Dessins d'enfants and Origami curves

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Contents

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1 Introduction

In this chapter, we give an introduction to the theory of dessins d'enfants. They provide a charming concrete access to a special topic of arithmetic geometry: Curves defined over number fields can be described by such simple combinatorial objects as graphs embedded into topological surfaces. Dessins d'enfants are in some sense an answer of Grothendieck to the beautiful Theorem of Belyi, which characterises curves defined over number fields by the existence of certain coverings of the projective line. Grothendieck was fascinated by the fact that such a covering is completely determined by the preimage of the real interval $[0, 1]$ and called this a dessin d'enfants. As one consequence that especially attracted people one has an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of dessins which is faithful. Therefore in principle all the information on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is hidden in some mysterious way in these combinatorial objects. The study of dessins d'enfants leads to the Grothendieck-Teichmüller group in which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ injects. It is still an open question whether these two groups are equal or not.

In the next three sections we introduce dessins d'enfants and the Galois action on them. We begin in Section 2 with a review of the correspondence between closed Riemann surfaces and regular complex projective curves. Since the link between these two fields is an essential tool which is used throughout the whole chapter, we provide a sketch of the proof. In Section 3 we give characterisations of dessins d'enfants in terms of Belyi pairs, graphs embedded into surfaces, ribbon graphs, monodromy homomorphisms and subgroups of the free group on two generators and explain how to get from one of these descriptions to the other. Section 4 is devoted to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins d'enfants. We review some of the known results on faithfulness and Galois invariants and explain how it gives rise to an action on the algebraic fundamental group $\widehat{F}_2$ of the three-punctured sphere. The explicit description of how $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the topological generators leads us to the definition of the Grothendieck-Teichmüller group. We finish the section by indicating how Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$ embeds into this group.

In the second part of the chapter we turn to connections between origamis and dessins d'enfants. Similar to the latter, origamis are given by combinatorial data and define arithmetic objects, more precisely curves in moduli space which are defined over $\overline{\mathbb{Q}}$. Following the same approach as for dessins, one can study the action of the absolute Galois group on them. Besides these analogies, origamis and dessins are linked by several explicit constructions.

Section 5 gives an introduction to origamis and explains how they define curves in the moduli space M_g of smooth algebraic curves of genus g. We call them origami curves; they are in fact special examples of Teichmüller curves. In Section 6 we describe the action of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ on them and state some known results. The last two sections present two explicit constructions of dessins d'enfants to a given origami. Section 7 interprets the origami curve itself as a dessin. In Section 8 we associate a dessin to every cusp of an origami curve. We illustrate these constructions by several nice examples.

The subject of dessins d'enfants has been treated from different points of view in several survey articles, as e.g. [33], [39] and [18, Chap. 2] to mention only a few. A collection of articles on dessins d'enfants including many explicit examples is contained in [32]. More on origamis can be found e. g. in [19] and [30] and the references therein. Almost all results in this chapter were known previously, with the exception of the examples in the last sections.

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2 From Riemann surfaces to algebraic curves

One fascinating aspect of the theory of dessins d'enfants is that it touches two different fields of mathematics, namely algebraic geometry and complex geometry. The bridge between these two fields is built on the following observation which was already understood by Riemann himself: Closed Riemann surfaces and regular complex projective curves can be considered to be the same. More precisely, we have an equivalence between the following three categories (see e.g. [27, Thm. 7.2]) and [12, Cor. 6.12]):

- Closed Riemann surfaces with non-constant holomorphic maps;
- Function fields over $\mathbb{C}$ of transcendence degree 1 with $\mathbb{C}$-algebra homomorphisms;
- Regular complex projective curves with dominant algebraic morphisms.

Recall that a *function field* over a field k is a finitely generated extension field of k. We give here only a brief outline of the above equivalences and refer for further readings to literature in complex geometry (e.g. [7, §16], [23, IV,1]) and algebraic geometry (e.g. [12]).

In a first step we describe how to get from the category of closed Riemann surfaces to the category of function fields over $\mathbb{C}$ of transcendence degree 1. Let X be a Riemann surface and $\mathbb{C}(X)$ the field of meromorphic functions from X to $\mathbb{C}$. Then $\mathbb{C}(X)$ is a function field: The fact that $\mathbb{C}(X)$ has transcendence degree 1 essentially follows from the Riemann-Roch theorem. Recall that the theorem determines for a divisor D on X the dimension of the complex vector space $L(D) = H^0(X, \mathcal{O}_D)$ of meromorphic functions f satisfying $\mathrm{div}(f) \geq -D$. It states in particular that if the divisor D is effective, i.e. $D = \sum_i a_i P_i$ with $a_i \geq 0$, then $\dim(L(D)) \leq 1 + \deg D$.
Suppose now that the degree of $\mathbb{C}(X)$ were greater or equal to 2. Then there would exist two algebraically independent meromorphic functions f and g. Let $P_1, \ldots, P_k$ be the poles of f and $Q_1, \ldots, Q_m$ be the poles of g, with degrees $a_1, \ldots, a_k$ and $b_1, \ldots, b_m$ respectively. One picks the divisor $D = \sum a_i P_i + \sum_j b_j Q_j$. By the definition of D we have, for $i+j \leq n$, $f^i g^j \in L(nD)$. Since f and g are algebraically independent, we have that all the $f^i g^j$ are linearly independent. Therefore $\dim(L(nD)) \geq (n^2 + 3n + 2)/2$. On the other hand, one obtains from the Riemann-Roch theorem that $\dim(L(nD)) \leq 1 + \deg(nD) = 1 + n\deg(D)$. These two inequalities give a contradiction for n large enough. Hence the transcendence degree of $\mathbb{C}(X)$ is ≤ 1. Equality follows from the fact that each compact Riemann surface admits a non-constant meromorphic function. The Riemann-Roch theorem precisely guarantees the existence of meromorphic functions. E.g. if we fix a divisor of degree greater or equal to $g+1$, then $\dim(L(D)) \geq 2$, therefore we have a non-constant meromorphic function in $L(D)$. Altogether we have seen that each closed Riemann surface defines a function field of transcendence degree 1. Furthermore a non-constant holomorphic function defines a morphism of $\mathbb{C}$-algebras between the function fields by pulling back the rational functions. We have thus constructed a contravariant functor from the category of closed Riemann surfaces to the category of function fields of transcendence degree 1.

The equivalence between function fields and regular projective complex curves is described e.g. in [12, Chap. I]. In fact the statement holds a bit more generally. One may replace the field $\mathbb{C}$ by any algebraically closed field k. Similarly as before one obtains a function field $k(C)$ of degree 1 starting from an algebraic curve C over k. In this case $k(C)$ is the field of all rational functions from C to $\mathbb{P}^1(k)$, the projective line over k. Two algebraic varieties are birationally equivalent if and only if they have the same function field [12, I Cor. 4.5] and nonsingular curves are birationally equivalent if and only if

they are isomorphic [12, Prop. 6.8]. Hence it remains to show that for each function field K of degree 1 over k one can construct a projective regular curve C whose function field is K. This construction is described in [12, I,§6]. It is based on the following observation: Each point p on an algebraic curve C defines a discrete valuation ring whose quotient field is $K = k(C)$, namely the local ring $\mathcal{O}_p$ of germs of regular functions on C near p. The main idea is to identify the points of the curve with the valuation rings, which they induce, in order to reconstruct the projective curve C from its function field $k(C)$.
Hence, given a function field K, we take the set C_K of discrete valuation rings of K. We want to think of its elements R as points of the algebraic curve that we are going to construct. First, C_K becomes a topological space by taking the finite sets and the whole space to be the closed sets. Next, we can cover C_K by affine regular curves as follows. Suppose that R is a point of C_K, i.e. R is a discrete valuation ring in K. Hence K is the quotient field of R and R is a local ring of dimension 1. Let m_R be its maximal ideal. We want to define an affine curve Y together with an embedding of Y into C_K, such that the image contains the point R. We pick an arbitrary $y \in R\backslash k$ and define B to be the integral closure of $k[y]$ in K. It follows from commutative algebra that B is contained in R, it is a Dedekind domain and a finitely generated k-algebra. Thus B is in particular the affine coordinate ring of an affine regular curve Y. Finally, we want to construct an injective continuous map from Y to C_K. Recall from algebraic geometry that the points in Y correspond to the maximal ideals of B. Let Q be in Y and n_Q the corresponding maximal ideal of B. Then B_{n_Q} is a local ring in K, and indeed a discrete valuation ring. Hence we may map $Q \in Y$ to $B_{n_Q} \in C_K$. This gives a continuous map from Y to C_K. Let m_Q be the unique maximal ideal in B_{n_Q}. Then $n_Q = m_Q \cap B$. Hence the map is injective. Furthermore R is in the image, since $m_R \cap B$ is a maximal ideal in B.
One then shows that C_K with this structure is a regular projective curve.

In order to close the circle between the three categories, one constructs for each regular complex projective curve C a closed Riemann surface X with the same function field: Suppose that C lies in $\mathbb{P}^n(\mathbb{C})$. $\mathbb{P}^n(\mathbb{C})$ becomes a complex manifold using the natural cover by $\mathbb{A}^n(\mathbb{C})$'s for charts to $\mathbb{C}^n$. (Be aware that $\mathbb{A}^n(\mathbb{C})$ and $\mathbb{C}^n$ do not have the same topology. Therefore one speaks of the *Zariski topology* and the *complex topology* of $\mathbb{P}^n(\mathbb{C})$ and later also of the curve C). C is the zero set of finitely many homogeneous polynomials f_1, $\ldots$, f_m. Since C is regular, the Jacobian matrices $(\frac{\delta f_i}{\delta x_j}(p))_{i,j}$ are invertible for all points p on C. The implicit functions theorem together with the fact that the complex dimension of C is 1, provides us locally with a function from $\mathbb{C}$ to C, which is invertible. Its inverse map is a chart for C. C becomes a closed Riemann surface X with these chart maps. Finally, one shows that the function fields are the same by checking that rational functions on C become meromorphic functions on the Riemann surface X and vice versa.

We will use this equivalence between the category of closed Riemann surfaces and the category of regular complex projective curves throughout the whole chapter. Observe in particular that the Riemann sphere corresponds to the projective line $\mathbb{P}^1(\mathbb{C})$ under this identification.

3 Dessins d'enfants

In this section we give a brief introduction to dessins d'enfants. They are a nice way to describe coverings $\beta : X \to \mathbb{P}^1(\mathbb{C})$ from a closed Riemann surface X to the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are ramified at most over the three points 0, 1 and ∞. Such coverings are called *Belyi morphisms*. One reason why they are particularly interesting is the famous Theorem of Belyi. This theorem establishes a connection between complex Riemann surfaces X, which allow a Belyi morphism, and projective algebraic curves C which are defined over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. As described in Section 2 we identify the closed Riemann surface X with the corresponding projective regular curve C defined over $\mathbb{C}$. C is defined over $\overline{\mathbb{Q}}$ if it can be described as the zero set of polynomials whose coefficients lie in $\overline{\mathbb{Q}}$. Observe that in this case the curve C actually is defined over a number field, since a curve can be defined by finitely many polynomials and therefore there exits a finite field extension of $\mathbb{Q}$ which contains all coefficients. Therefore Belyi morphisms provide a tool for studying complex curves over number fields.

Theorem 3.1. *(Theorem of Belyi, [3]) Let X be a projective complex regular curve. Then X is defined over $\overline{\mathbb{Q}}$ if and only if there exists a finite morphism $\beta : X \to \mathbb{P}^1(\mathbb{C})$ from X to the projective line $\mathbb{P}^1(\mathbb{C})$ which is ramified at most over 0, 1 and ∞.*

It follows from the proof of the theorem that if the condition of the theorem holds, we can choose the morphism β such that it is defined over $\overline{\mathbb{Q}}$. Therefore in the following, if we call β a *Belyi morphism*, we will always assume that it is defined over $\overline{\mathbb{Q}}$.

The surprising part of Belyi's result was the only-if direction. Belyi gave an elementary but tricky algorithm for how to calculate the morphism β. For the if part of the proof, Belyi referred to a very general result of A. Weil. Later on, more direct proofs were given by B. Köck in [17] in the language of algebraic geometry and by J. Wolfart in [38] using uniformisation theory. We shall sketch the main idea of the proof in Section 4.2

The theorem makes it particularly desirable to describe Belyi morphisms β as simple as possible. Fortunately, this can be done using "objects so simple that a child learns them while playing" (Grothendieck in [9]). In the following we present several methods on how to describe β and give an idea of the proofs why they are all equivalent and how one can retrieve β from them.

Let (X, β) be a *Belyi pair*, i.e. a closed Riemann surface X together with a Belyi morphism $\beta : X \to \mathbb{P}^1(\mathbb{C})$. We say that two Belyi pairs (X_1, β_1) and (X_2, β_2) are *equivalent*, if there exists an isomorphism $f : X_1 \to X_2$ such that $\beta_2 \circ f = \beta_1$. We consider Belyi pairs up to this equivalence relation.

Proposition 3.2. *A Belyi pair (X, β) is up to equivalence uniquely determined by:*

- *a dessin d'enfants (defined below) up to equivalence;*
- *a bipartite connected ribbon graph up to equivalence;*
- *a monodromy map $\alpha : F_2 \to S_d$, i.e. a transitive action of F_2 on $\{1, \ldots, d\}$, up to conjugation in S_d;*
- *a finite index subgroup of F_2 up to conjugation.*

Here F_2 denotes the free group on two generators. The first part of Proposition 3.2 is often called the *Grothendieck correspondence*. In the following we will sketch the proof of the proposition by explaining how to pass from one description to the next.

From the Belyi pair to a dessin One starts from the observation that a Belyi pair (X, β) naturally defines a bipartite graph G on the surface X: Let I be the closed segment on the real line $\mathbb{R}$ between 0 and 1. Then its preimage $\beta^{-1}(I)$ is a graph on X. Its vertices are the preimages of the two points 0 and 1. It carries a natural bipartite structure: we may colour all preimages of 0 with one colour (e.g. black) and all preimages of 1 with another colour (e.g. white). It is a striking fact which we will see in the rest of this section that the graph embedded into the topological surface carries already enough information. It uniquely determines the Belyi pair (X, β) up to equivalence and thus in particular the complex structure on X.
Furthermore, one observes that $X - G$ decomposes into components each of them containing precisely one preimage of ∞. The holomorphic map β restricted to one of the components is therefore ramified at most in one point and hence at this point is locally of the form $z \mapsto z^n$. Its image is an open cell. Therefore the component itself is an open cell and thus holomorphically equivalent to the open unit disk. Altogether, the graph G decomposes the surface into open cells containing precisely one preimage of ∞.

