H isomorphic to U/L_H.
This can be seen e. g. by looking at the Fermat curve C_{000}: it is mapped isomorphically onto $C_{0,3,0}$ by the transformation $(x : y : z) \mapsto (x + z : \sqrt[4]{8}\, y : x - z)$, which does not extend to an automorphism of $\mathcal{C}$ since we have seen in Proposition 13 d) that they all fix C_{000}.

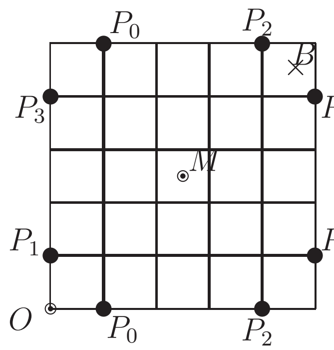
5 The Veech groups to the origamis D_P

In this section we finally study the Veech groups of the origamis D_P introduced in Section 4.1 (with P an n-torsion point of E_{-1}). Recall that a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is called *congruence group of level* n, if it contains the principal congruence group $\Gamma(n) = \{A \in \mathrm{SL}_2(\mathbb{Z}) | \, A \equiv I \mod n\}$ and n is minimal with this property. As before I denotes the identity matrix. We show that the Veech group $\Gamma(D_P)$ is a subgroup of index 3 of a congruence subgroup of level n (see Proposition 20). For n odd and P in a somehow general position, we write the Veech group down explicitly (see Theorem 2). It is a congruence group of level $2n$.[2]

Recall from Section 3.5 and Section 4.1 that the origami D_P is given by a degree $2n^2$ covering $p := d_P : X \to E_{-1}$, where E_{-1} is the torus endowed with the translation structure of $\mathbb{C}/\Lambda_I$ with $\Lambda_I = \mathbb{Z} \oplus \mathbb{Z}i$. The covering p splits into a covering $\kappa : X \to E_{-1}$ of degree 2 and the multiplication by n: $[n] : E_{-1} \to E_{-1}$.

[2]Using similar methods one can calculate the Veech group for n even and P in general position and obtains a congruence group of level n. The proof will be carried out elsewhere.

κ is the quotient by the translation σ; it is ramified over four points $P = P_0$, P_1, P_2 and P_3 on E_{-1}. They are symmetric with respect to a rotation $\bar{c}$ of angle $\frac{\pi}{2}$. Furthermore, $\bar{c}$ can be lifted to an automorphism c on X that has four fixed points.



$\bar{c}$ acts on E_{-1} as rotation of angle $\pi/2$ with fixed points O and M, it preserves $\{P_0, P_1, P_2, P_3\}$ and lifts to c on X.

We call μ the translation structure on $X \backslash \{\text{ramification points of } p\}$ that is obtained by lifting the one on $E_{-1} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ via p.

We start from the following two observations (proved in Lemma 17):

- Each diffeomorphism $f : X \to X$ which is affine with respect to μ descends via κ to an affine diffeomorphism $\bar{f} : E_{-1} \to E_{-1}$ which one may write as

$$\bar{f} : E_{-1} \to E_{-1}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + e \quad \text{with } A \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } e \in \mathbb{Z}^2. \tag{14}$$

  In order to make (14) well defined we have to choose some point to be $O = (0, 0)$. We choose one of the two fixed points of $\bar{c}$. Furthermore, we identify the n-torsion points with $(\mathbb{Z}/n\mathbb{Z})^2$ in the natural way . Since the four ramification points are symmetric with respect to $\bar{c}$, we may write them as $P_0 = (p, q)$, $P_1 = (-q, p)$, $P_2 = (-p, -q)$ and $P_3 = (q, -p)$ with $p, q \in \mathbb{Z}/n\mathbb{Z}$.

- Since $\bar{f}$ lifts to f via κ, it has to respect the ramification points, i.e.

$$\bar{f}(S) = S \quad \text{with } S := \{(p, q), (-q, p), (-p, -q), (q, -p)\} \tag{15}$$

We then show in Corollary 19, that the group of affine diffeomorphisms on X projects by $f \mapsto \bar{f}$ to a subgroup of index 3 of the group of diffeomorphisms on E_{-1} that fulfill (15). If n is odd, the affine diffeomorphisms have to fix O (Lemma 17) and fulfill an additional condition (Lemma 21). Using this we write down the

Veech groups explicitly for n odd and S in "general position".

