

On the boundary of Teichmüller disks in Teichmüller and in Schottky space

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Abstract

We study the boundary of Teichmüller disks in \overline{T}_g , a partial compactification of Teichmüller space, and their image in Schottky space.

We give a broad introduction to Teichmüller disks and explain the relation between Teichmüller curves and Veech groups.

Furthermore, we describe Braungardt's construction of \overline{T}_g and compare it with the Abikoff augmented Teichmüller space. Following Masur, we give a description of Strebel rays that makes it easy to understand their end points on the boundary of \overline{T}_g . This prepares the description of boundary points that a Teichmüller disk has, with a particular emphasis to the case that it leads to a Teichmüller curve.

Further on we turn to Schottky space and describe two different approaches to obtain a partial compactification. We give an overview how the boundaries of Schottky space, Teichmüller space and moduli space match together and how the actions of the diverse groups on them are linked. Finally we consider the image of Teichmüller disks in Schottky space and show that one can choose the projection from Teichmüller space to Schottky space in such a manner that the image of the Teichmüller disk is a quotient by an infinite group.

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1 Introduction

One of the original motivations that led to the discovery of Teichmüller space was to better understand the classification of Riemann surfaces. Riemann himself already saw that the compact Riemann surfaces of genus g with n marked points on it depend on $3g - 3 + n$ complex parameters (if this number is positive). More precisely, there is a complex analytic space $M_{g,n}$ whose points correspond in a natural way to the isomorphism classes of such Riemann surfaces. $M_{g,n}$ is even an algebraic variety, but its geometry is not easy to understand. Most of the basic properties are known today, but many questions on the finer structure of $M_{g,n}$ are still open.¹

Many classification problems become more accessible if the objects are endowed with an additional structure or *marking*. The general strategy is to first classify the marked objects and then, in a second step, to try to understand the equivalence relation that forgets the marking. The markings that Teichmüller introduced for a compact Riemann surface X consist of orientation preserving diffeomorphisms $f : X_{\text{ref}} \rightarrow X$ from a reference Riemann surface X_{ref} to X . Markings (X, f) and (X', f') are considered the same if $f' \circ f^{-1}$ is homotopic to a biholomorphic map. Thus different markings of a fixed Riemann surface differ by a homotopy class of diffeomorphisms of X_{ref} . In other words the *mapping class group* (or *Teichmüller modular group*)

$$\Gamma_g = \text{Diffeo}^+(X_{\text{ref}})/\text{Diffeo}^0(X_{\text{ref}}) \quad (1)$$

acts on the set T_g of all marked Riemann surfaces of genus g , and the orbit space T_g/Γ_g is equal to M_g (here $\text{Diffeo}^+(X_{\text{ref}})$ denotes the group of orientation preserving diffeomorphisms of X_{ref} and $\text{Diffeo}^0(X_{\text{ref}})$ the subgroup of those that are homotopic to the identity).

Teichmüller discovered that in each homotopy class of diffeomorphisms between compact Riemann surfaces X and X' there is a unique “extremal mapping”, i. e. a quasiconformal map with minimal dilatation. The logarithm of this dilatation puts a metric on T_g , the *Teichmüller metric*. With it T_g is a complete metric space, diffeomorphic to \mathbb{R}^{6g-6} , and Γ_g acts on T_g by isometries. There is also a structure as complex manifold on T_g , for which the elements of Γ_g act holomorphically and thus make the quotient map $T_g \rightarrow M_g$ into an analytic map between complex spaces.

That the complex structure on T_g is the “right one” for the classification problem can be seen from the fact that there is a family \mathcal{C}_g of Riemann surfaces over T_g which in a very precise sense is universal. This family can be obtained as follows: By the uniformization theorem, the universal covering of a compact Riemann surface X of genus $g \geq 2$ is (isomorphic to) the upper half plane \mathbb{H} .

¹Although we consider this general setting in a large part of this paper, we shall restrict ourselves in this introduction to the case $n = 0$ and write, as usual, M_g instead of $M_{g,0}$ (and later T_g instead of $T_{g,0}$).

Any marking $f : X_{\text{ref}} \rightarrow X$ induces an isomorphism f_* from $\pi_g = \pi_1(X_{\text{ref}})$, the fundamental group of the reference surface, to $\pi_1(X)$. We may obtain a holomorphic action of π_g on $T_g \times \mathbb{H}$ as follows: for $\gamma \in \Gamma_g$, $x = (X, f) \in T_g$ and $z \in \mathbb{H}$ put

$$\gamma(x, z) = (x, f_*(\gamma)(z)),$$

where we identify $\pi_1(X)$ with the group of deck transformations of the universal covering $\mathbb{H} \rightarrow X$. The quotient $\mathcal{C}_g = (T_g \times \mathbb{H})/\pi_g$ is a complex manifold with a natural projection $p : \mathcal{C}_g \rightarrow T_g$; the fibre $p^{-1}(X, f)$ is isomorphic to X . Moreover p is proper and therefore $p : \mathcal{C}_g \rightarrow T_g$ is a *family of Riemann surfaces*. The representation of \mathcal{C}_g as a quotient of a manifold by an action of π_g is called a *Teichmüller structure* on this family. It follows from results of Bers on the uniformization of families (see e. g. [4, Thm. XVII]) that this family is *universal*, i. e. every other family of Riemann surfaces of genus g with a Teichmüller structure can be obtained as a pullback from $p : \mathcal{C}_g \rightarrow T_g$. In a more fancy language: T_g is a fine moduli space for Riemann surfaces of genus g with Teichmüller structure.

It follows by the same arguments that for any family $\pi : \mathcal{C} \rightarrow S$ of Riemann surfaces (over some complex space S) there is an analytic map $\mu = \mu_\pi : S \rightarrow M_g$, which maps $s \in S$ to the point in M_g that corresponds to the isomorphism class of the fibre $\pi^{-1}(s)$. Unfortunately, Γ_g does not act freely on T_g ; therefore the quotient of \mathcal{C}_g by the action of Γ_g does not give a universal family over M_g : the fixed points of elements in Γ_g correspond to automorphisms of the Riemann surface, and the fibre over $[X] \in M_g$ in the family $\mathcal{C}_g/\Gamma_g \rightarrow M_g$ is the Riemann surface $X/\text{Aut}(X)$ (whose genus is strictly less than g if $\text{Aut}(X)$ is nontrivial). As a consequence, M_g is not a fine moduli space for Riemann surfaces, but only a “coarse” one (see e. g. [16, 1A] for a precise definition of fine and coarse moduli spaces).

There are several equivalent ways to define markings of Riemann surfaces and to describe Teichmüller space. Instead of classes of diffeomorphisms $f : X_{\text{ref}} \rightarrow X$ often conjugacy classes of group isomorphisms $\pi_g \rightarrow \pi_1(X)$ are used as markings. For the purpose of this paper the approach to Teichmüller space via Teichmüller deformations is very well suited; it is developed in Section 2.1. The starting point is the observation that a holomorphic quadratic differential q on a Riemann surface X defines a flat structure μ on $X^* = X - \{\text{zeroes of } q\}$. Composing the chart maps of μ with a certain (real) affine map yields a new point in T_g . Any point in T_g is in a unique way such a Teichmüller deformation of a given base point $(X_{\text{ref}}, \text{id})$, cf. Section 2.2.