Example 3.3. In Figure 1 we show the dessin on the elliptic curve $C : y^2 = x(x-1)(x-\lambda_0)$ with $\lambda_0 = 1/2 + (\sqrt{3}/2)i$. The curve C has an automorphism of order 3. The Belyi morphism β is the quotient map with respect to this automorphism.

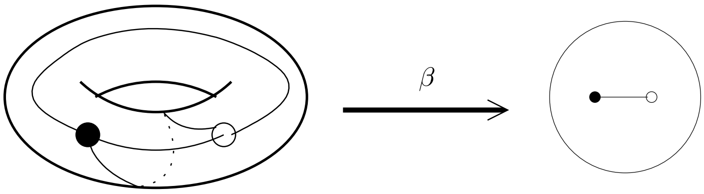


Figure 1: A Belyi morphism and its dessin d'enfants.

Definition 3.4. A *dessin d'enfants* is a bipartite connected graph G which is embedded into an orientable closed topological surface X, such that it fills the surface, i.e. $X \backslash G$ is a union of open cells. Two dessins d'enfants (X_1, G_1) and (X_2, G_2) are called *equivalent* if there exists a homeomorphism $f : X_1 \to X_2$ such that $f(G_1) = G_2$.

Dessins and bipartite ribbon graphs Ribbon graphs are a handy way to describe dessins. Let D be a dessin, i.e. $D = (G, i)$, where G is a connected graph and $i : G \hookrightarrow X$ is a continuous embedding of G into a closed topological surface X. We start from the observation that the abstract graph G does not uniquely determine the dessin. One can e.g. embed the same graph into surfaces of different genera, see Example 3.6. How much information do we have to add to the graph in order to nail down the dessin? It turns out that it suffices to assign to each vertex a cyclic permutation of the edges which are adjacent to the vertex. To simplify notations, we divide each edge into two half edges and number them with 1, ..., $2d$, where d is the number of edges of the graph. For each vertex v of G we take a chart (U, φ) of a small neighbourhood U of v in X to the plane $\mathbb{R}^2$ such that the image of $G \cap U$ is a star with the vertex $\varphi(v)$ as centre. Imagine we circle anticlockwise around the vertex in $\varphi(U)$. Let π_v be the cyclic permutation which denotes the order in which we meet the images of the half edges adjacent to v. Hence π_v is in the symmetric group S_{2d}. For the dessin in Figure 1 we obtain e.g. the cyclic permutations (1 3 5) and (2 4 6), if we label the half edges as in Figure 2.

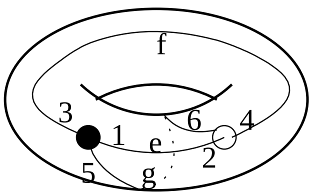


Figure 2: Labelling the half edges of the dessin in Figure 1.

Definition 3.5. A *ribbon graph* (G, O) – often also called *fat graph* – is a connected graph G together with a *ribbon structure* $O = \{\pi_v | v \text{ a vertex of } G\}$, which assigns to each vertex v of G a cyclic permutation π_v of the half edges adjacent to v. Two ribbon graphs (G_1, O_1) and (G_2, O_2) are called *equivalent* if there exists an isomorphism $h : G_1 \to G_2$ of graphs such that the pull back of O_2 is equal to O_1.

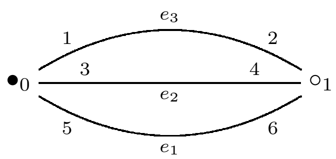
Let π be the product of all the π_v's and τ the transposition which maps each half edge to the other half edge that belongs to the same edge. Then the tuple (π, τ) determines the ribbon graph.

Recall that by the definition of dessins the graph G fills the surface X, i.e. $X \backslash G$ consists of disjoint open cells $C_1, \ldots, C_s$. Observe that we obtain the edges of the cycle bounding a cell C clockwise successively by taking the edges on which the half edges e, $\tau\pi(e)$, $(\tau\pi)^2(e)$, ... lie. Here e is the half edge at the beginning of an edge in the cycle, where the cycle carries the natural anti-clockwise orientation. For example, for the dessin in Figure 1 we obtain one cell which is bounded by the cycle $(e\ f\ g\ e\ f\ g)$.

One gets the dessin back from the ribbon graph doing the reverse procedure: Each cycle $(e, \tau\pi(e), \ldots, (\tau\pi)^k(e))$ defines a cycle in the graph which is the union of the corresponding edges. One glues a cell to each such cycle . Then each edge is on the boundary of precisely two cells (which may coincide) and one obtains a closed surface X in which G is embedded.

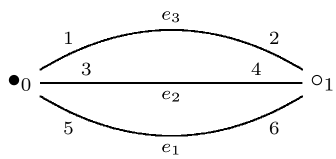
Example 3.6. In Figure 3 we show two ribbon graphs (G_1, O_1) and (G_2, O_2).

G_1:



with ribbon structure
$O_1 = \{\pi_1 = (5\ 3\ 1),\ \pi_2 = (2\ 4\ 6)\}$.

G_2:



with ribbon structure
$O_2 = \{\pi_1 = (5\ 3\ 1),\ \pi_2 = (2\ 6\ 4)\}$.

Figure 3: Two ribbon graphs.

Observe that the two ribbon graphs have the same underlying graph, but the ribbon structures are different and they define different surfaces. The second one is the ribbon graph from Figure 2.

For both ribbon graphs we have $\tau = (1\ 2)(3\ 4)(5\ 6)$. Hence for the first graph we obtain $\tau\pi = (1\ 6)(2\ 3)(4\ 5)$ and for the second graph we have $\tau\pi = (5\ 4\ 1\ 6\ 3\ 2)$. Thus, in the first case we obtain three cells: The first one is bounded by e_3 and e_1, the second one is bounded by e_3 and e_2 and the third one is bounded by e_2 and e_1. Gluing the disks along their edges gives a genus 0 surface. This can be checked with a short Euler characteristic calculation. As we already saw above, we obtain in the second case one cell bounded clockwise by the edges e_1, e_2, e_3, e_1, e_2 and e_3, and a surface of genus 1.

Hence we may equivalently talk about dessins or about bipartite ribbon graphs. One can check that the respective equivalence relations match each other.

Remark 3.7. The constructions above define a bijection between the set of equivalence classes of dessins and the set of equivalence classes of bipartite connected ribbon graphs. Furthermore we described a natural way to assign to each Belyi pair an equivalence class of dessins.

In order to see how we can retrieve the Belyi pair from a given dessin or a given ribbon graph, it is convenient to introduce monodromy maps.

Monodromy maps and subgroups of F_2 Recall that for an unramified degree d covering $p : X^* \to Y^*$ of surfaces we obtain the *monodromy homomorphism* $\alpha : \pi_1(Y^*) \to S_d$ to the symmetric group S_d on d letters as follows: Fix a point $y \in Y^*$. Call its d preimages $x_1, \ldots, x_d$. For $[c] \in \pi_1(Y^*, y)$ map $i \in \{1, \ldots, d\}$ to j, if x_j is the end point of the lift of c to X, which starts in x_i. The resulting map α is independent of the chosen point y and of the choice of the labelling of its preimage up to composition with a conjugation in S_d.

Let us now consider the natural embedding $\pi_1(X^*) \hookrightarrow \pi_1(Y^*)$ induced by p and let U be its image. U depends on the chosen base points of the fundamental groups only up to conjugation. Hence we may assume that the base point of $\pi_1(X^*)$ is the preimage of the base point of $\pi_1(Y^*)$ labelled by 1. Then the image of $\pi_1(X^*) \hookrightarrow \pi_1(Y^*) \xrightarrow{\alpha} S_d$ is the stabiliser $\mathrm{Stab}_{S_d}(1)$ of 1 in S_d and U is its full preimage in $\pi_1(Y^*)$. Hence one obtains U directly from α, namely $U = \alpha^{-1}(\mathrm{Stab}_{S_d}(1))$. Conversely given U one obtains α as follows: $\pi_1(Y^*)$ acts on the d cosets Ug_i of U in $\pi_1(Y^*)$ by multiplication from the right; α is the induced action on the indices.

Starting now from a Belyi pair (X,β), we obtain an unramified cover by removing the three ramification points 0, 1 and ∞ from $\mathbb{P}^1(\mathbb{C})$ and all their preimages from X. We denote the resulting punctured surfaces by $\dddot{\mathbb{P}}$ and X^*, respectively. We fix an isomorphism between $\pi_1(\dddot{\mathbb{P}})$ and F_2, the free group in two generators. Then $p : X^* \to \dddot{\mathbb{P}}$ is an unramified covering and defines a monodromy map from $F_2 \cong \pi_1(\dddot{\mathbb{P}})$ to S_d (where d is the degree of p) or equivalently a finite index subgroup U of $F_2 \cong \pi_1(\dddot{\mathbb{P}})$.

Finally we describe how to retrieve the Belyi pair from the subgroup U. The main ingredient that we use is the universal covering theorem. Let us choose a universal covering $u : \tilde{H} \to \dddot{\mathbb{P}}$. By the theorem we may identify $F_2 \cong \pi_1(\dddot{\mathbb{P}})$ with the group of deck transformations $\text{Deck}(\tilde{H}/\dddot{\mathbb{P}})$. By the same theorem each finite index subgroup of $\pi_1(\dddot{\mathbb{P}})$ defines an unramified covering β from some surface X^* to $\dddot{\mathbb{P}}$ such that it induces an embedding $\text{Deck}(\tilde{H}/X^*)$ $\hookrightarrow \text{Deck}(\tilde{H}/\dddot{\mathbb{P}})$ whose image is the subgroup U. There is a unique complex structure on X^* which makes β holomorphic, namely the lift of the complex structure on $\dddot{\mathbb{P}}$ via β. It follows from the classical theory of Riemann surfaces that there is a unique closed Riemann surface X which is the closure of X^*. It is obtained by filling in one point for each puncture. Furthermore β can be extended in a unique way to $\beta : X \to \mathbb{P}^1(\mathbb{C})$.
One can check that all this is independent of the choices that we did in between up to the equivalence relations, that the equivalence relations fit together and that the constructions are inverse to each other.

Remark 3.8. The above constructions define bijections between the set of equivalence classes of Belyi pairs, the set of conjugacy classes of group homomorphisms $F_2 \to S_d$ which are transitive actions, and the set of conjugacy classes of finite index subgroups of F_2.

As a last step, we have to show how we can relate dessins and ribbon graphs to monodromy maps or finite index subgroups of F_2.

From a dessin D to a Belyi pair (X, β) Let D be a dessin and $(G, O = \{\pi_1, \ldots, \pi_s\})$ the corresponding ribbon graph from Remark 3.7. How can we retrieve the monodromy of β from these data? Recall that G is bipartite and the vertices are coloured: the preimages of 0 are black and those of 1 are white. We may also colour the half edges used in the construction of Remark 3.7 with the colour of the vertex which lies on them. Observe that π acts on the set E_{black} of black half edges and the set E_{white} of white half edges separately. Thus we can decompose $\pi = \pi_{\text{black}} \circ \pi_{\text{white}}$ with $\pi_{\text{black}} \in \text{Perm}(E_{\text{black}})$ and $\pi_{\text{white}} \in \text{Perm}(E_{\text{white}})$.

Let us now choose a base point $y \in \dddot{\mathbb{P}}$ on the segment between 0 and 1 close to 0. Hence all its preimages x_i lie on black half edges. Furthermore we pick two curves c_1 and c_2 as generators of $\pi_1(\dddot{\mathbb{P}}) \cong F_2$, where c_1 is a simple closed circle around 0 and c_2 is a simple closed circle around 1; both starting in y and both anti-clockwise. By the definition of π (see Remark 3.7), the monodromy $\alpha(c_1)$ is the permutation π_{black} and the monodromy $\alpha(c_2)$ is the permutation $\tau\pi_{\text{white}}\tau$. Here we identify the point x_i with the black half edge on which it lies.

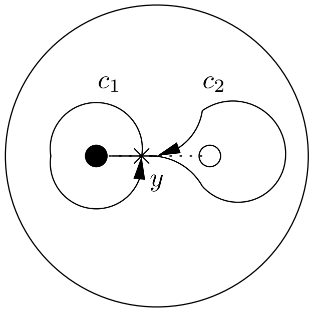


Figure 4: Generators of the fundamental group of $\dddot{\mathbb{P}}$.

Hence, we may assign to a dessin the monodromy map

$$F_2 \to S_n,\ x \mapsto \pi_{\text{black}},\ y \mapsto \tau\pi_{\text{white}}\tau.$$

Again one can check that this construction is inverse to the construction given in Remark 3.7 and the equivalence relations fit together.
Furthermore it follows from the above construction that given the monodromy map $\alpha : F_2 \to S_d$ of a Belyi pair (X, β), one obtains the corresponding bipartite ribbon graph (G, O) directly as follows: Label the black half edges with 1, ..., d and the white half edges with $d+1$, ..., $2d$. Then (G, O) is described by the two permutations

$$\begin{array}{ll} \tau : i \mapsto d+i \quad \text{and} \quad \pi = \pi_{\text{black}} \circ \pi_{\text{white}} \\ \text{with} \quad \pi_{\text{black}} : i \mapsto \alpha(x)(i) \quad \text{and} \quad \pi_{\text{white}} : d+i \mapsto d+\alpha(y)(i) \end{array}$$

Remark 3.9. The above construction defines a bijection between the set of equivalence classes of dessins and the set of equivalence classes of Belyi pairs. This map is the inverse map to the one described before Remark 3.7.

With Remark 3.9 we have finished the outline of the proof of Proposition 3.2.

It follows in particular that we can describe a Belyi pair (X, β) or equivalently the corresponding dessin D by a pair of permutations (σ_1, σ_2), namely $\sigma_1 = \alpha(c_1)$ and $\sigma_2 = \alpha(c_2)$, where $\alpha : F_2 \to S_d$ is the monodromy map. We will say the dessin *has monodromy* (σ_1, σ_2). This description is unique up to simultaneous conjugation with an element in S_d. Furthermore the group generated by σ_1 and σ_2 acts transitively on $\{1, \ldots, d\}$ and each pair of permutations with this property defines a Belyi pair.

The genus of a dessin Suppose that a dessin (X, β) of degree d has monodromy (σ_1, σ_2). The dessin naturally defines a two-dimensional complex. By the construction in Remark 3.9 we have:

- The black vertices are in one-to-one correspondence with the cycles in σ_1. Denote their number by s_1.
- The white vertices are in one-to-one correspondence with the cycles in σ_2. Denote their number by s_2.
- The faces of the complex are in one-to-one correspondence with the cycles in $\sigma_1 \circ \sigma_2$. Denote their number by f.