We will use the characterization of origami Veech groups in Theorem B for the proofs of our claims. Therefore we describe our setting in terms of subgroups of F_2. We consider the fundamental groups of the following punctured surfaces:

$$E^* := E_{-1} - \{\infty\}, \quad E^{[n]*} := E_{-1} - [n]^{-1}(\infty) \text{ and } X^* := X - p^{-1}(\infty)$$

Observe that $[n]^{-1}(\infty)$ are precisely the n-torsion points of E_{-1}.
From the morphisms κ, $[n]$ and $p = [n] \circ \kappa$, we get the following embeddings:

$$U := \pi_1(X^*) \;\subseteq\; H_n := \pi_1(E^{[n]*}) \;\subseteq\; \pi_1(E^*) = F_2$$

As in Theorem B the group $F_2 = F_2(x, y)$ is the free group on the two generators x and y. Observe that

$$H_n = \text{kernel}(\text{proj}_n) \quad \text{with} \quad \text{proj}_n : F_2 \to (\mathbb{Z}/n\mathbb{Z})^2, \quad x \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; y \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In particular, H_n is a characteristic subgroup of F_2.

Description of the n-torsion points on E_{-1} in terms of these groups:
The n-torsion points are precisely the cusps of $E^{[n]*} = \mathbb{H}/H_n$. Hence, they correspond to the equivalence classes of the elements $w[x, y]w^{-1} \in H_n$, with $w \in F_2$ and $w_1[x, y]w_1^{-1} \sim w_2[x, y]w_2^{-1} \;:\Leftrightarrow\; w_1 w_2^{-1} \in H_n \;\Leftrightarrow\; \text{proj}_n(w_1) = \text{proj}_n(w_2)$.
We may choose the base point B of the fundamental group $\pi_1(\mathbb{H}/H_n)$ such that $[x, y]$ is the loop around the cusp $(0, 0)$ (see the Figure of E_{-1} on page 24). Then $l_{a,b} := x^a y^b [x, y] y^{-b} x^{-a}$ with $a, b \in \{0, \ldots, n-1\}$ is the loop around the cusp $(\bar{a}, \bar{b})$, and these loops form a system of representatives for the equivalence classes.

Description of U by monodromy of κ:
Since $\kappa : X \to E_{-1}$ is a morphism of degree 2, U is a normal subgroup of H_n of index 2. Thus it is the kernel of a monodromy map $\mu : H_n \to S_2$.
From the ramification data of κ and the correspondence between the cusps of $E^{[n]*}$ and the equivalence classes of loops around cusps it follows that

$$\mu(l_{a,b}) = \begin{pmatrix} 1 & 2 \end{pmatrix} \iff (\bar{a}, \bar{b}) \in S = \{(p, q), (-q, p), (-p, -q), (q, -p)\}. \quad (16)$$

Here, $\bar{a}, \bar{b}$ are the images of a, b in $\mathbb{Z}/n\mathbb{Z}$. In the following we will also write $l_{a,b} \in S$ meaning that $(\bar{a}, \bar{b}) \in S$ and more generally $l \in S$, if l is a loop equivalent to some $l_{a,b} \in S$, i.e. if l is a loop around a ramification point.
Since $E^{[n]*}$ is a torus with n^2 punctures, its fundamental group H_n is generated by the simple loops x^n, y^n and the n^2 loops $l_{a,b}$ around the punctures. (More precisely the fundamental group is the free group in $n^2 + 1$ generators and we could omit one of the loops around a puncture. But since there is none better

than the other, we take all of them.)
There are precisely four maps μ_1, μ_2, μ_3, $\mu_4 : H_n \to S_2$ fulfilling (16): The images of the loops around the cusps are given by (16). For the other two generators x and y one has the following four possibilities

$$\begin{array}{ll} \mu_1 : \; x^n \mapsto \begin{pmatrix} 1 & 2 \end{pmatrix}, \;\; y^n \mapsto \begin{pmatrix} 1 & 2 \end{pmatrix}, & \mu_2 : \; x^n \mapsto \mathrm{id}, \;\; y^n \mapsto \mathrm{id}, \\ \mu_3 : \; x^n \mapsto \begin{pmatrix} 1 & 2 \end{pmatrix}, \;\; y^n \mapsto \mathrm{id}, & \mu_4 : \; x^n \mapsto \mathrm{id}, \;\; y^n \mapsto \begin{pmatrix} 1 & 2 \end{pmatrix} \end{array}$$