The main objects of interest in this article are *Teichmüller embeddings*, i. e. holomorphic isometric embeddings $\iota : \mathbb{H} \rightarrow T_g$ (or $\iota : \mathbb{D} \rightarrow T_g$), where \mathbb{H} (resp. \mathbb{D}) is given the hyperbolic metric and T_g the Teichmüller metric, see Definition 2.4. The restriction of ι to a hyperbolic geodesic line in \mathbb{H} (or \mathbb{D}) is then a (real) geodesic line in T_g in the usual sense. The image Δ_ι of such an embedding ι is called a *Teichmüller geodesic* or *Teichmüller disk* in

T_g . There are plenty of Teichmüller disks in T_g . To see this note first that the tangent space to T_g at a point $x = (X, f) \in T_g$ is naturally isomorphic to the vector space $Q_X = H^0(X, \Omega_X^{\otimes 2})$ of holomorphic quadratic differentials on X (this results from the Bers embedding of T_g as a bounded open subdomain of Q_X). We shall explain in Section 2.3 in three different ways how one can, for a given holomorphic quadratic differential q on a Riemann surface X , construct a Teichmüller embedding $\iota : \mathbb{D} \rightarrow T_g$ with $\iota(0) = x$ and $\iota'(0) = q$. This shows that for any $x \in T_g$ and any (complex) tangent vector at x there is a Teichmüller disk passing through x in direction of the given tangent vector.

There are several natural and closely related objects attached to a Teichmüller disk Δ_ι (or a Teichmüller embedding $\iota : \mathbb{H} \rightarrow T_g$): The first is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ called the (*projective*) *Veech group* $\bar{\Gamma}_\iota$, cf. Section 2.4.1. If q is the quadratic differential on the Riemann surface X by which ι is induced, $\bar{\Gamma}_\iota$ consists of the derivatives of those diffeomorphisms of X that are affine with respect to the flat structure defined by q . Veech showed that this subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is always discrete ([34, Prop. 2.7]).

A second group naturally attached to ι is the *stabilizer*

$$\mathrm{Stab}(\Delta_\iota) = \{\varphi \in \Gamma_g : \varphi(\Delta_\iota) = \Delta_\iota\}$$

of Δ_ι in the Teichmüller modular group. The pointwise stabilizer

$$\mathrm{Stab}^0(\Delta_\iota) = \{\varphi \in \Gamma_g : \varphi|_{\Delta_\iota} = \mathrm{id}_{\Delta_\iota}\}$$

is a finite subgroup of $\mathrm{Stab}(\Delta_\iota)$, and $\mathrm{Stab}(\Delta_\iota)/\mathrm{Stab}^0(\Delta_\iota)$ then is (via ι) a group of isometries of \mathbb{H} and thus a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. This subgroup coincides with the projective Veech group $\bar{\Gamma}_\iota$, see Section 2.4.3.

Given a Teichmüller embedding ι we are also interested in the image C_ι of Δ_ι in the moduli space M_g . The map $\Delta_\iota \rightarrow C_\iota$ obviously factors through the Riemann surface $\mathbb{H}/\bar{\Gamma}_\iota$ or rather through its mirror image V_ι , see Section 2.4 and in particular 2.4.3. The typical case seems to be $\mathrm{Stab}(\Delta_\iota) = \{\mathrm{id}\}$ (although it is not trivial to give explicit examples). Much attention has been given in recent years to the other extreme case that $\bar{\Gamma}_\iota$ is a lattice in $\mathrm{PSL}_2(\mathbb{R})$. Then V_ι is of finite hyperbolic volume and hence a Riemann surface of finite type, or equivalently an algebraic curve. In this case the induced map $V_\iota \rightarrow C_\iota$ is birational (see [11]), i. e. V_ι is the desingularization (or normalization) of C_ι . It follows from a result of Veech ([34]) that V_ι (and hence also C_ι) cannot be projective. If $\bar{\Gamma}_\iota$ is a lattice, the affine curve C_ι is called a *Teichmüller curve*, cf. Sect. 2.4. First examples were given by Veech [34]; in them, $\bar{\Gamma}_\iota$ is a hyperbolic triangle group. Later more examples with triangle groups as Veech groups were found, see [20] for a comprehensive overview and [6] for recent results. Explicit examples for Teichmüller curves also with non triangle groups as Veech groups can be found e. g. in [27], [9] and [23]. Möller has shown ([28]) that every Teichmüller curve is, as a subvariety of M_g , defined over a number field. This implies that there are at most countably many Teichmüller curves.

A special class of Teichmüller curves is obtained by *origamis* (or *square-tiled surfaces*). They arise from finite coverings of an elliptic curve that are ramified over only one point. Given such a covering $p : X \rightarrow E$, the quadratic differential $q = (p^*\omega_E)^2$ (where ω_E is the invariant holomorphic differential on E) induces a Teichmüller embedding whose Veech group is commensurable to $\mathrm{SL}_2(\mathbb{Z})$, see [15]. Lochak proposed in [23] a combinatorial construction for such coverings (which led to the name “origami”), and Schmithüsen [30] gave a group theoretic characterization of the Veech group. In [31], origamis and their Veech groups are systematically studied and numerous examples are presented. Origamis in genus 2 where q has one zero are classified in [19]. Using the description of origamis by gluing squares it is not difficult to see that there are, for any $g \geq 2$, infinitely many Teichmüller curves in M_g that come from origamis. In genus 3 there is even an explicit example of an origami curve that is intersected by infinitely many others, see [18].

We want to study boundary points of Teichmüller disks and Teichmüller curves; by this we mean, for a Teichmüller embedding $\iota : \mathbb{H} \rightarrow T_g$, the closures of Δ_ι and C_ι in suitable (partial) compactifications of T_g and M_g , respectively. For the moduli space we shall use the compactification \overline{M}_g by stable Riemann surfaces. Here a one-dimensional connected compact complex space X is called a *stable Riemann surface* if all singular points of X are ordinary double points, i. e. have a neighbourhood isomorphic to $\{(z, w) \in \mathbb{C}^2 : z \cdot w = 0, |z| < 1, |w| < 1\}$; moreover we require that every irreducible component L of X that is isomorphic to the projective line $\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ intersects $\overline{X} - L$ in at least three points. It was shown by Deligne and Mumford ([10]) that stable Riemann surfaces are classified by an irreducible compact variety \overline{M}_g that, like M_g , has the quality of a coarse moduli space. In fact, with the approach of Deligne-Mumford it is possible to classify stable algebraic curves over an arbitrary ground field: they construct a proper scheme over \mathbb{Z} of which \overline{M}_g is the set of complex-valued points. Some years later, Knudsen [22] showed that \overline{M}_g is a projective variety.

If $\iota : \mathbb{H} \rightarrow T_g$ is a Teichmüller embedding such that C_ι is a Teichmüller curve, the closure \overline{C}_ι of C_ι in \overline{M}_g is Zariski closed and therefore a projective curve. In particular, $\overline{C}_\iota - C_\iota$ consists of finitely many points, called the *cusps* of C_ι . It is very interesting to know, for a given Teichmüller curve C_ι , the number of cusps and the stable Riemann surfaces that correspond to the cusps. In the case that ι is induced by an origami there is an algorithm that determines (among other information) the precise number of cusps of C_ι , see [30].