Hence, we can calculate the genus as follows:

$$g = \frac{2 - \chi}{2} \quad \text{with } \chi = s_1 + s_2 - d + f$$

Definition 3.10. We call g as above the *genus of the dessin D*.

4 The Galois action on dessins d'enfants

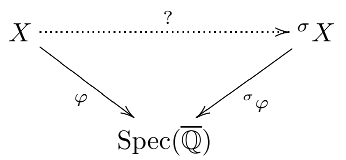
One of the original motivations to study dessins d'enfants was the hope to get new insights into the structure of the "absolute" Galois group $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ of the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$. This hope came from the fact that, as a consequence of the Grothendieck correspondence between dessins d'enfants and Belyi pairs explained in the previous section, $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of dessins d'enfants. We shall see that this action is faithful, so in principle, all information about $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ is somehow contained in the dessins d'enfants. Unfortunately, except for very special cases, it is so far not known how to describe the action of a Galois automorphism on a dessin in terms of the combinatorial data that determine the dessin. Nevertheless this approach led to many beautiful results concerning e. g. the faithfulness of the action on special classes of dessins d'enfants. Perhaps the most conceptual outcome of the investigation of the Galois action on dessins is the embedding of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ into the Grothendieck-Teichmüller group $\widehat{GT}$.

4.1 The action on dessins

In this section we explain the action of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins d'enfants by saying how it acts on Belyi pairs.

By the theorem of Belyi, every Riemann surface X that admits a Belyi morphism β is defined over a number field and thus in particular over $\overline{\mathbb{Q}}$. As explained in the first paragraph of Section 3, this means that, as an algebraic curve, X can be described as the zero set of polynomials with coefficients in $\overline{\mathbb{Q}}$. The fancier language of modern algebraic geometry expresses this property by saying that X admits a morphism of finite type $\varphi : X \to \mathrm{Spec}(\overline{\mathbb{Q}})$ to the one point scheme $\mathrm{Spec}(\overline{\mathbb{Q}})$. Such a φ is called a *structure morphism* of X.

Every Galois automorphism $\sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ induces an automorphism σ^* of $\mathrm{Spec}(\overline{\mathbb{Q}})$. Composing it with the structure morphism φ gives a new structure morphism ${}^\sigma\varphi := (\sigma^{-1})^* \circ \varphi : X \to \mathrm{Spec}(\overline{\mathbb{Q}})$. We call ${}^\sigma X$ the scheme X endowed with the structure morphism ${}^\sigma\varphi$. In more elementary language, ${}^\sigma X$ is obtained from X by applying σ to the polynomials defining X. In general, X and ${}^\sigma X$ are not isomorphic as $\overline{\mathbb{Q}}$-schemes or as Riemann surfaces, i. e. there is in general no isomorphism making the following diagram commutative:



Example 4.1. Let E be an elliptic curve over $\mathbb{C}$, in other words a Riemann surface of genus 1. E can be embedded into the projective plane as the zero set of a Weierstrass equation $y^2 = x^3 + ax + b$ (or rather its homogenisation). It is defined over a number field if and only if $a, b \in \overline{\mathbb{Q}}$. A Belyi map for E is obtained e. g. by applying Belyi's algorithm to the four critical values of the projection $\beta_0 : E \to \mathbb{P}^1, (x, y) \mapsto x$. The elliptic curve ${}^\sigma E$ is the zero set of $y^2 = x^3 + \sigma(a)x + \sigma(b)$. It is well known that Weierstrass equations define isomorphic Riemann surfaces if and only if their j-invariants agree. Thus ${}^\sigma E$ is isomorphic to E if and only if $j(E) = j(a, b) = \frac{a^3}{4a^3+27b^2}$ is fixed by σ.

To describe the Belyi map ${}^\sigma\beta : {}^\sigma X \to \mathbb{P}^1$ that gives the image of the Belyi pair (X, β) under σ, we first look at the characterisation of X as the zero set of polynomials $f_1, \ldots, f_k$ in variables $x_1, \ldots, x_n$: then β is, at least locally, also given as a polynomial in $x_1, \ldots, x_n$ with coefficients in $\overline{\mathbb{Q}}$, and ${}^\sigma\beta$ is obtained by applying σ to the coefficients of this polynomial.

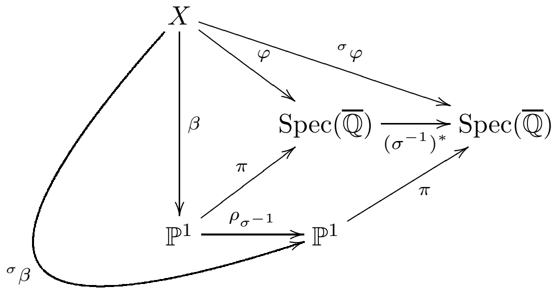
The description of ${}^\sigma\beta$ in terms of schemes is as follows: Let $\pi : \mathbb{P}^1 \to \mathrm{Spec}(\overline{\mathbb{Q}})$ denote the (fixed) structure morphism of the projective line $\mathbb{P}^1$; π is related to the structure morphism φ of X by the equation

$$\varphi = \pi \circ \beta.$$

Since $\mathbb{P}^1$ clearly is defined over $\mathbb{Q}$, for every $\sigma \in \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ the induced automorphism σ^* of $\mathrm{Spec}(\overline{\mathbb{Q}})$ lifts to an automorphism ρ_σ of $\mathbb{P}^1$. Then we have

$$^{\sigma}\beta = \rho_{\sigma^{-1}} \circ \beta.$$

This is summarised in the following commutative diagram:



4.2 Fields of definition and moduli fields

Before studying properties of the Galois action on dessins d'enfants, we shortly digress for the following question: Given a Riemann surface X, what is the smallest field over which X can be defined?

In general we say that a variety (or scheme) X/K over a field K can be defined over a subfield $k \subset K$ if there is a scheme X_0/k over k such that X is obtained from X_0 by extension of scalars: $X = X_0 \times_k K$. In this case, we call k a *field of definition* for X.

For example, a Riemann surface can always be defined over a field K which is finitely generated over $\mathbb{Q}$. Namely, considered as an algebraic curve, X is the zero set of finitely many polynomials, and we may take K to be the extension field of $\mathbb{Q}$ which is generated by the finitely many coefficients of these polynomials.

It is not true in general that there is a unique smallest subfield of K over which a given variety X/K can be defined. Therefore we cannot speak of "the field of definition" of X. But there is another subfield of K associated with X, called the *moduli field*, which is uniquely determined by X and turns out to be closely related to fields of definition:

Definition 4.2. Let $\mathrm{Aut}\,(\mathbb{C})$ be the group of all field automorphisms of $\mathbb{C}$. For a Riemann surface X denote by $U(X)$ the subgroup of all $\sigma \in \mathrm{Aut}\,(\mathbb{C})$ for which $^{\sigma}X$ is isomorphic to X. The fixed field $M(X) \subset \mathbb{C}$ of $U(X)$ is called the *moduli field* of X.

There are two rather straightforward observations about moduli fields:

Remark 4.3. Let X be a Riemann surface of genus g.

a) If $k \subset \mathbb{C}$ is a field of definition for X, then $M(X) \subseteq k$.

b) Suppose that X can be defined over $\overline{\mathbb{Q}}$ and let $[X]$ be the corresponding point in the moduli space $M_{g,\overline{\mathbb{Q}}}$ of regular projective curves defined over $\overline{\mathbb{Q}}$ (considered as a variety over $\overline{\mathbb{Q}}$). Recall that $M_{g,\overline{\mathbb{Q}}}$ is obtained from a variety $M_{g,\mathbb{Q}}$ which is defined over $\mathbb{Q}$ by extension of scalars. Then the orbit of $[X]$ under the action of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ on $M_{g,\overline{\mathbb{Q}}}$ gives a closed point $[X]_{\mathbb{Q}}$ in the variety $M_{g,\mathbb{Q}}$ whose residue field is isomorphic to $M(X)$.

Proof. a) If X is defined over k and if $\sigma \in \mathrm{Aut}\,(\mathbb{C})$ fixes k, then id_X is an isomorphism between ${}^{\sigma}X$ and X. Thus $\{\sigma \in \mathrm{Aut}\,(\mathbb{C}) : \sigma|_k = \mathrm{id}_k\} \subseteq U(X)$, hence $M(X) \subseteq k$.

b) (Sketch) Let $V \subset M_{g,\mathbb{Q}}$ be an affine neighbourhood of $[X]_{\mathbb{Q}}$ and let A be its affine coordinate ring. Then $[X]_{\mathbb{Q}}$ corresponds to a maximal ideal m in A, and $k = A/m$ is its residue field. In $A \otimes k$, m decomposes into maximal ideals $m_1, \ldots, m_d$ which are in bijection with the points in the Galois orbit of $[X]$. Thus the fixed field of the stabiliser of, say, m_1 in $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ is $A \otimes k/m_1 = k$. □

The relation between the field of moduli and fields of definition of a Riemann surface is much closer than indicated in part *a)* of the remark:

Proposition 4.4. *Any Riemann surface can be defined over a finite extension of its moduli field.*

This result is proved in [38]. for a proof in the language of algebraic geometry that holds for curves over any field, see [10]. Further results on moduli fields, in particular on the moduli field of a Belyi pair, can be found in [17]. There it is shown, among other nice properties, that for “most” curves, the moduli field is also a field of definition. The precise statement is that $X/\mathrm{Aut}\,(X)$ can be defined over $M(X)$ for any curve X of genus $g \geq 2$. This implies in particular that X can be defined over $M(X)$ if X admits no nontrivial automorphism. In this case, which holds for a generically chosen Riemann surface of genus ≥ 3, the moduli field is the unique smallest field of definition.

Proposition 4.4 plays a key role in the proof of the “if”-direction of Belyi's theorem. As explained in Section 3 one has to show that a Riemann surface can be defined over $\overline{\mathbb{Q}}$ if it admits a finite covering $\beta : X \to \mathbb{P}^1(\mathbb{C})$ which is ramified at most over 0,1 and ∞. Observe that, up to isomorphism, there are only finitely many coverings $Y \to \mathbb{P}^1(\mathbb{C})$ from some Riemann surface Y of a fixed degree that are unramified outside 0, 1, ∞ (see [17, Prop. 3.1] for an elementary proof of this fact). It follows that the moduli field of β and hence in particular that of X is a finite extension of $\mathbb{Q}$. From Proposition 4.4 we then conclude that X can be defined over a number field.

4.3 Faithfulness

We have established an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on Belyi pairs by defining $\sigma\cdot(X,\beta)$ to be the Belyi pair $({}^{\sigma}X, {}^{\sigma}\beta)$ for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Example 4.1 shows that this action is faithful, since for every Galois automorphism $\sigma \neq \mathrm{id}$ we can find $a, b \in \overline{\mathbb{Q}}$ such that $\sigma(j(a,b)) \neq j(a,b)$ and thus ${}^{\sigma}E$ is not isomorphic to E, where E is the elliptic curve with Weierstrass equation $y^2 = x^3 + ax + b$. Translating the Galois action to dessins d'enfants via the Grothendieck correspondence we deduce:

Proposition 4.5. *The action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on dessins d'enfants is faithful.*

Several nice examples for this Galois action on dessins are worked out in the manuscript [39] by J. Wolfart; he attributes the following one to F. Berg: Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be an element that maps the primitive 20th root of unity $\zeta = e^{\pi i/10}$ to ζ^3. Then σ maps the left hand dessin in Figure 5 to the right hand one:

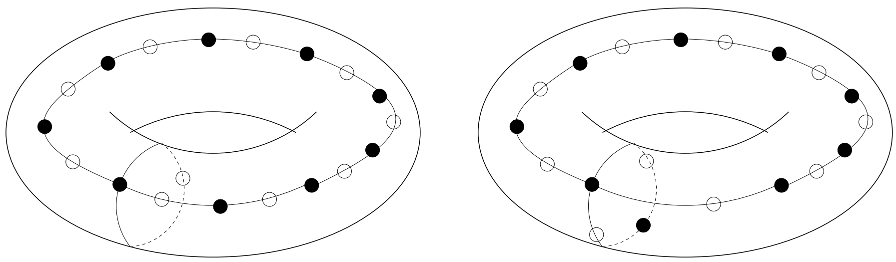


Figure 5: Two Galois equivalent dessins which are not isomorphic.

The dessin on the left lies on the elliptic curve $y^2 = (x+1)(x-1)(x-\cos\frac{\pi}{10})$, whereas the right hand dessin lies on $y^2 = (x+1)(x-1)(x-\cos\frac{3\pi}{10})$. The Belyi map is in both cases the composition of the projection $\beta_0(x,y) = x$ with the square $T_5^2(z)$ of the fifth Chebyshev polynomial.

In the proof of Proposition 4.5 we have shown more precisely that the action is faithful on dessins of genus 1. The same faithfulness result holds for the Galois action on dessins of any fixed genus $g \geq 1$. This can be seen for example using hyperelliptic curves: for mutually distinct numbers $a_1, \ldots, a_{2g}$ in $\mathbb{P}^1(\mathbb{C})$, the (affine) equation $y^2 = (x-a_1)\cdot\ldots\cdot(x-a_{2g})$ defines a nonsingular curve X of genus g. The automorphism $(x,y) \mapsto (x,-y)$ is called the *hyperelliptic involution* on X; the quotient map is the projection $(x,y) \mapsto x$. It is a covering $X \to \mathbb{P}^1$ of degree 2, ramified exactly over $a_1, \ldots, a_{2g}$. Two hyperelliptic curves with equations $y^2 = (x-a_1)\cdot\ldots\cdot(x-a_{2g})$ and $y^2 = (x-a'_1)\cdot\ldots\cdot(x-a'_{2g})$ are isomorphic if and only if there is a Möbius transformation that maps the set $\{a_1, \ldots, a_{2g}\}$ to the set $\{a'_1, \ldots, a'_{2g}\}$.

A hyperelliptic curve is defined over $\overline{\mathbb{Q}}$ if all the a_i are algebraic numbers. In this case, for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the curve ${}^{\sigma}X$ is given by the equation $y^2 = (x - \sigma(a_1)) \cdot \ldots \cdot (x - \sigma(a_{2g}))$. It is then easy, if $\sigma \neq \mathrm{id}$, to choose $a_1, \ldots, a_{2g}$ in such a way that there is no Möbius transformation that maps the a_i to the $\sigma(a_j)$. An explicit way to find suitable a_i's is explained in [1].