They correspond precisely to the four possible coverings ramified over the four given points in S that we described in Section 4.2. The coverings are the maps

$$\mathbb{H}/U_i \to \mathbb{H}/H_n \quad \text{with} \;\; U_i := \mathrm{kernel}(\mu_i), \tag{17}$$

induced by the inclusions $U_i \subseteq H_n$.

Affine diffeomorphisms:
Let us choose a universal covering $u : \mathbb{H} \to X^*$. We obtain the sequence:

$$\mathbb{H} \;\stackrel{u}{\to}\; X^* \;\stackrel{\kappa}{\to}\; E_{-1} \;\stackrel{[n]}{\to}\; E_{-1}.$$

Recall from the remark to Theorem B that for the group $\mathrm{Aff}(\mathbb{H}, u)$ of diffeomorphisms that are affine with respect to the translation structure lifted from X^* via u (which is the same as the one lifted from E^* via $[n] \circ \kappa \circ u$), one has the following correspondence:

$$\mathrm{Aff}(\mathbb{H}, u) \;\cong\; \mathrm{Aut}^+(F_2) \quad \text{by} \quad * : \tilde{f} \mapsto \tilde{f}^* := (x \mapsto \tilde{f} x \tilde{f}^{-1}) \in \mathrm{Aut}^+(F_2)$$

An affine diffeomorphism $\tilde{f}$ descends to a surface $\mathbb{H}/H$ $(H \subset F_2)$ if and only if $\tilde{f}^*(H) = H$. The Veech group of $\mathbb{H}/H$ is by Theorem B:

$$\Gamma(\mathbb{H}/H) = \hat{\beta}(\mathrm{Stab}(H)) \text{ with } \mathrm{Stab}(H) = \{\gamma \in \mathrm{Aut}^+(F_2) |\, \gamma(H) = H\}.$$

Suppose now, that f is an affine diffeomorphism of X^*.

Lemma 17. *f descends via κ to an affine diffeomorphism $\bar{f}$ on E_{-1} fixing the set S of ramification points of κ. If $n > 1$ is odd and $P = (p, q)$ is a primitive n-torsion point (i.e. n is the smallest number such that $n \cdot P = 0$), then $\bar{f}$ has a fixed point at the n-torsion point $O = (0, 0)$.*

Proof. Let $\tilde{f}$ be a lift of f to $\mathbb{H}$ via u. Since H_n is characteristic, one has: $\tilde{f}^*(H_n) = H_n$. Therefore, by the paragraph before Lemma 17, it follows that f descends via κ to $E^{[n]*} = \mathbb{H}/H_n$. We may extend this map to a diffeomorphism $\bar{f}$ of the whole surface E_{-1} and write it as

$$\bar{f} : z \mapsto Az + e \quad \text{with } A \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } e \in \mathbb{Z}^2.$$

$\bar{f}$ acts on the set of cusps, i.e. on the n-torsion points of E_{-1}. Since it can be lifted to X via κ, it maps the ramification points of κ to themselves, i. e. $\bar{f}(S) = S$. Let n be odd. We have to show that $e \equiv (0,0)^t \mod n$. We have:

$$\bar{f}(\begin{pmatrix} p \\ q \end{pmatrix}) + \bar{f}(\begin{pmatrix} -p \\ -q \end{pmatrix}) \equiv \bar{f}(\begin{pmatrix} -q \\ p \end{pmatrix}) + \bar{f}(\begin{pmatrix} q \\ -p \end{pmatrix}) \equiv 2e \mod \text{n}. \tag{18}$$

There are three possibilities to subdivide S into two pairs. By (18) the two sums should be equal mod n:

$$\begin{pmatrix} p-q \\ q+p \end{pmatrix} = \begin{pmatrix} -p+q \\ -q-p \end{pmatrix}, \quad \begin{pmatrix} p+q \\ q-p \end{pmatrix} = \begin{pmatrix} -p-q \\ p-q \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \mathbb{Z}/n\mathbb{Z}$$