The boundary $\partial M_g = \overline{M}_g - M_g$ is a divisor, i. e. a projective subvariety of (complex) codimension 1. It has irreducible components $D_0, D_1, \dots, D_{[\frac{g}{2}]}$; the points in D_0 correspond to irreducible stable Riemann surfaces with a double point, while for $i = 1, \dots, [\frac{g}{2}]$, D_i classifies stable Riemann surfaces consisting of two nonsingular irreducible components that intersect transversally, one of genus i and the other of genus $g - i$. The combinatorial structure of the intersections of the D_i is best described in terms of the *intersection graph*: For

a stable Riemann surface X , we define a graph $\Gamma(X)$ as follows: the vertices of $\Gamma(X)$ are the irreducible components of X , the edges are the double points (connecting two irreducible components of X which need not be distinct). For every graph Γ let $\overline{M}_g(\Gamma)$ be the set of points in \overline{M}_g corresponding to stable Riemann surfaces with intersection graph isomorphic to Γ . It is not hard to see that for a given genus g , there are only finitely many graphs Γ with nonempty $\overline{M}_g(\Gamma)$, and that the $\overline{M}_g(\Gamma)$ are the strata of a stratification of \overline{M}_g . This means that each $\overline{M}_g(\Gamma)$ is a locally closed subset of \overline{M}_g (for the Zariski topology), that \overline{M}_g is the disjoint union of the $\overline{M}_g(\Gamma)$, and that the closure of each $\overline{M}_g(\Gamma)$ is a finite union of other $\overline{M}_g(\Gamma')$. A natural question in our context is: which $\overline{M}_g(\Gamma)$ contain cusps of Teichmüller curves? In [24] Maier showed that if Γ has no “bridge”, i. e. no edge e such that $\Gamma - e$ is disconnected, the stratum $\overline{M}_g(\Gamma)$ contains points on a compactified Teichmüller curve \overline{C}_ι with a Teichmüller embedding ι that corresponds to an origami. Möller and Schmithüsen observed that this condition on the graph is necessary if the Teichmüller curve comes from a quadratic differential which is the square of a holomorphic 1-form (or equivalently from a translation structure on X^*).

Most of our knowledge about cusps of Teichmüller curves comes from studying boundary points of Teichmüller disks in a suitable extension of Teichmüller space. Several different boundaries for Teichmüller space with very different properties have been studied, like the Thurston boundary or the one coming from the Bers embedding. In the framework of this paper we look for a space \overline{T}_g in which T_g is open and dense such that the action of the group Γ_g extends to an action on \overline{T}_g , and the quotient space \overline{T}_g/Γ_g is equal to \overline{M}_g . Such a space is the “augmented” Teichmüller space \hat{T}_g introduced by Abikoff [1]. The points in \hat{T}_g are equivalence classes of pairs (X, f) , where X is a stable Riemann surface of genus g and $f : X_{\text{ref}} \rightarrow X$ is a *deformation*. This is a continuous surjective map such that there are finitely many loops c_1, \dots, c_k on X_{ref} with the property that f is a homeomorphism outside the c_i and maps each c_i to a single point P_i on X . Abikoff defined a topology on this space and showed that the quotient for the natural action of Γ_g on the pairs (X, f) is the moduli space \overline{M}_g as a topological space.

In his thesis [8], Braungardt introduced the concept of a covering of a complex manifold S with cusps over a divisor D . He showed that under mild assumptions on S there exists a universal covering \tilde{X} of this type which extends the usual holomorphic universal covering of $S - D$ by attaching “cusps” over D . \tilde{X} is no longer a complex manifold or a complex space, but Braungardt introduced a natural notion of holomorphic functions in a neighbourhood of a cusp and thus defined a sheaf $\mathcal{O}_{\tilde{X}}$ of rings (of holomorphic functions) on \tilde{X} . In this way \tilde{X} is a locally complex ringed space, and the quotient map $\tilde{X} \rightarrow \tilde{X}/\pi_1(S - D) = S$ is analytic for this structure. When applied to $S = \overline{M}_g$ and $D = \partial M_g$, Braungardt showed that the universal covering \overline{T}_g of \overline{M}_g with cusps over ∂M_g is, as a topological space with an action of Γ_g , homeomorphic to Abikoff’s augmented Teichmüller space. We shall reserve the symbol \overline{T}_g in

this article exclusively for this space (considered as a locally ringed space). In Chapter 3, we review Braungardt’s construction and results.

Our key technique to investigate boundary points of Teichmüller disks is the use of *Strebel rays*, see Definition 4.4. By this we mean a geodesic ray in T_g that corresponds by the construction in Section 2.2 to a Strebel quadratic differential on the Riemann surface X at the starting point of the ray. A Strebel differential decomposes X into cylinders swept out by horizontal trajectories. Mainly following [26] we give in Section 4.1 two explicit descriptions of the marked Riemann surfaces (X_K, f_K) (for $K > 1$) on a Strebel ray. This allows us to identify the boundary point (X_∞, f_∞) at the “end” of the ray as the stable Riemann surface that is obtained by contracting on X the core lines of the cylinders in a prescribed way, see Sections 4.1.5 and 4.1.6.

In the case that the Teichmüller embedding ι leads to a Teichmüller curve C_ι we show in Section 4.2 that all boundary points of Δ_ι are obtained in this way. This shows in particular that all cusps of Teichmüller curves are obtained by contracting, on a corresponding Riemann surface, the center lines of the cylinders of a Strebel differential. For the proof of this result we show that the Teichmüller embedding ι can be extended to a continuous embedding $\bar{\iota} : \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}_\iota^*\} \hookrightarrow \bar{T}_g$, see Prop. 4.13. Moreover, if the Veech group $\bar{\Gamma}_\iota$ is a lattice in $\text{PSL}_2(\mathbb{R})$, the image of $\bar{\iota}$ is the closure of Δ_ι in \bar{T}_g , see Prop. 4.14.

Since \bar{T}_g has these cusp singularities at the boundary that prevent it from being an ordinary complex space, whereas the boundary of \bar{M}_g is a nice divisor in a projective variety, it is interesting to look at spaces that lie properly between Teichmüller and moduli space and to ask for a boundary that fits somehow in between \bar{T}_g and \bar{M}_g . An example of such a space is provided by the *Schottky space* which goes back to the paper [32] of F. Schottky from 1887. He studied discontinuous groups that are freely generated by Möbius transformations $\gamma_1, \dots, \gamma_g$ (for some $g \geq 1$) chosen in such a way that there are disjoint closed Jordan domains $D_1, D'_1, \dots, D_g, D'_g$ such that γ_i maps D_i onto the complement of the interior of D'_i . The Riemann surface of such a *Schottky group* is compact of genus g . It can be shown that every compact Riemann surface X admits such a *Schottky uniformization* $X = \Omega/\Gamma$ (with $\Omega \subset \mathbb{P}^1(\mathbb{C})$ open and Γ a Schottky group), see Section 5.1. The covering $\Omega \rightarrow X$ is called a *Schottky covering*. It is minimal for the property that Ω is planar, i. e. biholomorphic to a subdomain of $\mathbb{P}^1(\mathbb{C})$; here minimality means that each unramified holomorphic covering $Y \rightarrow X$ with a planar manifold Y factors through Ω .

Schottky coverings are classified by a complex manifold S_g of dimension $3g - 3$, called the Schottky space. The natural map from T_g to M_g factors through S_g , therefore there is a subgroup $\Gamma_g(\alpha)$ of the mapping class group Γ_g such that $T_g/\Gamma_g(\alpha) = S_g$. Unfortunately the subgroup $\Gamma_g(\alpha)$ is not normal and depends on the choice of a certain group homomorphism α . As a consequence the induced map $S_g \rightarrow M_g$ is not the quotient for a group action. We

review this classical but not so widely known material in Sections 5.1 and 5.2.