With a bit more work, it is also possible to show that the Galois action on genus 0 dessins is faithful. Since all Riemann surfaces of genus zero are isomorphic to the projective line, it is not possible to find, as in the case of higher genus, a Riemann surface X such that ${}^{\sigma}X \not\cong X$. Rather one has to provide, for a given $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sigma \neq \mathrm{id}$, a rational function $\beta(z)$ such that ${}^{\sigma}\beta$ is not equivalent to β, i. e. not of the form $\beta \circ \rho$ for some Möbius transformation ρ. L. Schneps [33] showed that one can always find a suitable polynomial. The dessin d'enfants obtained from a polynomial is a planar graph whose complement in the plane is connected, hence the dessin is a tree. Schneps' result thus is

Proposition 4.6. *The Galois action on trees is faithful.*

Using a similar argument as for the hyperelliptic curves, F. Armknecht [1] gave an alternative proof of this result. L. Zapponi [41] improved the result to trees of diameter at most 4.

4.4 Galois invariants

To understand the Galois action on dessins d'enfants one can look for *Galois invariants*, i. e. properties of a dessin that remain unchanged under all Galois automorphisms. The idea, or rather the dream, is to find a complete list of invariants; then two dessins d'enfants would be Galois conjugate if and only if they agree on all the data from the list. Unfortunately such a list is not known up to now.

But several Galois invariants are known and can at least help distinguishing different orbits. The most fundamental invariants are derived from the correspondence of dessins d'enfants with Belyi pairs: If (X, β) is a Belyi pair and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a bijection between the ramification points of β on X and the ramification points of ${}^{\sigma}\beta$ on ${}^{\sigma}X$; moreover this bijection preserves the ramification orders. It therefore follows from the Riemann-Hurwitz formula that X and ${}^{\sigma}X$ have the same genus. Translating these remarks to the corresponding dessin D and observing that the ramification points of β over 0, 1 and ∞ correspond to the black vertices, the white vertices and the cells of D, respectively, we obtain:

Proposition 4.7. *The genus and the valency lists of a dessin d'enfants are Galois invariants.*

Recall that the genus of a dessin d'enfants $D = (G, i)$ is the genus of the surface onto which the dessin is drawn, see Definition 3.10. D has 3 valency lists: one for the black vertices, one for the white vertices, and one for the cells. These lists contain an entry for each vertex (resp. cell), and the entry is the valency of this vertex (resp. cell).

A famous example that these invariants do not suffice to separate Galois orbits is "Leila's flower", see [33], [40]. A few more subtle Galois invariants are known: the automorphism group of D, properties of the action of $\mathrm{Aut}(D)$ on vertices or edges (like "regularity"); Zapponi [40] introduced the *spin structure* of a dessin and showed that it is a Galois invariant and in particular that it separates the two non-equivalent versions of Leila's flower.

4.5 The action on $\widehat{\mathbf{F}}_2$

Recall that $\dddot{\mathbb{P}}$ is the projective line $\mathbb{P}^1(\mathbb{C})$ with the three points 0, 1 and ∞ removed. We saw in Proposition 3.2 that dessins d'enfants correspond bijectively to finite unramified coverings of $\dddot{\mathbb{P}}$ and thus to (conjugacy classes of) finite index subgroups of $F_2 = \pi_1(\dddot{\mathbb{P}})$. In this section we explain how the Galois action on dessins induces an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{F}_2$, the profinite completion of F_2, and thus an embedding of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ into $\mathrm{Aut}(\widehat{F}_2)$.

We restrict our attention to dessins for which the associated covering of $\dddot{\mathbb{P}}$ is Galois. The corresponding subgroup of F_2 is then normal, and we have no ambiguity "up to conjugation". Moreover the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on finite index subgroups of F_2 can also be interpreted as an action on the set of finite quotient groups F_2/N, where N runs through the normal subgroups of F_2. These finite quotient groups form a projective system of finite groups, with projections $F_2/N' \to F_2/N$ coming from inclusions $N' \subset N$. The inverse limit of this projective system is $\widehat{F}_2$, the profinite completion of F_2.

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{F}_2$ can be described quite explicitly. We sketch the approach by Y. Ihara, P. Lochak and M. Emsalem, see [16] and [6]; the details are worked out in [28]. Let x and y be generators of $F_2 = \pi_1(\dddot{\mathbb{P}})$ that correspond to loops around 0 and 1, resp. Their residue classes in the finite quotients F_2/N of F_2 define elements $(x \bmod N)_N$ and $(y \bmod N)_N$ of $\widehat{F}_2$, that we still denote by x and y. They are called *topological generators* since the subgroup they generate is dense in the profinite (or Krull) topology of $\widehat{F}_2$. Note that the group theoretical and the topological data are related as follows: if N is a finite index normal subgroup of F_2 which corresponds to the normal covering $p : Y \to \mathbb{P}^1$, then the order of $x \bmod N$ in F_2/N is the ramification index of p above 0, i. e. the l.c.m. of the ramification indices of the points in the fibre $p^{-1}(0)$; we denote this number by $e(N)$. With this notation at hand we can state the announced result:

Proposition 4.8. *For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and x and y the topological generators of $\widehat{F}_2$ described above we have:*

$$\sigma \cdot x = x^{\chi(\sigma)}$$
$$\sigma \cdot y = f_\sigma^{-1}\, y^{\chi(\sigma)} f_\sigma.$$

The element $f_\sigma \in \widehat{F}_2$ in the second formula will be explained later; $x^{\chi(\sigma)}$ is the element of $\widehat{F}_2$ defined by $x_N^{\chi(\sigma)} = (x \bmod N)^{\chi_{e(N)}(\sigma)}$, N running through the finite index normal subgroups of F_2, where for a positive integer e, $\chi_e : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to (\mathbb{Z}/e\mathbb{Z})^\times$ is the *cyclotomic character* , i. e. $\chi_e(\sigma) = n$ if $\sigma(\zeta_e) = \zeta_e^n$ for a primitive e-th root of unity ζ_e. Note that $\chi(\sigma) = (\chi_{e(N)}(\sigma))_N$ can be considered as an element of $\widehat{\mathbb{Z}}^\times$.

The starting point for the proof of Proposition 4.8 is the equivalence of the following categories:

- finite normal coverings of $\mathbb{P}^1(\mathbb{C})$ unramified outside 0, 1 and ∞;
- finite normal coverings of $\mathbb{P}^1(\overline{\mathbb{Q}})$ unramified outside 0, 1 and ∞;
- finite normal holomorphic unramified coverings of $\ddot{\mathbb{P}}$;
- finite Galois extensions of $\overline{\mathbb{Q}}(T)$ unramified outside T, $T-1$ and $\frac{1}{T}$.

The first equivalence is a consequence of Belyi's theorem, the others are standard results on Riemann surfaces and algebraic curves (cf. Section 2 and the paragraph before Remark 3.8).

A crucial technical tool in the proof is the notion of a *tangential base point* of a Riemann surface X. It consists of a point together with a direction in this point. For the fundamental group with respect to a tangential base point, only closed paths are considered that begin and end in the prescribed direction. For example, we denote by $\vec{01}$ the tangential base point of $\mathbb{P}^1(\mathbb{C})$ which is located at 0 and whose direction is the positive real axis. We take the element $x \in \pi_1(\ddot{\mathbb{P}}, \vec{01})$ to be a small loop around 0, that begins and ends in 0 in the direction towards 1:

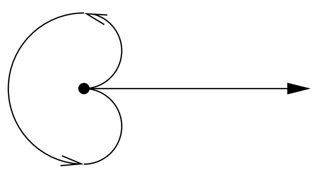


Figure 6: *The generator x in the tangential base point $\vec{u} = \vec{01}$.*

Another important tool is the field $\mathcal{P}_{\vec{u}}$ of convergent Puiseux series in a tangential base point $\vec{u}$. For $\vec{u} = \vec{01}$, these are series of the form $\sum_{n=k}^{\infty} a_n T^{\frac{n}{e}}$ for some integer k, some positive integer e and complex coefficients a_n, such that the series converges in some punctured neighbourhood of 0. These Puiseux series then define meromorphic functions in a neighbourhood of 0 that is slit along the real line from 0 to 1.

Given a covering $p : X \to \mathbb{P}^1$ that is possibly ramified over 0, the function field $\mathbb{C}(X)$ of X can be embedded into $\mathcal{P}_{\vec{u}}$ as follows: fix a point $v \in X$ above 0 and choose a local coordinate z in v such that p is given by $z \mapsto z^e$ in a neighbourhood of v. For a meromorphic function $f \in \mathbb{C}(X)$, let $f(z) = \sum a_n z^n$ be the Laurent expansion in v, and take $\sum a_n T^{\frac{n}{e}}$ to be its image in $\mathcal{P}_{\vec{u}}$, where e is the ramification index of p in v. If ζ is an e-th root of unity, we get another embedding by sending $\sum a_n z^n$ to $\sum a_n \zeta^n T^{\frac{n}{e}}$. These embeddings correspond bijectively to the tangential base points in v that are mapped to $\vec{u}$ by z.

Lifting the (small!) loop x via p to X with starting point v we again get a closed path, but it may end in v in a different direction from the starting one. In this way we get an action of x on the tangential base points over $\vec{u}$ and hence on the embeddings of $\mathbb{C}(X)$ into $\mathcal{P}_{\vec{u}}$.

Now let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. To describe $\sigma \cdot x$, we have to specify, for each Belyi pair (X, β), the embedding of $\mathbb{C}(X)$ into $\mathcal{P}_{\vec{u}}$ induced by $\sigma \cdot x$. By the above equivalences of categories, it suffices to take the function field $\overline{\mathbb{Q}}(X)$ and Puiseux series with coefficients in $\overline{\mathbb{Q}}$. Then σ acts on the coefficients of the Puiseux series, and an embedding that maps $f \in \overline{\mathbb{Q}}(X)$ to the series $\sum a_n T^{\frac{n}{e}}$ is transformed by $\sigma \cdot x$ into the embedding

$$f \mapsto \sum a_n T^{\frac{n}{e}} \mapsto \sum \sigma^{-1}(a_n) T^{\frac{n}{e}} \mapsto \sum \sigma^{-1}(a_n) \zeta^n T^{\frac{n}{e}} \mapsto \sum a_n \sigma(\zeta)^n T^{\frac{n}{e}}$$

where ζ is the root of unity corresponding to x. Since $\sigma(\zeta) = \zeta^{\chi_e(\sigma)}$, this shows the first formula of Proposition 4.8.

The second formula is proved similarly using a small loop y around 1. The difference is that here we need the path t from 0 to 1 along the real line to make y into a closed path around $\vec{u}$. But t can also be interpreted as acting on embeddings of $\overline{\mathbb{Q}}(X)$ into the field of Puiseux series. Working with fundamental groupoids instead of the fundamental group, we can calculate $\sigma \cdot t$ in a similar way as $\sigma \cdot x$. The element f_σ in the formula then turns out to be $t^{-1}\, \sigma \cdot t$.

4.6 The action on the algebraic fundamental group

At first glance the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{F}_2$ described in the previous section might look very special. But in fact it is an explicit example of the very general and conceptual construction of Galois actions on algebraic fundamental groups. We shall briefly explain this relation in this section.

The *algebraic fundamental group* $\pi_1^{\mathrm{alg}}(X)$ of a scheme X is defined as the projective limit of the Galois (or deck transformation) groups of the finite normal étale coverings of X. In general, a morphism of schemes is called *étale* if it is "smooth of relative dimension 0". If X is an algebraic curve over the complex numbers, this property is equivalent to the usual notion of an unramified covering. So in this case the projective system defining $\pi_1^{\mathrm{alg}}(X)$ is the system of the finite quotient groups of the topological fundamental group $\pi_1(X)$. It follows that the algebraic fundamental group of a Riemann surface is the profinite completion of its topological fundamental group, cf. [26, p. 164].

In the proof of Proposition 4.8 we used the equivalence of four categories, namely the normal coverings of $\mathbb{P}^1$ as a variety over $\mathbb{C}$ resp. $\overline{\mathbb{Q}}$ that are unramified outside 0, 1 and ∞, the unramified coverings of $\dddot{\mathbb{P}}$, and the suitably ramified Galois extensions of the function field $\overline{\mathbb{Q}}(T)$ of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$. In all four categories, to every object there is associated a finite group (the Galois group of the covering resp. the field extension). The morphisms in the respective category make these groups into a projective system. The inverse limits of these systems are resp. the algebraic fundamental groups of $\dddot{\mathbb{P}}_{\mathbb{C}}$ and $\dddot{\mathbb{P}}_{\overline{\mathbb{Q}}}$, the profinite completion $\widehat{F}_2$ of the topological fundamental group $\pi_1(\dddot{\mathbb{P}}) = F_2$, and the Galois group of $\Omega/\overline{\mathbb{Q}}(T)$, where Ω is the maximal Galois field extension of $\overline{\mathbb{Q}}(T)$ which is unramified outside T, $T-1$ and $\frac{1}{T}$. As a corollary to Proposition 4.8 we thus obtain:

Remark 4.9. We have the following chain of group isomorphisms:

$$\pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\mathbb{C}}) \cong \pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\overline{\mathbb{Q}}}) \cong \widehat{F}_2 \cong \mathrm{Gal}\,(\Omega/\overline{\mathbb{Q}}(T)).$$

From the chain of Galois extensions $\mathbb{Q}(T) \subset \overline{\mathbb{Q}}(T) \subset \Omega$ we obtain the exact sequence

$$1 \to \mathrm{Gal}\,(\Omega/\overline{\mathbb{Q}}(T)) \to \mathrm{Gal}\,(\Omega/\mathbb{Q}(T)) \to \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

of Galois groups (since $\mathrm{Gal}\,(\overline{\mathbb{Q}}(T)/\mathbb{Q}(T)) \cong \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$). Using the isomorphisms of Remark 4.9, we obtain the following special case of Grothendieck's exact sequence of algebraic fundamental groups, cf. [26, Thm. 8.1.1] :

$$1 \to \pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\overline{\mathbb{Q}}}) \to \pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\mathbb{Q}}) \to \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

This exact sequence provides us a priori with an outer action of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\overline{\mathbb{Q}}})$, i. e. a group homomorphism from $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ to the outer automorphism group $\mathrm{Out}(\widehat{F}_2) = \mathrm{Aut}\,(\widehat{F}_2)/\mathrm{Inn}\,(\widehat{F}_2)$ of $\widehat{F}_2 \cong \pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\overline{\mathbb{Q}}})$. The additional information that we obtain from the explicit results in Section 4.5 is that the sequence splits, and that the outer action thus is in fact a true action. In other words, the construction in Section 4.5 corresponds to a particular splitting homomorphism $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to \pi_1^{\mathrm{alg}}(\dddot{\mathbb{P}}_{\overline{\mathbb{Q}}})$.