In the first two cases one obtains: $2(p-q) \equiv 0$ and $2(p+q) \equiv 0$. Hence, $4p \equiv 0$ and $4q \equiv 0$. But $P = (p,q)$ is a primitive n-torsion point and n is odd, therefore only the third case is possible and $2e \equiv (0,0)^t \mod n$. Since n is odd, $e \equiv (0,0)^t \mod n$. □

It follows from Lemma 17 that in order to obtain the Veech group of D_P, we may restrict to affine diffeomorphisms of E_{-1} that fix the set S. By the correspondence between loops around the cusps of $\mathbb{H}/H_n$ and the n-torsion points of E_{-1}, we have:

$$\begin{aligned} &\tilde{f} \in \mathrm{Aff}(\mathbb{H}, u) \text{ descends via } \kappa \circ u \text{ to an affine diffeomorphism } \bar{f} \text{ on } E_{-1} \text{ that fixes } S \\ &\iff \quad \tilde{f}^* \in \tilde{G} := \{\gamma \in \mathrm{Aut}^+(F_2) |\ \gamma(l_{a,b}) \in S \ \Leftrightarrow\ l_{a,b} \in S\} \end{aligned} \tag{19}$$

Proposition 18. $\mathrm{Stab}(U)$ *is a subgroup of* $\tilde{G}$ *of index* 3.

Proof. We first show that $\mathrm{Stab}(U) \subseteq \tilde{G}$:
Let γ be in $\mathrm{Stab}(U)$. With (16), it follows for all loops around cusps $l_{a,b}$:

$$\begin{aligned} l_{a,b} \in S \quad &\iff \quad \mu(l_{a,b}) \neq \mathrm{id} \iff l_{a,b} \notin U \overset{\gamma(U)=U}{\iff} \gamma(l_{a,b}) \notin U \\ &\iff \quad \mu(\gamma(l_{a,b})) \neq \mathrm{id} \iff \gamma(l_{a,b}) \in S \end{aligned}$$

Thus $\gamma \in \tilde{G}$ and $\mathrm{Stab}(U) \subseteq \tilde{G}$. In fact, this was a group theoretical confirmation for the fact that $\bar{f}$ has to respect the ramification points, if it lifts via κ to X.

In order to obtain the index, we study the action of $\tilde{G}$ on the set $\{U_1, U_2, U_3, U_4\}$ given by $U_i \mapsto \gamma(U_i)$. By the definition of $\tilde{G}$, it acts on these four groups, since they are precisely those groups that have the same monodromy on the loops around cusps as U.
$\mathrm{Stab}(U) \subseteq \tilde{G}$ is the stabilizing group of $U \in \{U_1, U_2, U_3, U_4\}$. Therefore the index $[\tilde{G} : \mathrm{Stab}(U)]$ of $\mathrm{Stab}(U)$ in $\tilde{G}$ is equal to the length of the orbit $\tilde{G} \cdot U$. We will see that there are two orbits: one consisting of a single subgroup, the other one consisting of the remaining three subgroups including U. This then proves the claim.

Let us consider the Riemann surfaces $\mathbb{H}/U_1$, $\mathbb{H}/U_2$, $\mathbb{H}/U_3$ and $\mathbb{H}/U_4$. Recall that they are the four possible degree 2 coverings of $E_{-1} = \mathbb{H}/H_n$ with ramification locus equal to S. Recall furthermore from 4.2 that precisely one of them, let us say $\mathbb{H}/U_j$, is hyperelliptic. But for each γ the surface $\mathbb{H}/\gamma(U_j)$ is again hyperelliptic: By the remark to Theorem B, γ defines an affine diffeomorphism $\tilde{f}$ on $\mathbb{H}$ with $\tilde{f}^* = \gamma$. Then $\tilde{f}$ descends to an affine diffeomorphism $f : \mathbb{H}/U_j \to \mathbb{H}/\gamma(U_j)$. Let $A \in \mathrm{SL}_2(\mathbb{Z})$ be its derivative. We may conjugate the hyperelliptic involution τ on $\mathbb{H}/U_j$ with f and obtain the affine map $f\tau f^{-1}$ of derivative $A \cdot (-I) \cdot A^{-1} = -I$ on $\mathbb{H}/\gamma(U_j)$, where I is the identity matrix. Since the derivative is $-I$, it is an automorphism. Furthermore, the degree and number of fixed points are the same as those of τ on $\mathbb{H}/U_j$, therefore it is also a hyperelliptic involution.
But only for $i = j$, $\mathbb{H}/U_i$ is hyperelliptic. Therefore $\gamma(U_j) = U_j$.
It remains to show that the other three groups form only one orbit. Let γ_1 and γ_2 be the automorphisms