The concept of Schottky coverings can be extended to stable Riemann surfaces. If the analogous construction as for ordinary Riemann surfaces is applied to a surface X with nodes, we obtain a covering space Ω which is not planar, but on which nevertheless a free group Γ acts by holomorphic automorphisms with quotient space $\Omega/\Gamma = X$. Although the groups Γ are no longer subgroups of $\mathrm{PSL}_2(\mathbb{C})$, it is possible to find parameters for them in almost the same way as for Schottky groups, namely by cross ratios of fixed points. It then turns out that these generalized Schottky coverings are classified by a complex manifold \overline{S}_g (which contains S_g as an open dense subset), see Section 5.3. This result was originally proved in [13]; here we show that it can easily be derived from Braungardt's characterization of \overline{T}_g as the universal covering of \overline{M}_g with cusps over ∂M_g , see Section 5.4.

Finally we wonder what the image in S_g of a Teichmüller disk Δ_ι in T_g might look like. In the general case we have no idea. Of course, the image may depend on the choice of the subgroup $\Gamma_g(\alpha)$ that gives the map $T_g \rightarrow S_g$. In the special situation that C_ι is a Teichmüller curve we prove that for suitable choice of α , the image of Δ_ι in S_g is not a disk, see Prop. 5.21.

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2 Geodesic rays, Teichmüller disks and Teichmüller curves

The aim of this section is to introduce Teichmüller disks and Teichmüller curves. We start by recalling in 2.1 the concept of Teichmüller deformations and using them we give a definition for the Teichmüller space T_g alternative to the one we gave in the introduction. This will help us to define geodesic rays in the Teichmüller space in 2.2. In 2.3 we introduce Teichmüller disks as complex version of geodesic rays giving different alternative definitions. Finally in 2.4 we introduce the Veech group and Teichmüller curves and summarize some facts about the interrelation between these objects.

2.1 Teichmüller deformations

As one of numerous possibilities, one can define the Teichmüller space as the space of Teichmüller deformations. We briefly recall this concept here. It is described e.g. in [2, Ch I, §3].

At the end of this subsection we extend it to the corresponding concept for punctured Riemann surfaces and their Teichmüller space $T_{g,n}$, cf. [2, Chapter II, §1].

Let $X = X_{\text{ref}}$ be a fixed Riemann surface of genus $g \geq 2$ and q be a holomorphic quadratic differential on X . We refer to the zeros of q as *critical points*, all other points are *regular points*. Then on the surface

$$X^* = X - \{P \in X \mid P \text{ is a critical point of } q\}$$

the differential q naturally defines a *flat structure* μ , i.e. an atlas such that all transition maps are of the form $z \mapsto \pm z + c$, with some constant $c \in \mathbb{C}$. The charts of μ in regular points z_0 are given as

$$z \mapsto \int_{z_0}^z \sqrt{q(\xi)} d\xi. \quad (2)$$

One may deform this flat structure by composing each chart with the map

$$x + iy \mapsto Kx + iy = \frac{1}{2}(K+1)z + \frac{1}{2}(K-1)\bar{z}, \quad (x, y \in \mathbb{R}) \quad (3)$$

with K an arbitrary real number > 1 . This defines a new flat structure on X^* which can uniquely be extended to a holomorphic structure on X .

We call X_K the Riemann surface that we obtain this way, $X_1 = X$ the surface with the original complex structure and $f_K : X_1 \rightarrow X_K$ the map that is topologically the identity. The map f_K is a Teichmüller map and has constant complex dilatation

$$k(z) = \frac{(f_K)_{\bar{z}}}{(f_K)_z} = \frac{K-1}{K+1}.$$

Its maximal dilatation $\sup_{z \in X} \frac{1+|k(z)|}{1-|k(z)|}$ (as a quasiconformal map) is equal to K .

Definition 2.1. Let q be a holomorphic quadratic differential on X and $K \in \mathbb{R}_{>1}$. The pair (X_K, f_K) as defined above is called the Teichmüller deformation of X of constant dilatation K with respect to q .

The pair (X_K, f_K) defines a point in the Teichmüller space T_g which for simplicity we also denote as (X_K, f_K) . Since the constant dilatation of f_K is equal to K , the Teichmüller distance between the points (X_1, id) and (X_K, f_K) of T_g is $\log(K)$.

If two holomorphic quadratic differentials on X are positive scalar multiples of each other, they define, for each K , the same point in T_g . Thus one restricts to differentials with norm 1. By Teichmüller's existence and uniqueness theorems, see e.g. [2, Chapter I, (3.5), (4.2)], one can show that each point in T_g is uniquely obtained as a Teichmüller deformation. If Q_X is the vector space of all holomorphic quadratic differentials on X and if Σ_X is the unit sphere in Q_X , one may thus write

$$\{(X, q, k) \mid q \in \Sigma_X, k \in (0, 1) \cup \{0\}\} = T_g. \quad (4)$$

and the identification of the two sets is done by the map:

$$\begin{aligned} (X, q, k) &\mapsto (X_K, f_K) \quad \text{with } K = \frac{1+k}{1-k} \Leftrightarrow k = \frac{K-1}{K+1} \quad \text{and} \\ 0 &\mapsto \text{the base point } (X, \text{id}) \end{aligned} \quad (5)$$

(X_K, f_K) depends of course by its definition on the differential q . In the following we will denote the base point also by $(X, q, 0)$.

The map (5) is a homeomorphism. Here on the left hand side of (4) one takes the topology obtained by identifying it with the open unit ball in Q_X . It follows in particular, that T_g is contractible.

Teichmüller deformations can be understood as *affine deformations* in the following sense: Let us here and in the rest of the article identify \mathbb{C} with \mathbb{R}^2 by the \mathbb{R} -linear map sending $(1, i)$ to the standard basis of \mathbb{R}^2 . Then the map in (3) is equal to the affine map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Since composing charts with a biholomorphic map does not change the point in Teichmüller space, one obtains the same point (X_K, f_K) in T_g if one composes each chart of the flat structure μ on X with the affine map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto D_K \cdot \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with } D_K = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \quad (6)$$

We will use the following notations which are compatible with those in Section 2.3.2 where we introduce the general concept of affine deformations.

Definition 2.2. Let X be a compact Riemann surface of genus g , q a holomorphic quadratic differential, μ the flat structure defined by q . We call the flat structure defined by (6) μ_{D_K} and denote $(X, \mu) \circ D_K = (X, \mu_{D_K})$.

Note that (X, μ_{D_K}) is as Riemann surface isomorphic to X_K . Thus the point $[(X, \mu_{D_K}), \text{id}]$ in T_g defined by the marking $\text{id} : X \rightarrow (X, \mu_{D_K})$ is equal to (X_K, f_K) .

Finally, let us turn to *Teichmüller deformations of punctured Riemann surfaces*: The definition is done almost in the same way as in the case without punctures, see [2, Chapter II, §1]. Suppose that g and n are natural numbers with $3g - 3 + n > 0$. Let X be a Riemann surface of genus g with n marked points P_1, \dots, P_n , and $X_{\text{ref}} = X_0 = X - \{P_1, \dots, P_n\}$.