4.7 The Grothendieck-Teichmüller group

In the last two sections we established a group homomorphism $\tau : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\widehat{F}_2)$ coming from the Galois action on dessins. It follows from Proposition 4.5 that τ is injective. In this way, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is embedded into a group whose definition does not refer to field extensions or number theory; but since $\mathrm{Aut}(\widehat{F}_2)$ is a very large group that is not well understood, there is not much hope that this embedding alone can shed new light on the structure of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

In his paper [4], V. Drinfel'd defined a much smaller subgroup of $\mathrm{Aut}(\widehat{F}_2)$, which still contains the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ under τ. He called this group the *Grothendieck-Teichmüller group* and denoted it by $\widehat{GT}$. It is still an open question whether $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is equal to $\widehat{GT}$. In this section we present the definition of $\widehat{GT}$ and indicate how $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is embedded into $\widehat{GT}$.

We saw in Proposition 4.8 that for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the automorphism $\tau(\sigma) \in \mathrm{Aut}(\widehat{F}_2)$ is completely determined by the "exponent" $\lambda_\sigma = \chi(\sigma) \in \widehat{\mathbb{Z}}^\times$ and the "conjugator" $f_\sigma \in \widehat{F}_2$. The explicit knowledge of f_σ makes it possible to show that it acts trivially on abelian extensions of $\overline{\mathbb{Q}}(T)$ and therefore that f_σ is contained in the (closure of the) commutator subgroup $\widehat{F}_2'$ of $\widehat{F}_2$, see [16, Prop. 1.5] or [28, Sect. 4.4]. The composition of automorphisms implies that pairs in $\widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$ that come from Galois automorphisms, are multiplied according to the rule

$$(\lambda, f) \cdot (\mu, g) = (\lambda\mu, f F_{\lambda,f}(g)), \tag{4.1}$$

where $F_{\lambda,f}$ is the endomorphism of $\widehat{F}_2$ which is induced by $x \mapsto x^\lambda$ and $y \mapsto f^{-1}y^\lambda f$. Motivated by his investigations of braided categories Drinfel'd found some natural conditions to impose on such pairs (λ, f):

Definition 4.10. ***a)*** Let $\underline{\widehat{GT}}_0$ be the set of pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^\times \times \widehat{F}_2'$ that satisfy

$$\begin{array}{ll} (\mathrm{I}) & \theta(f)\, f = 1 \\ (\mathrm{II}) & \omega^2(fx^m)\,\omega(fx^m)\, fx^m = 1 \end{array}$$

where $m = \frac{1}{2}(\lambda - 1)$ and θ resp. ω are the automorphisms of $\widehat{F}_2$ defined by $\theta(x) = y$, $\theta(y) = x$ resp. $\omega(x) = y$, $\omega(y) = (xy)^{-1}$.

b) Let $\widehat{GT}_0$ be the group of elements in $\underline{\widehat{GT}}_0$ that are invertible for the composition law (4.1).

c) The *Grothendieck-Teichmüller group* $\widehat{GT}$ is the subgroup of $\widehat{GT}_0$ of elements that satisfy the further relation

$$(\mathrm{III}) \qquad \rho^4(\tilde{f})\,\rho^3(\tilde{f})\,\rho^2(\tilde{f})\,\rho(\tilde{f})\,\tilde{f} = 1$$

which takes place in the profinite completion $\widehat{K}_5$ of the pure braid group K_5 on five strands. This group is generated by elements $x_{i,i+1}$ for $i \in \mathbb{Z}/5\mathbb{Z}$, and ρ is the automorphism that maps $x_{i,i+1}$ to $x_{i+3,i+4}$; finally $\tilde{f} = f(x_{1,2}, x_{2,3})$.

It is not obvious from the definition that $\widehat{GT}_0$ is a group, more precisely the group of all elements in $\underline{\widehat{GT}}_0$ that induce automorphisms on $\widehat{F}_2$. The proofs of these facts can be found in [20] and [34]; a careful proof with all details is contained in [8].

The relation between the Galois group and the Grothendieck-Teichmüller group is stated in

Theorem 4.11. *Via the homomorphism τ,* $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ *becomes a subgroup of* $\widehat{GT}$.

That the pairs $(\lambda_\sigma, f_\sigma)$ coming from Galois automorphisms satisfy the first two relations (I) and (II) can be shown using the explicit computations of the action on $\widehat{F}_2$, see [28, Sect. 5.1]. The proof of the third relation is a bit more complicated; we refer to [16].

5 Origamis

5.1 Introduction to origamis

In the world of dessins d'enfants we study finite unramified holomorphic coverings $\beta : X^* \to \dddot{\mathbb{P}}$ between Riemann surfaces. We have seen in Section 3 that such a covering is up to equivalence completely determined by the covering $R^* \to \dddot{S}$ of the underlying topological surfaces. It is very tempting to generalise this and look at general finite unramified coverings between *punctured closed surfaces*, i.e. closed surfaces with finitely many points removed. It turns out that choosing the once punctured torus $E^* = E\backslash\infty$ (∞ some point on the torus E) as base surface instead of $\dddot{S}$ is in some sense the next "simplest" case. Following the spirit and the denominations of [19], we call a covering $p : R \to E$ ramified at most over the point ∞ an *origami*. Note that this defines the unramified covering $R^* \to E^*$, where $R^* = R\backslash p^{-1}(\infty)$, and conversely each finite unramified cover of E^* is obtained in this way. Similarly as for Belyi pairs we call two origamis $O_1 = (p_1 : R_1 \to E)$ and $O_2 = (p_2 : R_2 \to E)$ *equivalent*, if there exists some homeomorphism $f : R_1 \to R_2$ such that $p_2 \circ f = p_1$.

The first observation is that the different combinatorial descriptions of a topological covering $R^* \to \dddot{S}$ explained in Section 3 smoothly generalise to arbitrary unramified coverings of punctured closed surfaces. In the case of origamis we obtain the equivalent descriptions stated in Proposition 5.1. The generalisation of a dessin d'enfants can be done as follows: In the case of the three punctured sphere $\dddot{S}$, we used that we obtain a cell if we remove the interval $I = [0, 1]$ from $\dddot{S}$. For the once-punctured torus E^* we remove two simply closed curves x and y starting in the puncture as shown in Figure 7.

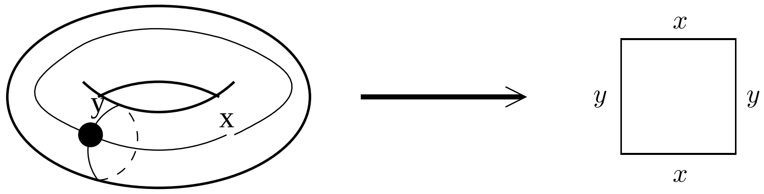


Figure 7: Removing two simply closed curves from the torus E gives a cell.

The cell that we obtain in this way is bounded in E by four edges labelled with x and y. We identify it with a quadrilateral. Similarly as described in Section 3, we have for an origami $p : R^* \to E^*$ that $R^* \backslash (p^{-1}(x) \cup p^{-1}(y))$ decomposes into a finite union of quadrilaterals. Unlike the case of dessins, the map p restricted to $R^* \backslash (p^{-1}(x) \cup p^{-1}(y))$ is unramified and the number of quadrilaterals is the degree d of p. We retrieve the surface R^* by gluing the quadrilaterals. Hereby only edges labelled with the same letter x or y may be glued. Furthermore we have to respect orientations. Altogether this leads to the following "origami-rules": Glue finitely many copies of the Euclidean unit square such that

- Each left edge is glued to a unique right edge and vice versa.
- Each upper edge is glued to a unique lower one and vice versa.
- We obtain a connected surface R.

R has a natural covering map $p : R \to E$ by mapping each square to one square which forms the torus E. The map p is unramified except possibly above ∞, which is the one point on E that results from the vertices of the square. Thus $p : R \to E$ is an origami. Note that for the moment we are only interested in the topological covering p and it would not be necessary to take Euclidean unit squares which endows R in addition with a metric.

It is remarkable that by some fancy humour of nature the fundamental groups of $\dddot{\mathbb{P}}$ and E^* are both the same abstract group, the free group F_2 in two generators.

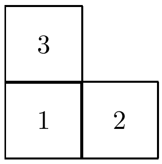
Proposition 5.1. *An origami $p : R^* \to E^*$ is up to equivalence uniquely determined by*

- *a surface obtained from gluing Euclidean unit squares according the "origami rules" (see above).*

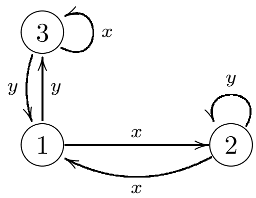
- *a finite oriented graph whose edges are labelled with x and y such that each vertex has precisely one incoming edge and one outgoing edge labelled one with x and one with y, respectively.*
- *a monodromy map $\alpha : F_2 \to S_d$ up to conjugation in S_d.*
- *a finite index subgroup U of F_2 up to conjugation in F_2.*

The equivalences stated in Proposition 5.1 are carried out in detail e.g. in [31, Sect.1]. Thus we restrict here to giving the different descriptions for an example.

Example 5.2. In the following we describe the origami, commonly known as $L_{2,2}$, in the different ways assembled in Proposition 5.1.



Gluing squares according to the origami-rules: Opposite edges age glued.



The finite graph which describes the origami.

The monodromy map is the map $\rho : F_2 \to S_3$ which is given by $x \mapsto (1\ 2)$ and $y \mapsto (1\ 3)$ and a corresponding subgroup of F_2 is $U = < x^2, y^2, xyx^{-1}, yx^{-1}y >$.

A short Euler characteristic calculation shows that for this example the surface R has genus 2. The covering map $p : R \to E$ has degree 3 and the puncture ∞ has one preimage on R.

5.2 Teichmüller curves

So far, we have only considered coverings between topological surfaces. A crucial point of the theory of dessins d'enfants is that the three-punctured sphere $\dddot{S}$ has a unique complex structure as a Riemann surface. Therefore choosing a finite unramified covering $\beta : R^* \to \dddot{S}$ defines a closed Riemann surface of genus $g = \text{genus}(R)$: Take the unique complex structure on the sphere and lift it via p to R^*. For the so obtained Riemann surface X^* there is a unique closed Riemann surface X into which we can embed X^* holomorphically. Hence, β defines the point $[X]$ in M_g, respectively $[X^*]$ in $M_{g,n}$, where M_g is the moduli space of regular complex curves of genus g, $M_{g,n}$ is the moduli space of regular complex curves with n marked points and n is the number of points in $X\backslash X^*$. Recall from algebraic geometry that M_g and $M_{g,n}$ are themselves complex varieties. In fact they are obtained by base change from schemes defined over $\mathbb{Z}$. By Belyi's Theorem the image points $[X] \in M_g$, respectively $[X^*] \in M_{g,n}$ are points defined over $\overline{\mathbb{Q}}$.

How can we generalise this construction for origamis? Since we have a one-dimensional family of complex structures on the torus E, an origami $O = (p : S \to E)$ will define a collection of Riemann surfaces depending on one complex parameter. More generally, an unramified cover $p : R_1^* \to R_2^*$ between punctured closed surfaces naturally defines the holomorphic and isometric embedding

$$\iota_p : \; T(R_2^*) \; \hookrightarrow \; T(R_1^*), \quad [\mu] \; \mapsto \; [p^*\mu], \tag{5.1}$$

from the Teichmüller space $T(R_2^*)$ to the Teichmüller space $T(R_1^*)$, which maps a complex structure μ on R_2^* to the complex structure $p^*\mu$ on R_1^* obtained as pull back via p. We now project the image $B := \iota_p(T(R_2^*))$ to $M_{g,n}$ and further to M_g. How do the images in the moduli spaces look like? Can we describe their geometry based on the combinatorial data of the map p with which we started?

In the following we restrict to the case of origamis. Thus we obtain an embedding $\iota_p : \mathbb{H} \cong T_{1,1} \hookrightarrow T_{g,n}$ which is holomorphic and isometric. Such a map is called a *Teichmüller embedding* and its image in Teichmüller space is called a *Teichmüller disk*. Teichmüller disks arise in general from the following construction, which is described in detail and with further hints to literature e.g. in [14]: Let X be a Riemann surface together with a flat structure ν on it; i.e. we have an atlas on $X\backslash\{P_1, \dots, P_n\}$ for finitely many points P_i such that all transition maps are locally of the form $z \mapsto \pm z + c$ with some constant c. Suppose furthermore that the P_i's are cone singularities of ν. Then each matrix $A \in \mathrm{SL}_2(\mathbb{R})$ induces a new flat structure ν_A by composing each chart with the affine map $z \mapsto A \cdot z$. This defines a map

$$\iota_\nu : \; \mathbb{H} \; \cong \; \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \; \to \; T_g, \quad [A] \; \mapsto [\nu_A] \tag{5.2}$$

which is in fact a holomorphic and isometric embedding, i.e. it is a Teichmüller embedding. It is a nice feature that for an origami $O = (p : S \to E)$ the surface S comes with a flat structure: One identifies E with $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}i)$. This quotient carries a natural flat structure induced by the Euclidean structure on $\mathbb{C}$. It is actually a translation structure, i.e. the transition maps are of the form $z \mapsto z + c$. Note that in the description of origamis with the "origami-rules" we obtain the translation structure for free, if we glue the edges of the unit squares via translations. The translation surfaces arising in this way are often called *square tiled surfaces*. It is not hard to see that for an origami O the induced maps ι_ν defined in (5.2) and $\text{proj}_{g,n} \circ \iota_p$ (with ι_p from (5.1)) from $\mathbb{H}$ to T_g are equal (see e.g. [30, p.11]); here $\text{proj}_{g,n} : T_{g,n} \to T_g$ is the natural projection obtained by forgetting the marked points. In the following we will therefore denote the map $\iota_p = \iota_\mu$ just by ι_O.