$$\gamma_1 : F_2 \to F_2,\ \ x \mapsto xyx^{-1},\ y \mapsto x^{-1} \quad \text{and} \quad \gamma_2 : F_2 \to F_2,\ \ x \mapsto x,\ y \mapsto yx^n$$

One can check that they are both in $\tilde{G}$. A more geometrical argument is given as follows: Recall that we had chosen the identification of the n-torsion points with $(\mathbb{Z}/n\mathbb{Z})^2$ such that $(0,0)$ is a fixed point of $\bar{c}$ and therefore also of $[-1] = \bar{c}^2$.
By the remark to Theorem B and the fact that H_n is characteristic, γ_1 and γ_2 define affine diffeomorphisms $\bar{f}_1$, resp. $\bar{f}_2$, on $\mathbb{H}/H_n = E^{[n]*}$ with derivative

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{respectively} \quad A_2 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

which we may again extend to diffeomorphisms of E_{-1}.
Observe that both γ_1 and γ_2 map $[x, y]$ to itself, thus $\bar{f}_1$ and $\bar{f}_2$ have a fixed point in $(0,0)$. It follows that the action of γ_1 and γ_2 on the equivalence classes of loops $w[x,y]w^{-1}$ is equal to the action of the derivatives A_1 (resp. A_2) on $(\mathbb{Z}/n\mathbb{Z})^2$.
A_1 and A_2 both fix S and thus γ_1 and γ_2 are in $\tilde{G}$ (see (19)).
In the following we determine the orbits of γ_1 and γ_2 on $M := \{U_1, U_2, U_3, U_4\}$.
We have:

$$\gamma_1(x^n) = xy^nx^{-1},\ \ \gamma_1(y^n) = x^{-n} \quad \text{and} \quad \gamma_2(x^n) = x^n,\ \ \gamma_2(y^n) = (yx^n)^n$$

For $i \in \{1, \dots, 4\}$, one obtains: $\mu_i(\gamma_1(y^n)) = \mu_i(x^{-n}) \overset{\text{in} S_2}{=} \mu_i(x^n)$.
For the calculation of $\mu_i(\gamma_1(x^n))$, one has to decompose xy^nx^{-1} into the chosen generators: $\mu_i(\gamma_1(x^n)) = \mu_i(xy^nx^{-1}) = \mu_i(l_{0,0}l_{0,1} \dots l_{0,n-1}y^n)$.
Recall that $\mu_i(l_{a,b}) = \begin{pmatrix} 1 & 2 \end{pmatrix}$ if and only if (a,b) is a ramification point. But since the four ramification points are symmetric with respect to $(0,0)$, either zero or two of them are of the form $(0,b)$. Therefore the monodromy of the loops around cusps in $\mu_i(\gamma_1(x^n))$ adds up to id and we have: $\mu_i(\gamma_1(x^n)) = \mu_i(y^n)$.

From the calculations of $\mu_i(\gamma_1(x^n))$ and $\mu_i(\gamma_1(y^n))$ it follows that γ_1 acts in the following way:

$$\gamma_1: \quad U_1 \mapsto U_1, \quad U_2, \mapsto U_2, \quad U_3 \mapsto U_4, \quad U_4 \mapsto U_3$$

Let us now study γ_2: Obviously one has $\mu_i(\gamma_2(x^n)) = \mu_i(x^n)$. For calculating $\mu_i(\gamma_2(y^n))$, one observes similarly as above that $(yx^n)^n$ can be written as product, where the factors are x^n, n times y^n, and loops around cusps. Here the sum of the monodromies of the cusps depends on the position of P, therefore one obtains two cases. If the sum is id one has

$$\mu_i(\gamma_2)(y^n) = \mu_i(x^n) + n \cdot \mu_i(y^n) \overset{n \text{ odd}}{=} \mu_i(x^n) + \mu_i(y^n)$$

and the action of γ_2 on M is given by:

$$\gamma_2: \quad U_1 \mapsto U_3, \quad U_2, \mapsto U_2, \quad U_3 \mapsto U_1, \quad U_4 \mapsto U_4$$