In this case, one uses *admissible holomorphic quadratic differentials* on X_0 . They are by definition those meromorphic quadratic differentials on X that restrict to a holomorphic quadratic differential on X_0 and have at each puncture either a simple pole or extend holomorphically across the puncture, see [2, Chapter II, (1.4)]. The vector space of these differentials is called Q_{X_0} . For $q \in Q_{X_0}$ we define the *critical points* to be the marked points and all zeroes of q ; the remaining points are called *regular*. Now, the definition of Teichmüller deformation is done exactly as before, just always replacing Q_X by Q_{X_0} . One obtains in the same way:

$$\{(X, q, k) \mid q \in \Sigma_{X_0}, k \in (0, 1) \} \cup \{0\} = T_{g,n}. \quad (7)$$

Here Σ_{X_0} is the unit ball in Q_{X_0} .

2.2 Geodesic rays

Let $X = X_{\text{ref}}$ be a Riemann surface of genus g . A holomorphic quadratic differential q on X naturally defines a geodesic embedding of $\mathbb{R}_{\geq 0}$ into T_g with respect to the Teichmüller metric on T_g as is described in the following.

Definition 2.3. Let q be a holomorphic quadratic differential on X and γ the map:

$$\gamma = \gamma_q : \begin{cases} [0, \infty) & \rightarrow T_g \\ t & \mapsto (X_K, f_K) = (X, \mu_{D_K}) = (X, q, k) \\ & \text{with } K = e^t, \quad k = \frac{K-1}{K+1} \end{cases} \quad (8)$$

The image of γ is called the geodesic ray in T_g in direction of q (or with respect to) q starting at (X, id) .

Here we use the notation of the last section:

$$(X_K, f_K) \stackrel{\text{Def. 2.2}}{=} (X, \mu_{D_K}) \stackrel{(5)}{=} (X, q, k)$$

is the point in T_g defined by the Teichmüller deformation of X of dilatation K with respect to q . Recall from the last section that the distance between

the two points (X_K, f_K) and (X, id) in T_g is $\log(K)$. Thus γ is an isometric embedding.

In fact, from the description of T_g given in (4) one observes that all points in T_g which have distance $\log(K)$ to the base point (X, id) are Teichmüller deformations of X of constant dilatation K with respect to a holomorphic quadratic differential. It follows that each isometric embedding of $[0, \infty)$ into T_g is of the form (8).

2.3 Teichmüller disks

In this section we define Teichmüller disks. They can be found defined under this name e.g. in [29, p. 149/150] and [12, 8.1-8.2]. One may find comprehensive overviews e. g. in [34] and [11], or more recently [27] and [23], to pick only a few of numerous references where they occur. We introduce them here in detail comparing three different ways how to construct them. For completeness we have included most of the proofs.

Definition 2.4. *Let $3g - 3 + n > 0$. A Teichmüller disk Δ_ι is the image of a holomorphic isometric embedding*

$$\iota : \mathbb{D} \hookrightarrow T_{g,n}$$

of the complex unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ into the Teichmüller space. Here we take the Poincaré metric of constant curvature -1 on \mathbb{D} and the Teichmüller metric on $T_{g,n}$. The embedding ι is also called Teichmüller embedding.

Instead of the unit disk \mathbb{D} one may take as well the upper half plane \mathbb{H} with the hyperbolic metric. We will switch between these two models using the holomorphic isometry

$$f : \mathbb{H} \rightarrow \mathbb{D}, \quad t \mapsto \frac{i-t}{i+t}. \quad (9)$$

Thus Teichmüller disks are obtained equivalently as images of holomorphic isometric embeddings $\mathbb{H} \hookrightarrow T_{g,n}$ of the upper half plane \mathbb{H} into the Teichmüller space $T_{g,n}$.

How does one find such embeddings? Similarly as for geodesic rays, each holomorphic quadratic differential q on a Riemann surface X defines a Teichmüller disk. In the following we describe three alternative constructions starting from such a differential q that all lead to the same Teichmüller disk Δ_q . For simplicity we only consider the case $n = 0$ and $g \geq 2$. However the same constructions can be done in the general case of punctured surfaces.

2.3.1 Teichmüller disks as a collection of geodesic rays

Definition 2.5. *Let q be a holomorphic quadratic differential on a Riemann surface X of genus g . Let ι_1 be the map*

$$\iota_1 : \begin{cases} \mathbb{D} & \rightarrow T_g \\ z = r \cdot e^{i\varphi} & \mapsto (X, e^{-i\varphi} \cdot q, r). \end{cases}$$

Here we use the definition of T_g given by (4). Hence, $(X, e^{-i\varphi} \cdot q, r)$ is the point defined by the Teichmüller deformation of X of dilatation $K = \frac{1+r}{1-r}$ with respect to $q_{-\varphi} = e^{-i\varphi} \cdot q$.

We shall show in Proposition 2.8 that ι_1 is an isometric holomorphic embedding, thus the image Δ_{ι_1} of ι_1 is a Teichmüller disk.

The map ι_1 may be considered as a collection of geodesic rays in the following sense: Let τ_φ be the geodesic ray in \mathbb{D} starting from 0 in direction φ , i.e.:

$$\tau_\varphi : \begin{cases} [0, \infty) & \rightarrow \mathbb{D} \\ t & \mapsto r(t) \cdot e^{i\varphi} \end{cases} \quad \text{with } r(t) = \frac{e^t - 1}{e^t + 1}$$

Then $\iota_1 \circ \tau_\varphi : [0, \infty) \rightarrow T_g$ is equal to the map given in (8) that defines the geodesic ray to the holomorphic quadratic differential $q_{-\varphi} = e^{-i\varphi} \cdot q$ on X . Thus the Teichmüller disk Δ_{ι_1} is the union of all geodesic rays defined by the differentials $e^{i\varphi} \cdot q$ with $\varphi \in [0, 2\pi)$. Furthermore, $\iota_1 \circ \tau_\varphi$ is the parameterization by length of the restriction $\iota_1|_{R_\varphi}$ of ι_1 to the ray $R_\varphi = \{r \cdot e^{i\varphi} \mid r \in [0, 1)\}$.

2.3.2 Teichmüller disks by affine deformations

We now describe a second approach that starting from a holomorphic quadratic differential q leads to the same Teichmüller disk as in 2.3.1.

Recall from Section 2.1 that a holomorphic quadratic differential q defines on $X^* = X - \{\text{zeroes of } q\}$ a flat structure μ . The group $\text{SL}_2(\mathbb{R})$ acts on the flat structures of X^* (as topological surface) in the following way: Let $B \in \text{SL}_2(\mathbb{R})$ and μ be a flat structure on X^* . Composing each chart of μ with the affine map $z \mapsto B \cdot z$ gives a new flat structure on X^* which we denote $B \circ (X, \mu)$ or (X, μ_B) . In the special case $B = D_K$ we obtain the Teichmüller deformation of dilatation K , cf. Definition 2.2.

Definition 2.6. *We call $(X, \mu_B) = B \circ (X, \mu)$ affine deformation of (X, μ) by the matrix B .*

Note that for B_1, B_2 in $\text{SL}_2(\mathbb{R})$ one may write

$$B_1 \circ (B_2 \circ (X, \mu)) = B_1 \circ (X, \mu_{B_2}) = (X, \mu_{B_1 B_2}) = B_1 \cdot B_2 \circ (X, \mu).$$

The flat structure μ_B defines in particular a complex structure on X . We identify here the complex plane \mathbb{C} with \mathbb{R}^2 as we already did in Section 2.1. In general the new complex structure will be different from the one defined by μ . Taking the identity $\text{id} : (X, \mu) \rightarrow (X, \mu_B)$ on X as marking, we obtain a point $P_B = [(X, \mu_B), \text{id}]$ in the Teichmüller space T_g . By abuse of notation we will sometimes denote this point also just as (X, μ_B) .