The study of Teichmüller disks has lead to vivid research activities connecting different mathematical fields such as dynamical systems, algebraic geometry, complex analysis and geometric group theory. Many different authors have contributed to this field in the last years with a multitude of interesting results (see e.g. [25] in this volume or [14] for comments on literature). Important impacts to this topic were already given in [35]. An important tool for the study of Teichmüller disks is the *Veech group*, which was introduced in [36]. For a translation surface (X, ν) one takes the *affine group* $\text{Aff}(X, \nu)$ of diffeomorphisms which are locally affine. The Veech group $\Gamma(X, \nu)$ is its image in $\text{SL}_2(\mathbb{R})$ under the derivative map D, which maps each affine diffeomorphism to its linear part. The article [25] in this volume gives a more detailed introduction to Veech groups, an overview on recent results and hints to more literature. In Theorem 5.3 we list the properties of Veech groups that we will use. It is a collection of results contributed by different authors, which we have learned mainly from [36], [5] and [22]. Section 2.4 in [14] contains a quite detailed summary of them and further references. An important ingredient is the fact that if we have a translation structure and pull it back by an affine diffeomorphism f, it is changed by composing each chart with the affine map $z \mapsto Az$, where A is the inverse of the derivative of f. Therefore the elements in the mapping class group which come from affine diffeomorphisms stabilise the image Δ of the Teichmüller embedding ι_ν. One shows that in fact, they form the full stabiliser of Δ. Furthermore the group $\text{Trans}(X, \mu) = \{f \in \text{Aff}(X, \mu) | \; D(f) = \text{identity matrix}\}$ acts trivially on Δ and $\Gamma(X, \nu) \cong \text{Aff}(X, \nu)/\text{Trans}(X, \nu)$.

Theorem 5.3. *Let X be a holomorphic surface and ν a translation structure on X with finitely many cone singularities. Let $\iota = \iota_\nu : \mathbb{H} \hookrightarrow T_g$ be the corresponding Teichmüller embedding, Δ its image in T_g, $p_g : T_g \to M_g$ the natural projection and Γ_g the mapping class group for genus g. Then we have:*

- $\text{Stab}_{\Gamma_g}(\Delta) \cong \text{Aff}(X, \nu)$.
- *$p_g|_\Delta$ factors through the quotient map $q : \Delta \to \Delta/\Gamma(X, \nu)$, i.e. we obtain a map $n : \Delta/\Gamma(X, \nu) \to M_g$ with $p|_\Delta = n \circ q$.*
- *The image of Δ in M_g is an algebraic curve C if and only if the Veech group $\Gamma(X, \nu)$ is a lattice in $\text{SL}_2(\mathbb{R})$. If this is the case, C is called a* Teichmüller curve*, and n is a birational map. Therefore it is the normalisation of C. C is birationally equivalent to a mirror image of $\mathbb{H}/\Gamma(X, \nu)$.*

In the following we will only consider Teichmüller embeddings coming from origamis. In this case, the Veech group is commensurable to $\mathrm{SL}_2(\mathbb{Z})$ and thus a lattice in $\mathrm{SL}_2(\mathbb{R})$. It turns out to be useful given an origami $O = (p : X \to E)$, to consider only affine diffeomorphisms which preserve $p^{-1}(\infty)$. The image of this group is in fact a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Following the notations in [30], we denote it by $\Gamma(O)$ and call it *the Veech group of the origami O*. If we replace T_g by $T_{g,n}$ and M_g by $M_{g,n}$ in Theorem 5.3, then $\Gamma(O)$ becomes the effective stabilising group of $\Delta \subseteq T_{g,n}$. [29] describes an algorithm which computes $\Gamma(O)$.

Coming back to the question asked at the beginning of this section, we state that in the case of an origami the image of the map ι_p in M_g is an algebraic curve which comes from a Teichmüller disk. In the following sections we study these curves, which we call *origami curves*. More precisely we point out some explicit relations between them and dessins d'enfants.

6 Galois action on origamis

In [19], Lochak suggested to study the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on origamis in some sense as generalisation to the action on dessins d'enfants following the spirit of Grothendieck's *Esquisse d'un programme*. Recall from Section 4 that for each $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and each projective curve X defined over $\overline{\mathbb{Q}}$, we obtain a projective curve ${}^{\sigma}X$. This actually defines an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $M_{g,\overline{\mathbb{Q}}}$, the moduli space of regular projective curves which are defined over $\overline{\mathbb{Q}}$.

In the following we want to make the definition of an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on origamis more precise: Let $O = (p : R \to E)$ be an origami with $\mathrm{genus}(R) = g$. Recall that O defines a whole family of coverings $p_A : X_A \to E_A$ ($A \in \mathrm{SL}_2(\mathbb{R})$) between Riemann surfaces. It follows from Theorem 5.3 that two coverings p_A and $p_{A'}$ are equivalent, if and only if A and A' are mapped to the same point on $\tilde{C}(O) = \mathbb{H}/\Gamma(O)$, where $\Gamma(O)$ is the Veech group; furthermore X_A and $X_{A'}$ are isomorphic, if and only if the two matrices are mapped to the same point on the possibly singular curve $C(O)$. In particular we may parameterise the family of coverings by the elements t of $\tilde{C}(O)$ and denote them as $p_t : X_t \to E_t$. In the following we will restrict to those t for which $p_t : X_t \to E_t$ is defined over $\overline{\mathbb{Q}}$. We denote this subset of $\tilde{C}(O)$ by $\tilde{C}_{\overline{\mathbb{Q}}}(O)$ and similarly we write $C_{\overline{\mathbb{Q}}}(O)$.

Let us now pick some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One immediately has two ideas how σ could act on origami curves; both lead at first glance to a problem:

- $C = C_{\overline{\mathbb{Q}}}(O)$ is mapped to its image ${}^\sigma C_{\overline{\mathbb{Q}}} = \{{}^\sigma(X_t) |\, t \in \tilde{C}_{\overline{\mathbb{Q}}}(O)\}$. Is the image again an origami curve; or more precisely is there some origami ${}^\sigma O$ such that ${}^\sigma C_{\overline{\mathbb{Q}}} = C_{\overline{\mathbb{Q}}}({}^\sigma O)$?
- For $p_t : X_t \to E_t$ (defined over $\overline{\mathbb{Q}}$) define ${}^\sigma p_t$ similarly as ${}^\sigma\beta$ in Section 4. Each ${}^\sigma p_t$ defines an origami. Do they all lead to the same origami curve?

In [24, Prop. 3.2] Möller showed that the two approaches lead to the same unique origami curve ${}^\sigma C$. We denote the corresponding origami by ${}^\sigma O$, i.e. ${}^\sigma C = C({}^\sigma O)$.

The basic ingredient of the proof in [24] is to consider the Hurwitz space of all coverings with the same ramification behaviour as p for a given origami $O = (p : X \to E)$. By a result of Wewers in [37], one obtains a smooth stack over $\mathbb{Q}$. The covering p lies in a connected component of it, whose image in moduli space is the origami curve $C(O)$. Möller deduces from this that $C(O)$ is defined over a number field and that one has the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ described above.

The Galois action on origamis is faithful in the following sense: For each σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ there exists an origami O such that $C(O) \neq {}^\sigma C(O)$. This is shown in [24, Theorem 5.4]. The proof uses the faithfulness of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins of genus 0 (see Prop. 4.6). Starting with a Belyi morphism $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with ${}^\sigma\beta \ncong \beta$, one takes the fibre product of β with the degree 2 morphism $E \to \mathbb{P}^1(\mathbb{C})$, where E is an elliptic curve which is defined over the fixed field of σ in $\overline{\mathbb{Q}}$. Precomposing the obtained morphism with the normalisation and postcomposing with multiplication by 2 on E, gives an origami as desired.

This is a nice example for some interplay going on between origamis and dessins in the way it was proposed in [19]. In the next two sections we describe two further ways, how origamis and dessins can be related.

7 A dessin d'enfants on the origami curve

Let $O = (p : R \to E)$ be an origami and let $\Gamma(O)$ be its Veech group. In this section we consider the corresponding Teichmüller curve in the moduli space $M_{g,n}$ of n-punctured curves and denote it by $C(O)$. As always, g is the genus of R and n is the number of preimages of the ramification point $\infty \in E$. Let $\tilde{C}(O)$ be the quotient $\mathbb{H}/\Gamma(O)$. Recall from Theorem 5.3 that $\tilde{C}(O)$ is the normalisation of $C(O)$.

The quotient $\tilde{C}(O)$ naturally defines a dessin d'enfants, as it was pointed out in [19, Proof of Prop.3.2].: $\Gamma(O)$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Thus we obtain a finite covering $q : \mathbb{H}/\Gamma(O) \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{A}^1(\mathbb{C})$. We may fill in cusps and extend q to a finite covering $q : X \to \mathbb{P}^1(\mathbb{C})$ of closed Riemann surfaces. This covering has ramification at most above three points of $\mathbb{P}^1(\mathbb{C})$: the two ramification points of the map $\mathbb{H} \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ and the cusp $\infty = \mathbb{P}^1(\mathbb{C})\backslash\mathbb{A}^1(\mathbb{C})$. Hence q is a Belyi morphism. Applying once more the Theorem of Belyi, one obtains for free that the complex curve $\tilde{C}(O)$ is defined over $\overline{\mathbb{Q}}$.

The dessin corresponding to q is obtained quite explicitly from this description, as we explain in the following. Recall that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We take our favourite fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, namely the ideal triangle with vertices $P = \zeta_3 = e^{\frac{2\pi i}{3}}$, $Q = \zeta_3 + 1$ and the cusp $R = \infty$, see Figure 8. Recall that P is a fixed point of the matrix $S \circ T$, which is of order 3 in $\mathrm{PSL}_2(\mathbb{Z})$. Furthermore, i is a fixed point of the order 2 matrix S and thus a further hidden vertex of the fundamental domain. Finally, the transformation T maps the edge PR to QR and S maps the edge Pi to Qi. We obtain $\mathbb{P}^1(\mathbb{C})$ by "gluing" PR to QR and Pi to Qi and filling in the cusp at ∞.

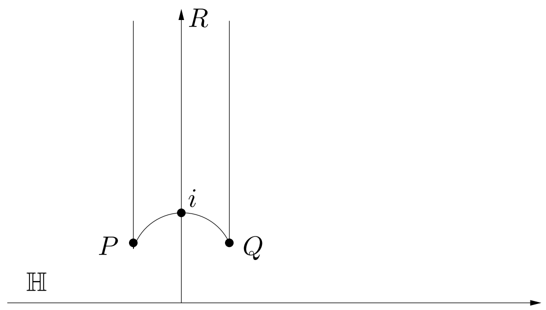


Figure 8: *Fundamental domain of* $\mathrm{SL}_2(\mathbb{Z})$.

In order to make the dessin explicit, we identify the image of P on $\mathbb{P}^1(\mathbb{C})$ with 0, the image of i with 1 and the image of the cusp with ∞. The geodesic segment PQ is then mapped to our interval I; its preimage $q^{-1}(I)$ on X is the dessin.

The algorithm in [29] gives the Veech group Γ by a system G of generators and a a system C of coset representatives. C is in fact a Schreier-transversal with respect to the generators S and T of $\mathrm{SL}_2(\mathbb{Z})$, i.e. each element in C is given as a word in S and T such that each prefix of it is also in C. Therefore C defines a connected fundamental domain F of Γ which is the union of translates of the triangle PQR; for each coset we obtain one translate. The identification of the boundary edges of F are given by the generators in G. Thus the fundamental domain F is naturally tessellated by triangles, which indicate the Belyi morphism. The dessin is the union of all translates of the edge PQ.

In the following we describe the dessin for an example. We take the origami D drawn in Figure 9, which is studied in [31].

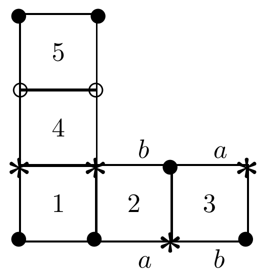


Figure 9: The origami D. Edges with the same label and unlabelled edges that are opposite are glued.

The Veech group $\Gamma = \Gamma(D)$ and the fundamental domain of Γ are given in Section 3 of [31]. The index of Γ in $\mathrm{SL}_2(\mathbb{Z})$ is 24 and the quotient $\mathbb{H}/\Gamma$ is a surface of genus 0 with six cusps. Figure 10 shows a fundamental domain of $\Gamma(D)$.

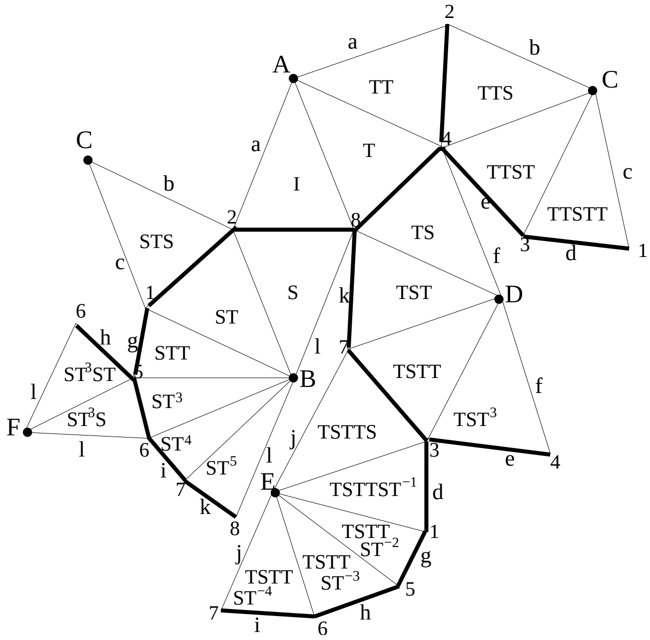


Figure 10: *The fundamental domain for the Veech group* $\Gamma(D)$.

We use a schematic diagram: Each triangle represents a translate of the triangle PQR. The vertices labelled with $A, \ldots, F$ are the cusps. The thickened edges form the dessin. The planar graph is redrawn in Figure 11. This picture matches its embedding into $\mathbb{P}^1(\mathbb{C})$.

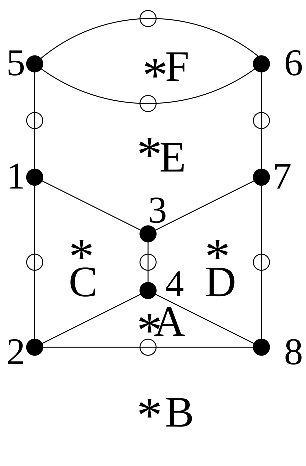


Figure 11: *The dessin on the origami curve* $\tilde{C}(D)$.

8 Dessins d'enfants related to boundary points of origami curves

Let $O = (p : X \to E)$ be an origami of genus $g \geq 2$ and $C(O)$ the corresponding origami curve in the moduli space M_g. Recall that the algebraic variety M_g can be compactified by a projective variety $\overline{M}_g$, the *Deligne-Mumford compactification*, which classifies stable Riemann surfaces, i. e. surfaces with "nodes" (see below for a precise definition). The closure $\overline{C(O)}$ of $C(O)$ in $\overline{M}_g$ is a projective curve; its boundary $\partial C(O) = \overline{C(O)} - C(O)$ consists of finitely many points, called the *cusps* of the origami curve.