If the sum of the monodromies of the loops around cusps in $(yx^n)^n$ is $\begin{pmatrix}1 & 2\end{pmatrix}$, then

$$\mu_i(\gamma_2)(y^n) = \mu_i(x^n) + n \cdot \mu_i(y^n) + \begin{pmatrix}1 & 2\end{pmatrix} \overset{n \text{ odd}}{=} \mu_i(x^n) + \mu_i(y^n) + \begin{pmatrix}1 & 2\end{pmatrix}$$

and the action of γ_2 is given by:

$$\gamma_2: \quad U_1 \mapsto U_1, \quad U_2, \mapsto U_4, \quad U_3 \mapsto U_3, \quad U_4 \mapsto U_2.$$

In both cases the action of $\tilde{G}$ has an orbit of 3 elements. This finishes the proof. □

From the proposition we obtain the following two conclusions.

Corollary 19. *The group of affine diffeomorphisms on E_{-1} that lift via κ to X is a subgroup of index 3 of the group of affine diffeomorphisms on E_{-1} that fix S.*

Proof. The isomorphism $\star : \mathrm{Aff}(\mathbb{H}, u) \xrightarrow{\sim} \mathrm{Aut}^+(F_2)$ induces the following isomorphisms:

$$\begin{aligned} \tilde{G} &\cong \{\tilde{f} \in \mathrm{Aff}(\mathbb{H}, u) |\ \bar{f} \text{ fixes } S\} \text{ and} \\ \mathrm{Stab}(U) &\cong \{\tilde{f} \in \mathrm{Aff}(\mathbb{H}, u) |\ \bar{f} \text{ lifts to } X\}. \end{aligned} \tag{20}$$

We denote as before by $\bar{f}$ the affine diffeomorphism on E_{-1} to which $\tilde{f}$ descends via $\kappa \circ u$. Observe that, furthermore, the group of deck transformations of $\kappa \circ u : \mathbb{H} \to E_{-1}$ corresponds by $\star$ to the group $\mathrm{C}_{H_n} \subseteq \mathrm{Aut}^+(F_2)$ of conjugations with elements in H_n. Taking in (20) the quotient by C_{H_n}, the claim follows from the proposition. □

Corollary 20. *$\Gamma(D_P)$ is a subgroup of index 3 in $\hat{\beta}(\tilde{G})$, where $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is as in Theorem B. If n is odd, then $\hat{\beta}(\tilde{G}) = \mathrm{Stab}(S)$, with*

$$\mathrm{Stab}(S) = \{A \in \mathrm{SL}_2(\mathbb{Z}) | \, A \cdot S = S\}.$$

Proof. The first claim follows from Proposition 18, Theorem B, and the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Inn}(F_2) \cong F_2 & \longrightarrow & \mathrm{Aut}^+(F_2) & \xrightarrow{\hat{\beta}} & \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_n & \longrightarrow & \tilde{G} & \longrightarrow & \hat{\beta}(\tilde{G}) & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_n & \longrightarrow & \mathrm{Stab}(U) & \longrightarrow & \Gamma(D_P) & \longrightarrow & 1
\end{array}$$

$\tilde{G}$ projects to the group of affine diffeomorphisms $\bar{f} : z \mapsto Az + e$ on E_{-1} that map S to itself. If n is odd, $e = 0$ by Lemma 17. Therefore A fixes S itself. □

In order to obtain a further condition for an affine diffeomorphism f on X, we study how the affine diffeomorphism $\bar{f} : E_{-1} \to E_{-1}$ to which f descends acts on the 2-torsion points of E_{-1}. The four 2-torsion points are $O = (0, 0)$, $M = (\frac{n}{2}, \frac{n}{2})$, $(\frac{n}{2}, 0)$ and $(0, \frac{n}{2})$. If n is odd, then O is a fixed point of $\bar{f}$ by Lemma 17. For this condition, we just needed that $\bar{f}$ stabilizes S. In the following Lemma we actually have to use that $\bar{f}$ lifts to X.