Thus one obtains the map

$$\hat{\iota}_2 : \text{SL}_2(\mathbb{R}) \rightarrow T_g, \quad B \mapsto P_B = [(X, \mu_B), \text{id}] = (X, \mu_B)$$

If however the matrix $A = U$ is in $\mathrm{SO}_2(\mathbb{R})$ the map $\mathrm{id} : (X, \mu_B) \rightarrow (X, \mu_{U \cdot B})$ is holomorphic, thus the point in Teichmüller space is not changed, i. e.

$$U \in \mathrm{SO}_2(\mathbb{R}) \Rightarrow P_{UA} = P_A \text{ for all } A \in \mathrm{SL}_2(\mathbb{R}) \quad (10)$$

Hence $\hat{\iota}_2$ induces a map

$$\iota_2 : \mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R}) \rightarrow T_g, \quad \mathrm{SO}_2(\mathbb{R}) \cdot B \mapsto P_B = [(X, \mu_B), \mathrm{id}] = (X, \mu_B).$$

Please note: The action of $\mathrm{SL}_2(\mathbb{R})$ on the flat structures $\{(X, \mu_A) \mid A \in \mathrm{SL}_2(\mathbb{R})\}$ does not descend to the image set $\{P_A \mid A \in \mathrm{SL}_2(\mathbb{R})\}$ in T_g ; in particular: $P_U = P_I \not\equiv P_{AU} = P_A$!

The Teichmüller disk:

One may identify $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$ with the upper half plane \mathbb{H} in the following way: Let $\mathrm{SL}_2(\mathbb{R})$ act by Möbius transformations on the upper half plane \mathbb{H} . This action is transitive and $\mathrm{SO}_2(\mathbb{R})$ is the stabilizing group of i . Thus the map

$$p : \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathbb{H}, \quad A \mapsto -\overline{A^{-1}(i)} \quad (11)$$

induces a bijection $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R}) \rightarrow \mathbb{H}$. Its inverse map is induced by

$$\mathbb{H} \rightarrow \mathrm{SL}_2(\mathbb{R}), \quad t \mapsto \frac{1}{\sqrt{\mathrm{Im}(t)}} \begin{pmatrix} 1 & \mathrm{Re}(t) \\ 0 & \mathrm{Im}(t) \end{pmatrix}$$

Composing ι_2 from above with this bijection one obtains a map from \mathbb{H} to T_g which we also denote by ι_2 .

Definition 2.7. *Let q be a holomorphic quadratic differential on the Riemann surface X and μ the flat structure defined by q . Let ι_2 be the map*

$$\iota_2 : \mathbb{H} \rightarrow T_g, \quad t \mapsto P_{A_t} = [(X, \mu_{A_t}), \mathrm{id}]$$

with A_t chosen such that $-\overline{A_t^{-1}(i)} = t$.

Note that the identification of $\mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$ with \mathbb{H} given by p may seem a bit ponderous, but one has to compose $A \mapsto A^{-1}(i)$ with the reflection at the imaginary axis in order that ι_2 becomes holomorphic. We will see this later in 2.3.3. In fact one has much more, as is stated in the following proposition.

Proposition 2.8. *The maps ι_1 and ι_2 are Teichmüller embeddings. They define the same Teichmüller disk*

$$\Delta_q = \Delta_{\iota_1} = \iota_1(\mathbb{D}) = \Delta_{\iota_2} = \iota_2(\mathbb{H}). \quad (12)$$

Proof. The proof is given in the rest of Subsection 2.3.2 and in 2.3.3:

In Proposition 2.12 we show that $\iota_2 = \iota_1 \circ f$ with f from (9) (see also Figure 1); thus it is sufficient to show only for one of them that it is isometric, and in the same manner for being holomorphic.

In Proposition 2.11 it is shown that ι_2 is isometric. In Subsection 2.3.3, it

is shown that ι_1 is holomorphic (see Corollary 2.15). For this purpose we introduce an embedding $\iota_3 : \mathbb{D} \rightarrow T_g$, using Beltrami differentials, for which it is not difficult to see that it is holomorphic, and show that it is equal to ι_1 . That ι_1 and ι_2 define the same Teichmüller disks then also follows also from Proposition 2.12. \square

In fact the described constructions do not only give some special examples but all Teichmüller disks are obtained as follows: Each Teichmüller disk is equal to Δ_q as in (12) for some holomorphic quadratic differential q . And all Teichmüller embeddings are of the form $\iota_1 : \mathbb{D} \hookrightarrow T_g$ or equivalently $\iota_2 : \mathbb{H} \hookrightarrow T_g$, see [12, 7.4].

In order to see that ι_2 from Definition 2.7 is isometric we first calculate the Teichmüller distance between two affine deformations.

Teichmüller distance between two affine deformations:

In what follows we will constantly use the following fact about matrices in $\mathrm{SL}_2(\mathbb{R})$:

Remark 2.9. *Each matrix $A \in \mathrm{SL}_2(\mathbb{R})$ with $A \notin \mathrm{SO}_2(\mathbb{R})$ can be decomposed uniquely up to the minus signs as follows:*

$$A = U_1 \cdot D_K \cdot U_2 \quad \text{with } U_1, U_2 \in \mathrm{SO}_2(\mathbb{R}), \quad D_K = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix}, \quad K > 1.$$

$$\text{We may denote: } U_2 = U_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \quad (13)$$

This fact can e.g. be seen geometrically as follows: $\mathrm{SL}_2(\mathbb{R})$ acts transitively on the upper half plane \mathbb{H} by Möbius transformations. The point $i \in \mathbb{H}$ can be mapped to $A(i) \neq i$ by first doing a stretching along the imaginary axis in direction ∞ and afterwards a rotation around i , i. e. $A(i) = U_1(D_K(i))$ with suitably chosen $U_1 \in \mathrm{SO}_2(\mathbb{R})$ and D_K with $K > 1$ as in the remark. Since the stabilizer of i in $\mathrm{SL}_2(\mathbb{R})$ is $\mathrm{SO}_2(\mathbb{R})$, one has $A = U_1 \cdot D_K \cdot U_2$ with U_2 also in $\mathrm{SO}_2(\mathbb{R})$. A short calculation gives the uniqueness claim.

In the following proposition again let q be a holomorphic quadratic differential on $X = X_{\mathrm{ref}}$ and μ the flat structure that q defines.

Proposition 2.10. *Let A and B be in $\mathrm{SL}_2(\mathbb{R})$ with $A \cdot B^{-1} \notin \mathrm{SO}_2(\mathbb{R})$ and*

$$A \cdot B^{-1} = U_1 \cdot D_K \cdot U_2$$

with U_1, U_2 and D_K as in (13). Then the Teichmüller distance between the two points $P_A = [(X, \mu_A), \mathrm{id}]$ and $P_B = [(X, \mu_B), \mathrm{id}]$ in T_g is $\log(K)$.