In this section we shall associate in a natural way dessins d'enfants to the cusps of origami curves.

8.1 Cusps of origami curves

There is a general procedure to determine the cusps of algebraic curves in moduli space, called *stable reduction*. We first recall the notion of a stable Riemann surface:

Definition 8.1. A one-dimensional connected compact complex space X is called *stable Riemann surface* if

(i) every point of X is either smooth or has a neighbourhood which is analytically isomorphic to $\{(z, w) \in \mathbb{C}^2 : z \cdot w = 0\}$ (such a point is called a *node*) and

(ii) every irreducible component of X that is isomorphic to $\mathbb{P}^1(\mathbb{C})$ intersects the other components in at least three points.

Now let C_0 be an algebraic curve in M_g and $x \in \partial C$ a cusp of C. We may assume that C_0 is smooth (by removing the finitely many singular points of C_0) and that also $C = C_0 \cup \{x\}$ is smooth (by passing to the normalisation). Next we assume that over C_0 we have a family $\pi_0 : \mathcal{C}_0 \to C_0$ of smooth curves over C_0, i. e. a proper flat morphism π_0 such that the fibre $X_c = \pi^{-1}(c)$ over a point $c \in C_0$ is isomorphic to the compact Riemann surface which is represented by c (for this we may have to pass to a finite covering of C_0). The stable reduction theorem (see [11, Prop. 3.47]) states that, after passing to another finite covering C' of C (which can be taken totally ramified over x), the family $\mathcal{C}_0 \times_{C_0} C'_0$ extends to a family $\pi : \mathcal{C} \to C'$ of stable Riemann surfaces, and that the stable Riemann surface $X_\infty = \pi^{-1}(x)$, that occurs as fibre over the cusp x, is independent of the choice of C'.

Although the proof of the stable reduction theorem is constructive, this construction usually becomes quite involved: First examples are discussed in [11, Sect. 3C]; a particularly nice example for the cusp of an origami curve is worked out in [2].

If the algebraic curve C_0 in M_g is a Teichmüller curve, there is a much more direct way to find the stable Riemann surface to a cusp, avoiding the stable reduction theorem. This construction is based on the description of Jenkins-Strebel rays in [21] and worked out in detail in [14, Sect. 4.1]. The basic observation is that for every cusp x of a Teichmüller curve C there is a direction on the flat surface X defining C in which X is decomposed into finitely many cylinders; this direction is associated to a Jenkins-Strebel differential on X. The stable Riemann surface corresponding to the cusp is now obtained by contracting the core curves of these cylinders. See [14, Sect. 4.2] for a proof of this result.

In the special case of a Teichmüller curve coming from an origami, the construction is particularly nice: Let $O = (p : X \to E)$ be an origami as above. The squares define a translation structure on X and divide it into horizontal cylinders, which we denote by $C_1, \dots, C_n$. The core lines $c_1, \dots, c_n$ of these cylinders are the connected components of the inverse image $p^{-1}(a)$ of the horizontal closed path a on the torus E. Contracting each of the closed paths c_i to a point x_i turns X into a surface X'_∞ which is smooth outside $x_1, \dots, x_n$. It is shown in [14, Sect. 4.1] how to put, in a natural way, a complex structure on X'_∞. Then X'_∞ satisfies the above Definition 8.1, except perhaps (ii). If an irreducible component of X'_∞ violates (ii), we can contract this component to a single point and obtain a complex space which still satisfies (i). After finitely many such contractions we obtain a stable Riemann surface X_∞. This process of contracting certain components is called "stabilising". For simplicity we used here the horizontal cylinders. But the construction is the same for any direction in which there is a decomposition into cylinders.

If we apply this construction to the torus E itself, we obtain a surface E_∞ which has a single node and whose geometric genus is zero. This surface is known as *Newton's node* and can algebraically be described as the singular plane projective curve with affine equation $y^2 = x^3 - x^2$.

Note that in the above construction, the covering p naturally extends to a covering $p_\infty : X_\infty \to E_\infty$, which is ramified at most over the critical point ∞ of p (or, to be precise, the point on E_∞ that corresponds to ∞ on E), and over the node. This is illustrated in the following picture for the origami W from [15]:

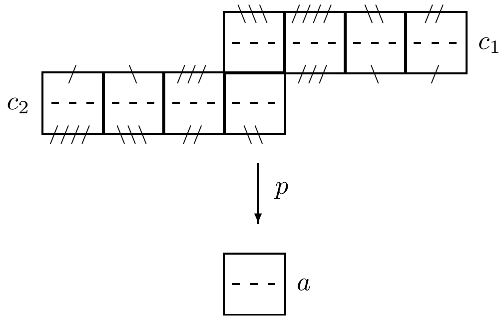


Figure 12: The origami covering for the cusp of W

8.2 The dessin d'enfants associated to a boundary point

The construction in 8.1 leads in a natural way to a dessin d'enfants, as was observed in [19, Sect. 3.1], where it is attributed to L. Zapponi. Let, as before, $O = (p : X \to E)$ be an origami of genus $g \geq 2$ and $C(O)$ the corresponding origami curve in M_g. Furthermore let $x \in \partial C(O)$ be a cusp and X_∞ the stable Riemann surface that is represented by x. Denote by $X_1, \ldots, X_n$ the irreducible components of X_∞ and by $p_\infty : X_\infty \to E_\infty$ the covering discussed at the end of the previous section. For each $i = 1, \ldots, n$, the restriction of p_∞ to X_i gives a finite covering $p_i : X_i \to E_\infty$. For the degrees d_i of p_i we have the obvious relation

$$\sum_{i=1}^{n} d_i = d = \deg(p).$$

Now let C_i be the normalisation of X_i $(i = 1, \ldots, n)$. Then p_i induces a covering $f_i : C_i \to \mathbb{P}^1(\mathbb{C})$ (which is the normalisation of E_∞).

Proposition 8.2. *For every boundary point x of the origami curve $C(O)$ and each irreducible component X_i of the stable Riemann surface X_∞, the covering $f_i : C_i \to \mathbb{P}^1(\mathbb{C})$ is a Belyi morphism.*

Proof. We already noticed in Section 8.1 that the covering $p_\infty : X_\infty \to E_\infty$ is ramified at most over the critical point ∞ of p and over the node. The normalisation map $\mathbb{P}^1(\mathbb{C}) \to E_\infty$ maps two different points to the node, so each f_i can be ramified over these two points, and otherwise only over the inverse image of ∞. □

In Section 3 we explained that a dessin d'enfants is completely determined by the monodromy map of the corresponding Belyi map β, i. e. two permutations σ_0 and σ_1 in S_d, where d is the degree of β.

Similarly, an origami $O = (p : X \to E)$ is also determined by two permutations σ_a and σ_b, see Section 5.1; they describe the gluing of the squares in horizontal resp. vertical direction: a horizontal cylinder consists of the squares in a cycle of σ_a, and the vertical ones are given by the cycle decomposition of σ_b. At the same time, σ_a and σ_b describe the monodromy of the covering p by looking at the lifts of the horizontal closed path a and the vertical closed path b on the torus E.

There is a nice relation between the permutations σ_a and σ_b of the origami O and the permutations σ_0 and σ_1 of the dessin d'enfants associated to the boundary point x on $\partial C(O)$ which is obtained by contracting the centre lines of the horizontal cylinders. It was first made explicit (but not published) by Martin Möller as follows:

Proposition 8.3. *Let $O = (p : X \to E)$ be an origami of degree d and σ_a, σ_b the corresponding permutations in S_d. Then the dessin d'enfants associated to the horizontal boundary point on $C(O)$ is defined by*

$$\sigma_0 = \sigma_a, \quad \sigma_1 = \sigma_b \sigma_a \sigma_b^{-1}.$$

In this proposition, the dessin d'enfants is not necessarily connected; it is the union of the dessins to the irreducible components described above in Proposition 8.2.

Proof. Recall the construction of the covering $p_\infty : X_\infty \to E_\infty$ and the Belyi map $f_\infty : \cup_{i=1}^n X_i \to \mathbb{P}^1(\mathbb{C})$: E_∞ is obtained from the torus E by contracting the horizontal path a to a single point, the node of E_∞. Let U be a neighbourhood of the node, analytically isomorphic to $\{(z, w) \in \mathbb{C}^2 : |z| \leq 1, |w| \leq 1, z \cdot w = 0\}$. U is the union of two closed unit disks U_0, U_1 which are glued together at their origins. In the normalisation $\mathbb{P}^1(\mathbb{C})$ of E_∞, the node has two preimages, and the preimage of U is the disjoint union of the two disks U_0 and U_1. The loops l_0 and l_1 can be taken as simple loops in U_0 resp. U_1 around the origin. On E_∞, l_0 and l_1 are the images of parallels a_0 and a_1 of a, one above a, the other below:

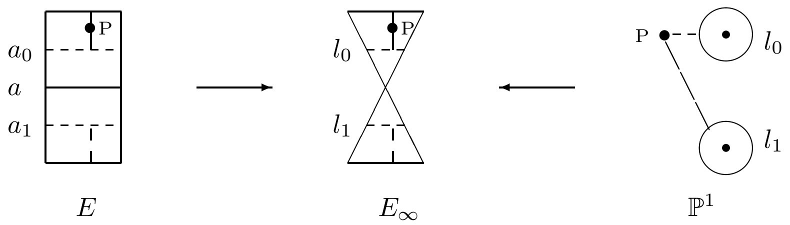


Figure 13: The loops on E, E_∞ and $\mathbb{P}^1$

Since all our loops have to be considered as elements of the resp. fundamental groups, we have to choose base points in $\mathbb{P}^1(\mathbb{C})$, E_∞, and E. Since l_0 and l_1 may not pass through the origin (resp. the node), a_0 and a_1 may not intersect a. Therefore, if we choose the base point as in the figure, a_0 is homotopic to a, but a_1 is homotopic to bab^{-1}.

Finally we have to lift a_0 and a_1 to X_∞ resp. $\cup_{i=1}^n X_i$ and write down the order in which we traverse the squares if we follow the irreducible components of these lifts. Thereby clearly the lift of a_0 induces σ_a, whereas the lift of a_1 induces $\sigma_b\sigma_a\sigma_b^{-1}$. □

The Belyi map $f_\infty : \cup_{i=1}^n X_i \to \mathbb{P}^1(\mathbb{C})$ can also be described directly in a very explicit way: In the above proof, $E_\infty - \{\text{ node}\}$ is obtained by gluing U_0 and U_1 along their boundaries (with opposite orientation). We may assume that the distinguished point ∞, over which the origami map p is ramified, lies on this boundary, and that, for the given Euclidean structure, the boundary has length 1. In this way we have described an isomorphism between $E_\infty - \{\text{ node}\}$ and $\dddot{\mathbb{P}}$.

Now let $C_1, \ldots, C_n$ be the horizontal cylinders of the origami surface X. Contracting the centre line c_i of C_i to a point turns $C_i - c_i$ into the union of two punctured disks $U_{0,i}$ and $U_{1,i}$. If C_i consists of d_i squares, the boundary of $U_{0,i}$ and $U_{1,i}$ has length d_i, and is subdivided by the squares into d_i segments of length 1. The Belyi map f_∞ is obtained by mapping each $U_{0,i}$ to U_0 and each $U_{1,i}$ to U_1 in such a way that the lengths are preserved. Thus in standard coordinates, the restriction of f_∞ to $U_{0,i}$ is $z \mapsto z^{d_i}$.

8.3 Examples

8.3.1 The origami $\mathbf{L_{2,2}}$ The smallest origami with a surface X of genus > 1 (actually 2) is the one called $L_{2,2}$ in Example 5.2; it is also the smallest one in the family $L_{n,m}$ of L-shaped origamis. The origami map $p : L_{2,2} \to E$ is of degree 3 and totally ramified over the point $\infty \in E$ (the vertex of the square). As explained in the previous section, the same holds for the covering $p_\infty : X_\infty \to E_\infty$ of the degenerate surfaces corresponding to the boundary points in the horizontal direction. As $X = L_{2,2}$ has 2 cylinders in the horizontal direction, X_∞ has 2 singular points which both are mapped by p_∞ to the node of E_∞. X_∞ is irreducible, and its geometric genus is 0. Thus the normalisation of X_∞ is $\mathbb{P}^1(\mathbb{C})$, and the induced map $f_\infty : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is of degree 3. The two points of the normalisation of E_∞, that lie over the node (and which we normalised to be 0 and 1), both have two preimages under f_∞, one ramified, the other not. Thus we obtain the following dessin for the Belyi map f_∞:

Figure 14: The dessin for a cusp of $L_{2,2}$

Since f_∞ is totally ramified over ∞, we can take it to be a polynomial. If we further normalise it so that 0 is a ramification point, we find that f_∞ is of the form

$$f_\infty(x) = x^2(x-a)$$

for some $a \in \mathbb{C}$. The derivative of f_∞ is

$$f'_\infty(x) = 3x^2 - 2ax = x(3x-2a),$$

thus the other ramification point of f_∞ in $\mathbb{C}$ is $\frac{2a}{3}$. Since the corresponding critical value is 1, we must have

$$1 = f_\infty(\frac{2a}{3}) = \frac{4a^2}{9}(-\frac{1}{3}a) = \left(\frac{a}{3}\right)^3 \cdot (-4)$$

$$\text{or} \qquad a = 3 \cdot \sqrt[3]{-\frac{1}{4}}.$$

All three choices of the third root lead to the same dessin, as can be seen from the following observation: The polynomial $f_a(x) = f_\infty(x) = x^2(x-a)$ has its zeroes at 0 and a, and takes the value 1 at $\frac{2}{3}a$ and $-\frac{a}{3}$, as can easily be checked. The cross ratio of these four points is -8, hence rational. This means that for all possible choices of a, the Belyi map f_a is equivalent to $f_a \circ \sigma_a$, where the Möbius transformation σ_a is determined by

$$\sigma_a(0) = 0, \qquad \sigma_a(1) = -\frac{1}{3}a, \qquad \sigma_a(\infty) = a,$$

and consequently $\sigma_a(-8) = \frac{2}{3}a$. An easy calculation shows

$$\sigma_a(x) = \frac{ax}{x-4} \qquad \text{and} \qquad f_a \circ \sigma_a(x) = -27\frac{x^2}{(x-4)^3}.$$

Note that $f_a \circ \sigma_a$ has a triple pole (at 4), a double zero at 0 (and another zero at ∞), and it takes the value 1 with multiplicity 2 at -8 (and a third time at 1).