Lemma 21. *Let $f \in \mathrm{Aff}(X, \mu)$. If n is odd, then $\bar{f}$ fixes M and we have:*

$$\Gamma(D_P) \;\subseteq\; \Gamma_{u,u} \;:=\; \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mod 2\}$$

Proof. Recall that $-I = \bar{c}^2$ is the rotation on E_{-1} of angle π and has fixed points M and O. It can be lifted to the automorphism τ on X for which the two preimages O_1 and O_2 of O and the two preimages M_1 and M_2 of M are fixed points (see Section 4.1). Geometrically τ is a rotation of degree π of the two squares that form X.
Let A be the derivative of f. Then $\tau^{-1}f^{-1}\tau f$ is an affine diffeomorphism of X with derivative $-IA^{-1}(-I)A = I$. Hence, it is in fact a translation.
Furthermore, $\bar{f}(O) = O$ by Lemma 17. Therefore f either fixes O_1 and O_2 or it interchanges them. Since they are both fixed points of τ, we get in both cases: $f^{-1}\tau^{-1}f\tau(O_1) = O_1$.
Hence $f^{-1}\tau^{-1}f\tau$ is a translation and has a fixed point that is not a ramification point; thus: $f^{-1}\tau^{-1}f\tau = \mathrm{id}$. It follows that $\tau(f(M_1) = f(\tau(M_1)) = f(M_1)$. Hence, $f(M_1)$ is also a fixed point of τ and thus one of the points O_1, O_2, M_1 and

M_2. But $\bar{f}(O) = O$, therefore $f(M_1) \in \{M_1, M_2\}$ and $\bar{f}(M) = M$. This proves the first part of the claim.
Let us now write $\bar{f}$ as $\bar{f} : z \mapsto Az + e \stackrel{n \text{ odd}}{=} Az$ with $A \in \mathrm{SL}_2(\mathbb{Z})$. The coordinates of M are $(\frac{n}{2}, \frac{n}{2})$. It follows that

$$\begin{pmatrix} \frac{n}{2} \\ \frac{n}{2} \end{pmatrix} \equiv A \cdot \begin{pmatrix} \frac{n}{2} \\ \frac{n}{2} \end{pmatrix} = \frac{n}{2} \cdot \begin{pmatrix} a+b \\ c+d \end{pmatrix} \bmod n \quad \text{with } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus $a+b$ and $c+d$ have to be odd. This is equivalent with A being in $\Gamma_{u,u}$. □

We may now use these conditions in order to calculate the Veech groups explicitly if n is odd and $P = P_0 = (p, q)$ and $P_1 = (-q, p)$ are in general position, i.e.:

$$\bar{B} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \text{ is invertible over } \mathbb{Z}/n\mathbb{Z}. \tag{21}$$

Theorem 2. *Let n be odd and $P = (p, q)$ be an n-torsion point such that P and $P_1 = \bar{c}(P) = (q, -p)$ are in general position in the sense explained above. Then*

$$\begin{aligned}
\Gamma(D_P) = \quad & \{A \in \mathrm{SL}_2(\mathbb{Z}) |\, A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A \equiv \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod n\} \\
& \cap \; \{A \in \mathrm{SL}_2(\mathbb{Z}) |\, A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A \equiv \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod 2\} \\
= \quad & \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) |\, a+c \text{ and } b+d \text{ odd and} \\
& \qquad (A \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } A \equiv \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod n)\}
\end{aligned}$$

Proof. We proceed in two steps:

First step:
We first determine $\mathrm{Stab}(S)$. We have:

$$\begin{aligned}
& A \in \mathrm{Stab}(S) \\
& \quad \Leftrightarrow A \cdot S = S \text{ with } S = \{b_1 := \begin{pmatrix} p \\ q \end{pmatrix},\, b_2 := \begin{pmatrix} -q \\ p \end{pmatrix},\, b_3 := \begin{pmatrix} -p \\ -q \end{pmatrix},\, b_4 := \begin{pmatrix} q \\ -p \end{pmatrix}\} \\
& \quad \Leftrightarrow \bar{A} \cdot S = S, \text{ where } \bar{A} \text{ is the image of } A \text{ in } \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}).
\end{aligned}$$