Proof. We will proceed in three steps:

a) Suppose B is the identity matrix I and

$$A = D_K = \begin{pmatrix} \sqrt{K} & 0 \\ 0 & \frac{1}{\sqrt{K}} \end{pmatrix} \quad \text{for some } K \in \mathbb{R}_{>1}.$$

Thus we have in fact that $P_A = [(X, \mu_{D_K}), \text{id}]$ is the point in T_g defined by the Teichmüller deformation of dilatation K with respect to q , see Definition 2.2. Hence the distance between P_A and the base point $(X_{\text{ref}}, \text{id}) = P_I$ is $\log(K)$.

b) Suppose again that $B = I$, but A is an arbitrary matrix in $\text{SL}_2(\mathbb{R})$.

Thus $A = U_1 \cdot D_K \cdot U_2$ and the map $\text{id} : (X, \mu) \rightarrow (X, \mu_A)$ is the composition of three maps:

$$(X, \mu) \xrightarrow{\text{id}} (X, \mu_{U_2}) \xrightarrow{\text{id}} (X, \mu_{D_K U_2}) \xrightarrow{\text{id}} (X, \mu_{U_1 D_K U_2})$$

Since the first and the third map are biholomorphic the Teichmüller distance is again $\log(K)$.

More precisely, write $U_2 = U_\theta$ as in (13). Then μ_{U_2} is the flat structure obtained by composing each chart with $z \mapsto e^{i\theta} \cdot z$. This is equal to the flat structure defined by the quadratic differential $q_{2\theta} = (e^{i\theta})^2 \cdot q$ which is holomorphic on the Riemann surface X .

Now, $\text{id} : (X, \mu_{U_2}) \rightarrow (X, \mu_{D_K U_2})$ is (up to the stretching $z \mapsto \sqrt{K} \cdot z$) the Teichmüller deformation of dilatation K with respect to the holomorphic quadratic differential $q_{2\theta}$. Thus the distance between $P_A = P_{U_1 D_K U_2} \stackrel{(10)}{=} P_{D_K U_2}$ and the base point $P_B = P_I$ is $\log(K)$.

c) Let now A, B be arbitrary in $\text{SL}_2(\mathbb{R})$. The Teichmüller metric does not depend on the chosen base point. Thus we may consider P_B as base point and P_A as coming from the affine deformation defined by the matrix $A \cdot B^{-1}$. Then with the given decomposition $A \cdot B^{-1} = U_1 \cdot D_K \cdot U_2$ the distance is as in b) equal to $\log(K)$. \square

Proposition 2.11. ι_2 is an isometric embedding

Proof. We denote by ρ the Poincaré distance in \mathbb{H} and by d_T the Teichmüller distance in T_g . Let t_1 and t_2 be arbitrary distinct points in \mathbb{H} . We may write $t_1 = p(A)$ and $t_2 = p(B)$ with A, B in $\text{SL}_2(\mathbb{R})$, p as in (11). Let $AB^{-1} = U_1 D_K U_2$ the decomposition of AB^{-1} from (13). ($AB^{-1} \notin \text{SO}_2(\mathbb{R})$ because $t_1 \neq t_2$)

$$\begin{aligned} \rho(t_1, t_2) &= \rho(\overline{-B^{-1}(i)}, \overline{-A^{-1}(i)}) = \rho(B^{-1}(i), A^{-1}(i)) = \rho(AB^{-1}(i), i) \\ &= \rho(U_1 D_K U_2(i), i) = \rho(U_1 D_K(i), i) \stackrel{\star}{=} \rho(D_K(i), i) \\ &= \rho(Ki, i) = \log(K) \stackrel{\text{Prop. 2.10}}{=} d_T(P_B, P_A) = d_T(\iota_2(t_1), \iota_2(t_2)) \end{aligned}$$

The equality \star is given since U_1 is a hyperbolic rotation with center i and thus does not change the distance to i . \square

Now we show that ι_1 and ι_2 are “almost” the same map.

Proposition 2.12. ι_1 and ι_2 fit together. More precisely: $\iota_1 \circ f = \iota_2$, with the isomorphism $f : \mathbb{H} \rightarrow \mathbb{D}$ from (9).

The following diagram may be helpful while reading the proof. Some parts will be explained only after the proof; in particular the space $B(X)$ of Beltrami differentials will be introduced in 2.3.3.

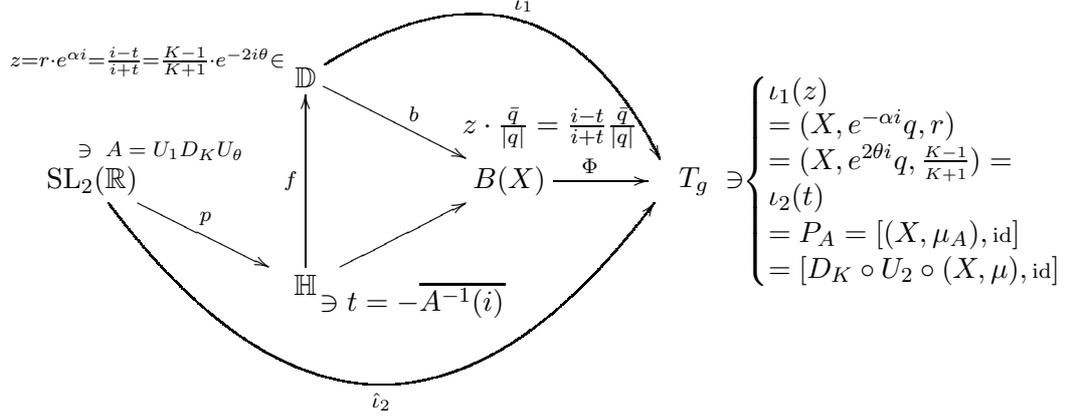


Figure 1: Diagram for alternative definitions of the Teichmüller disk Δ_q

Proof. We proceed in two steps:

1. Let $A \in \mathrm{SL}_2(\mathbb{R})$ be decomposed as in (13): $A = U_1 \cdot D_K \cdot U_2$, $U_2 = U_\theta$. We show that $(f \circ p)(A) = r \cdot e^{-2i\theta}$ with $r = \frac{K-1}{K+1}$.
2. We show that $\iota_1(r \cdot e^{-2i\theta}) = \hat{\iota}_2(A)$.

Step 1: One may express $t := p(A)$ in terms of K and θ as follows:

$$\begin{aligned} t &= -\overline{A^{-1}(i)} = -U_2^{-1} D_K^{-1}(\bar{i}) = -U_2^{-1}\left(-\frac{i}{K}\right) = -\frac{\cos(\theta) \cdot \frac{-i}{K} + \sin(\theta)}{-\sin(\theta) \cdot \frac{-i}{K} + \cos(\theta)} \\ &= \frac{i \cos(\theta) - K \sin(\theta)}{i \sin(\theta) + K \cos(\theta)} \end{aligned}$$

Now one has:

$$\begin{aligned} f(p(A)) &= f(t) = \frac{-t+i}{t+i} = \frac{-i \cos(\theta) + K \sin(\theta) + i(i \sin(\theta) + K \cos(\theta))}{i \cos(\theta) - K \sin(\theta) + i(i \sin(\theta) + K \cos(\theta))} \\ &= \frac{(K-1)[\sin(\theta) + i \cos(\theta)]}{(K+1)[- \sin(\theta) + i \cos(\theta)]} = \frac{K-1}{K+1} \cdot \frac{-(\sin(\theta) + i \cos(\theta))^2}{(\sin(\theta) - i \cos(\theta))(\sin(\theta) + i \cos(\theta))} \\ &= \frac{K-1}{K+1} (\cos(\theta) - i \sin(\theta))^2 = \frac{K-1}{K+1} \cdot e^{-2i\theta} \end{aligned}$$