It was shown in [30] that the origami curve $C(L_{2,2})$ has only one further cusp besides the one just discussed. It corresponds to cylinders in the "diagonal" direction $(1,1)$. In fact, there is only one cylinder in this direction (of length 3), and by taking this direction to be horizontal, the origami looks like

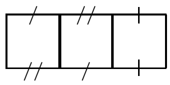


Figure 15: Another view on $L_{2,2}$

where as usual edges with the same marking are glued.

The corresponding singular surface X_∞ has one irreducible component with one singular point. Its normalisation is an elliptic curve E_0 which admits an automorphism of order 3 (induced by the cyclic permutation of the three "upper" and the three "lower" triangles of X_∞). This property uniquely determines E_0: It is the elliptic curve with Weierstrass equation $y^2 = x^3 - 1$ and j-invariant 0.

The corresponding dessin d'enfants is

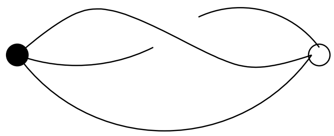


Figure 16: The dessin of a boundary point of $C(L_{2,2})$

It is the same ribbon graph as G_2 in Example 3.6. The Belyi map f for this dessin is, up to normalisation, the quotient map for the automorphism of order 3. If E_0 is given in Weierstrass form as above, this automorphism is the map $(x,y) \mapsto (\zeta_3 x, y)$, where $\zeta_3 = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$ is a primitive third root of unity. Such a quotient map is $(x,y) \mapsto y$. It is easily seen to be totally ramified over i, $-i$ and ∞. To make the critical values 0, 1 and ∞, we have to compose with the linear map $z \mapsto \frac{i}{2}(z - i)$. This shows that our Belyi map is

$$f(x,y) = \frac{i}{2}(y - i).$$

8.3.2 General L-shaped origamis Denote by $L_{n,m}$ the L-shaped origami with n squares in the horizontal and m squares in the vertical direction:

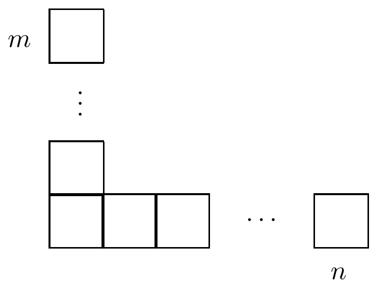


Figure 17: The origami $L_{n,m}$; opposite edges are glued

These origamis have been studied from several points of view by Hubert and Lelièvre, Schmithüsen, and others. The genus of $L_{n,m}$ is 2, independent of n and m. The Veech group in general gets smaller if n and m increase, and the genus of $C(L_{n,m})$ can be arbitrarily large. Also the number of cusps of $C(L_{n,m})$ grows with n and m.

In this section we only discuss the cusp of $L_{n,m}$ which is obtained by contracting the core lines of the horizontal cylinders. The resulting singular surface X_∞ has $m-1$ irreducible components: there is one component that contains the cylinder of length n and also the upper half of the top square. All other components consist of the upper half of one square, together with the lower half of the next square. Each such component is a projective line that intersects two of the other components. Moreover such a component contains a vertex, i. e. a point which is mapped to ∞ by f_∞ (but not ramified). It follows that the Belyi map f_i corresponding to such a component is the identity map $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.

Thus the only interesting irreducible component of X_∞ is the one that contains the "long" horizontal cylinder. For simplicity we only discuss the case where there are no components of the other type, i. e. $m = 2$. In this case we have, as for the $L_{2,2}$, exactly 2 singular points on X_∞. They are both mapped to the node of E_∞ by p_∞, one unramified, the other with ramification order n (note that the degree of p_∞ is $n+1$, the number of squares of $L_{n,2}$). As in the previous subsection, this picture is preserved if we pass to the normalisation. Thus $f_\infty^{-1}(0)$ and $f_\infty^{-1}(1)$ both consist of 2 points, one unramified, the other ramified of order n, and the dessin looks as follows:

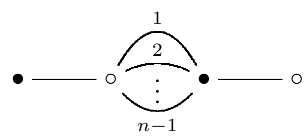


Figure 18: The dessin at the cusp of $L_{n,2}$

The $n-1$ cells of the dessin correspond to the fact that $L_{n,2}$ has $n-1$ different vertices: one of order 3 and $n-2$ of order 1. Therefore the two vertices of order one of the dessin lie in the same cell.

Note that there is only one dessin of genus 0 with these properties, namely two vertices of order n and two vertices of order one, which are in the same cell. Hence our dessin is completely determined by its Galois invariants. This implies in particular that the moduli field of the dessin is $\mathbb{Q}$.

It is also possible to determine explicitly the associated Belyi map: To simplify the calculation, we first exhibit a rational function with a zero and a pole of order n, and in addition a simple zero and a simple pole; later we shall change the roles of 1 and ∞ to get the proper Belyi map. So we begin with a rational function of the type

$$f_0(x) = x^n \cdot \frac{x-1}{x-c}.$$

The condition that f_0 has a further ramification point of order 3 implies that f_0' has a double zero somewhere. A straightforward calculation shows that this happens if and only if the parameter c has the value

$$c = c_n = \Big(\frac{n-1}{n+1}\Big)^2.$$

The corresponding ramification point is

$$v_n = \sqrt{c_n} = \frac{n-1}{n+1}.$$

Since we want the critical value in this point to be 1, we have to replace f_0 by

$$f_1(x) = b_n^{-1} \cdot f_0(x) \qquad \text{with} \qquad b_n = -\Big(\frac{n-1}{n+1}\Big)^{n-1} = -c^{\frac{n-1}{2}}.$$

Now we interchange 1 and ∞ (keeping 0 fixed); to give the final function a nicer form, we bring the zeroes to 0 and ∞, and the places where the value 1 is taken to 1 and a forth point which is determined by the cross ratio of the zeroes and poles of f_1, i. e. 0, 1, ∞ and c; it turns out to be $d_n = 1 - \frac{1}{c_n}$. Altogether we replace f_1 by

$$f_n = \beta \circ f_1 \circ \sigma, \quad \text{where} \quad \beta(x) = \frac{x}{x-1} \quad \text{and} \quad \sigma(x) = \frac{x}{x-d_n}.$$

The final result is

$$f_n(x) = \frac{\gamma_n x^n}{\gamma_n x^n - (x-d_n)^n(x-1)}, \quad \text{with} \quad \gamma_n = \Big(\frac{n+1}{n-1}\Big)^{n+1}.$$

By construction, f_n has a triple pole; it turns out to be $p_n = \frac{2n}{n-1}$. Putting in the values of the constants we find e. g.

$$f_2(x) = \frac{-27x^2}{(x-4)^3} \quad \text{and} \quad f_3(x) = \frac{-16x^3}{(x-3)^3(x+1)}.$$

8.3.3 The quaternion origami Let W be the quaternion origami which was illustrated at the end of Section 8.1 and studied in detail in [15]. It has genus 3, and the origami map $p : W \to E$ is a normal covering of degree 8 with Galois group Q_8, the classical quaternion group. Its Veech group is $\mathrm{SL}_2(\mathbb{Z})$, which implies that the origami curve $C(W)$ in $\overline{M}_3$ has only one cusp. As indicated in Figure 12 this cusp corresponds to a stable curve W_∞ with two irreducible components, both nonsingular of genus 1; the components intersect transversely in two points. Both components of W_∞ admit an automorphism of order 4 and are therefore isomorphic to the elliptic curve E_{-1} with Weierstrass equation $y^2 = x^3 - x$. The normalisation of W_∞ then consists of two copies of E_{-1}. On each of them, p induces a Belyi map $f : E_{-1} \to \mathbb{P}^1(\mathbb{C})$ of degree 4, which is totally ramified over the 2 points that map to the node of E_∞ (these are the points of intersection with the other component). Over ∞ we have two points on E_{-1}, both ramified of order 2.

Thus the corresponding dessin d'enfants is

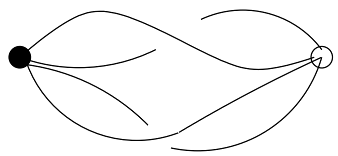


Figure 19: The dessin of the boundary point of $C(W)$.

The Belyi map in this case is a quotient map for the automorphism c of order 4, which acts by $(x, y) \mapsto (-x, iy)$. Such a quotient map is $(x, y) \mapsto x^2$; it is ramified in the four 2-torsion points of E_{-1}: two of them are the fixed points of c, the other two are exchanged by c. The critical values are 0, 1 and ∞, but not in the right order: To have the values 0 and 1 in the fixed points of c we have to change the roles of 1 and ∞ in $\mathbb{P}^1(\mathbb{C})$, and then obtain the Belyi map $f : E_{-1} \to \mathbb{P}^1(\mathbb{C})$ as

$$f(x, y) = \frac{x^2}{x^2 - 1}$$

or, in homogeneous coordinates,

$$f(x : y : z) = (x^2 : x^2 - z^2) = (y^2 + xz : y^2).$$

8.3.4 The characteristic origami of order 108 Our last example in this section is the origami B with 108 squares which corresponds to a normal origami covering $p : B \to E$ with Galois group

$$G = \{(\sigma_1, \sigma_2, \sigma_3) \in S_3 \times S_3 \times S_3 : \prod_{i=1}^{3} \operatorname{sign}(\sigma_i) = 1\}.$$

As for W in the previous section, the Veech group of B is $\mathrm{SL}_2(\mathbb{Z})$. It was the first normal origami of genus > 1 that was discovered to have the full group $\mathrm{SL}_2(\mathbb{Z})$ as Veech group. It is studied in detail in [2] and also (more shortly) in [13].

The genus of B is 37; the horizontal cylinders all have length 6. Contracting their core lines gives a stable curve B_∞ with 6 irreducible components, each nonsingular of genus 4. Each of the irreducible components intersects three others in two points each. The intersection graph of B_∞ is

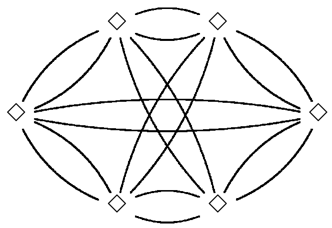


Figure 20: The intersection graph of the 108 origami.

Since the group G acts transitively on the irreducible components of B_∞, they are all isomorphic. Let us denote by C one of them. The stabiliser of C in G is a subgroup H of order 18. The quotient map $f : C \to C/H = \mathbb{P}^1(\mathbb{C})$ is the Belyi map corresponding to this (unique) cusp of the origami curve $C(B)$.

The ramification of f over ∞ comes from the fixed points of the elements of H. There are two different subgroups of order 3 that have 3 fixed points each, and no other fixed points. The other ramification points lie over the two points in $\mathbb{P}^1(\mathbb{C})$, that are mapped to the node of E_∞. Hence they are the 6 points where C meets other components, and each of them has ramification order 6. These considerations show by the way that the genus of C is in fact 4, since by Riemann-Hurwitz we have

$$2g - 2 = 18 \cdot (-2) + 6 \cdot (6 - 1) + 6 \cdot (3 - 1) = -36 + 42 = 6.$$

On the original origami B, the component C corresponds to 36 half squares. The 18 upper halves among them are the lower halves of three horizontal cylinders, and in the same way, the 18 lower halves contributing to C are the upper halves of three other cylinders. The core lines of these six cylinders give the six ramification points of f that lie over 0 and 1. The precise picture looks as follows:

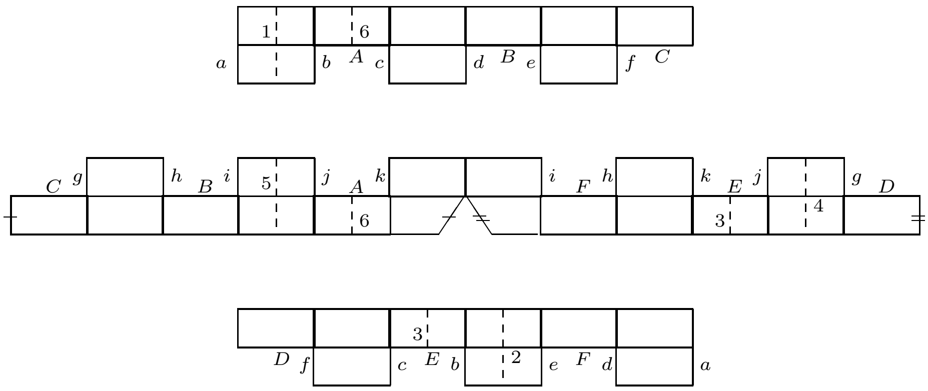


Figure 21: The 36 half squares of which a component of the curve B_∞ is composed. Vertical gluings are indicated by capital letters, horizontal gluings by small letters. The dashed lines are, in the order $1, \ldots, 6$*, the boundary of one of the six cells of the dessin.*

In each row of the figure, the upper horizontal edges give one vertex of the dessin (corresponding to a point lying over 0). The lower edges of the second row give two vertices over 1, and the third comes from the six lower edges in the first and the last row.

The 18 edges of the dessin are vertical centre lines of the squares; some of them are shown in the figure. Each of the three "upper" vertices is connected to two of the "lower" vertices by three edges each, and not connected to other vertices. The order in which the edges leave the vertices is determined by the horizontal gluing of the squares.

One way of describing the resulting dessin d'enfants is to consider its cells and their gluing. Since $f^{-1}(\infty)$ consists of 6 points of ramification order 6, our dessin has 6 cells, and each of them is a hexagon. In the origami, these hexagons are found as follows: begin with an arbitrary edge (i. e. a vertical centre line of a square); at its end point, go one square to the right and continue with the edge that starts at its centre. Go on like this until you reach the first edge again. The figure shows one example for this. Note that the 6 vertices of this hexagon are all different. By symmetry this holds for all 6 hexagons. The way how these hexagons have to be glued can be read off from the origami. Thus finally we find the following dessin, in which, as in the pictures of origamis, edges with the same label have to be glued:

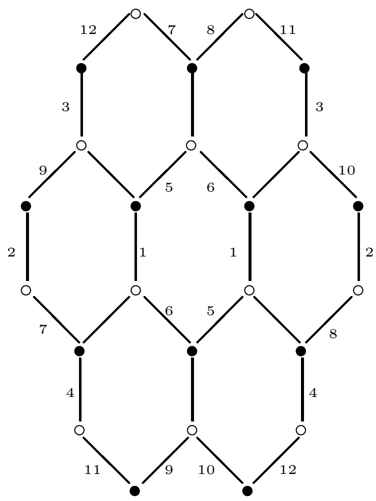


Figure 22: The dessin d'enfants to the cusp of the 108 origami. The surface consists of the six outer hexagons, with edges glued as indicated by the labels.

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