Therefore $\mathrm{Stab}(S)$ is a congruence group of level n. We shall show that its image in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is equal to:

$$\bar{\Gamma}_S := \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} \subseteq \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$$

Observe that these four matrices map S to itself. Hence $\bar{\Gamma}_S$ is contained in the image of $\mathrm{Stab}(S)$. Let now $\bar{A}$ be a matrix in the image, i.e. $\bar{A} \cdot S = S$. We have to show that $\bar{A}$ is in $\bar{\Gamma}_S$.
By the definition of $\bar{B} \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ in (21), we have $\bar{A} \cdot \bar{B} = (\bar{A} \cdot b_1, \bar{A} \cdot b_2)$, where $(\bar{A} \cdot b_1, \bar{A} \cdot b_2)$ is the matrix whose first column is $\bar{A}b_1$ and whose second column is $\bar{A}b_2$. Its determinant is equal to $\det(\bar{A}) \cdot \det(\bar{B}) = \det(\bar{B})$. Define $d := \det(\bar{B}) = p^2 + q^2 \in (\mathbb{Z}/n\mathbb{Z})^\times$. We consider now the following four cases:

Case 1: $\bar{A}b_1 = b_1$.
b_2 is the only element of S such that the determinant of the matrix (b_1, b_2) is equal to d. Hence it follows from the above that $\bar{A} \cdot b_2 = b_2$. Thus:

$$\bar{A} \cdot \bar{B} = \bar{B} \overset{\bar{B} \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})}{\Longrightarrow} \bar{A} = \bar{I}, \quad \text{with } \bar{I} \text{ the identity matrix in } \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}).$$

In the *other three cases*, namely $\bar{A} \cdot b_1 = b_2$, $\bar{A} \cdot b_1 = b_3$ and $\bar{A} \cdot b_1 = b_4$, we similarly obtain:

$$\bar{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \bar{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \bar{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hence, $\bar{\Gamma}_S$ is the full image of $\mathrm{Stab}(S)$ in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ and $\mathrm{Stab}(S)$ is the preimage of $\bar{\Gamma}_S$ in $\mathrm{SL}_2(\mathbb{Z})$.

Second step: For any $k \in \mathbb{N}$ let $p_k : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/k\mathbb{Z})$ be the natural projection. We determine the Veech group $\Gamma(D_P)$ as subgroup of $\Gamma_S := p_n^{-1}(\bar{\Gamma}_S)$.

From Corollary 20 and Lemma 21 it follows that

$$\Gamma(D_P) \subseteq \mathrm{Stab}(S) \cap \Gamma_{u,u} = \Gamma_S \cap \Gamma_{u,u} =: \Gamma. \tag{22}$$

We show that the index $[\Gamma_S : \Gamma]$ of Γ in $\Gamma_S = \mathrm{Stab}(S)$ is equal to 3. It then follows that we have equality in (22), since $[\Gamma_S : \Gamma(D_P)]$ is also equal to 3 by Proposition 18. This finishes the proof.
Γ_S and Γ are both congruence groups of level $2n$. Hence we may as well show that their images in $\mathrm{SL}_2(\mathbb{Z}/2n\mathbb{Z})$ differ by index 3, i.e. that $[p_{2n}(\Gamma_S) : p_{2n}(\Gamma)] = 3$.
Recall that $\mathrm{SL}_2(\mathbb{Z}/2n\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, since $\gcd(2, n) = 1$. Thus we have:

$$\begin{aligned} p_{2n}(\Gamma_S) &= \{A \in \mathrm{SL}_2(\mathbb{Z}/2n\mathbb{Z}) |\ A \equiv \pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod n\} \\ &\cong p_n(\Gamma_S) \times \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \\ p_{2n}(\Gamma) &= \{A \in \mathrm{SL}_2(\mathbb{Z}/2n\mathbb{Z}) |\ A \equiv \pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod n \text{ and} \\ &\qquad A \equiv I, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bmod 2\} \\ &\cong p_n(\Gamma_S) \times \{I, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} \subseteq \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

Thus, $[\Gamma_S : \Gamma] = [p_{2n}(\Gamma_S) : p_{2n}(\Gamma)) = |\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})|/2 = 3.$ □

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