Step 2: $\iota_1(r \cdot e^{-2i\theta}) = (X, e^{2i\theta} \cdot q, r) \in T_g$ is the point in the Teichmüller space that is obtained as Teichmüller deformation of dilatation $\frac{1+r}{1-r} = K$

with respect to the quadratic differential $e^{2i\theta} \cdot q$. Recall from the proof of Proposition 2.10 that this is precisely the point in T_g defined by the affine deformation $D_K \circ U_\theta \circ (X, \mu) = (X, \mu_{D_K U_\theta}) = (X, \mu_{D_K U_2})$. Thus

$$(X, e^{2i\theta} \cdot q, r) = P_{D_K U_\theta} = P_{D_K U_2} \stackrel{(10)}{=} P_{U_1 D_K U_2} = P_A = \hat{t}_2(A). \quad (14)$$

□

Using (14) one may also describe the geodesic rays $\iota_1 \circ \tau_\varphi$ from 2.3.1 in the Teichmüller disk $\Delta_q = \Delta_{\iota_1} = \Delta_{\iota_2}$ as follows.

Corollary 2.13. *Define D_K, U_θ as in (13). The map*

$$[0, \infty) \rightarrow T_g, \quad t \mapsto P_{D_K U_\theta} = [D_K \circ (X, \mu_{U_\theta}), \text{id}] \quad \text{with } K = e^t$$

is equal to $\iota_1 \circ \tau_{-2\theta}$.

It is thus by 2.3.1 the geodesic ray in direction of the quadratic differential $q_{2\theta} = e^{2\theta i} q$.

Proof. One has: $t \xrightarrow{\tau_{-2\theta}} r(t)e^{-2\theta i} \xrightarrow{\iota_1} (X, e^{2\theta i} \cdot q, r(t)) \stackrel{(14)}{=} P_{D_K U_\theta}$. □

Hence, geometrically one obtains the geodesic ray to q_φ by rotating the flat structure by $U_{\frac{\varphi}{2}}$ and then stretching in vertical direction with dilatation K .

2.3.3 Beltrami differentials

In order to see that ι_1 and ι_2 are holomorphic we introduce an alternative way to define ι_1 using Beltrami differentials. We keep this aspect short and refer to e.g. [29] for more details.

Let

$$M(X) = \{(X_1, f) \mid \begin{array}{l} X_1 \text{ Riemann surface,} \\ f : X \rightarrow X_1 \text{ is a quasiconformal homeomorphism} \end{array}\} / \approx$$

with $(X_1, f_1) \approx (X_2, f_2) \Leftrightarrow f_2 \circ f_1^{-1}$ is biholomorphic.

One has a natural projection $M(X) \rightarrow T_g$. Furthermore $M(X)$ can be canonically identified with the open unit ball $B(X)$ in the Banach space $L_{(-1,1)}^\infty(X)$ of $(-1, 1)$ -forms by the bijection:

$$M(X) \rightarrow B(X), \quad (X_1, f) \mapsto \mu_f,$$

where μ_f is the Beltrami differential (or complex dilatation) of f , cf. [29, 2.1.4].

Thus one obtains a projection $\Phi : B(X) \rightarrow T_g$. The map Φ is holomorphic ([29, 3.1]). Furthermore, for each quadratic differential q and for all $k \in (0, 1)$ the form $k \frac{\bar{q}}{|q|}$ is in $B(X)$ ([29, 2.6.3]) Thus one may define the map

$$\iota_3 : \left\{ \begin{array}{lll} \mathbb{D} & \xrightarrow{b} & B(X) & \xrightarrow{\Phi} & T_g \\ z & \mapsto & z \cdot \frac{\bar{q}}{|q|} & \mapsto & \Phi(z \cdot \frac{\bar{q}}{|q|}) \end{array} \right.$$

It is composition of two holomorphic maps and thus itself holomorphic. We will show in the following remark that $\iota_3 = \Phi \circ b = \iota_1$, cf. Figure 1.

Remark 2.14. For all $z_0 \in \mathbb{D} : \iota_3(z_0) = \iota_1(z_0)$.

Proof. Let $z_0 = r \cdot e^{i\alpha} \in \mathbb{D}$ and $A \in \mathrm{SL}_2(\mathbb{R})$ with $f(p(A)) = z_0$. Decompose $A = U_1 D_K U_2$ as in (13) with $U_2 = U_\theta$. Then by Step 1 of the proof of Proposition 2.12, $r = \frac{K-1}{K+1}$ and $\alpha = -2\theta$. Furthermore, by Proposition 2.12

$$\iota_1(z) = \hat{\iota}_2(A) = [(X, \mu_A), \mathrm{id}] = [(X, \mu_{D_K U_2}), \mathrm{id}]$$

Let us calculate the Beltrami differential of the Teichmüller deformation $f = \mathrm{id} : X \rightarrow (X, \mu_{D_K U_2})$. We will see that it is equal to $z_0 \cdot \frac{\bar{q}}{|q|}$. From this it follows that $\iota_1(z) = \iota_3(z)$.

One has $f = g \circ h$ with $h = \mathrm{id} : X \rightarrow (X, \mu_{U_2})$ and $g = \mathrm{id} : (X, \mu_{U_2}) \rightarrow (X, \mu_{D_K U_2})$. Locally in the charts of the flat structure defined by q , the maps g and h are given by

$$g : z \mapsto K \cdot \mathrm{Re}(z) + i \cdot \mathrm{Im}(z) \quad \text{and} \quad h : z \mapsto e^{i\theta} \cdot z.$$

Thus in terms of these charts one has:

$$\begin{aligned} f_z &= g_z \cdot h_z + g_{\bar{z}} \cdot \bar{h}_z = e^{i\theta} \cdot g_z & f_{\bar{z}} &= g_z \cdot h_{\bar{z}} + g_{\bar{z}} \cdot \bar{h}_{\bar{z}} = e^{-i\theta} \cdot g_{\bar{z}} \\ \Rightarrow \frac{f_{\bar{z}}}{f_z} &= e^{-2i\theta} \cdot \frac{g_{\bar{z}}}{g_z} = e^{-2i\theta} \cdot \frac{K-1}{K+1} = e^{i\alpha} \cdot r = z_0 \end{aligned}$$

Hence the Beltrami differential of f is $z_0 \cdot \frac{\bar{q}}{|q|}$. □

One obtains immediately the following conclusion.

Corollary 2.15. $\iota_1 = \iota_3$ is holomorphic. By Proposition 2.12 ι_2 is also holomorphic.

2.4 Teichmüller curves

In this section we introduce Teichmüller curves and recall some properties of them, in particular their relation to Veech groups. This was explored by Veech in his article [34] and has been studied by many authors since then. Overviews and further properties can be found e.g. in [27], [11] or [20].

Let $\iota : \mathbb{D} \hookrightarrow T_g$ be a Teichmüller embedding and $\Delta = \Delta_\iota = \iota(\mathbb{D})$ its image. We may consider the image of Δ_ι in the moduli space M_g under the natural projection $T_g \rightarrow M_g$, cf. Chapter 1. In general it will be something with a large closure. But occasionally it is an algebraic curve. Such a curve is called Teichmüller curve.

Definition 2.16. If the image of the Teichmüller disk Δ in the moduli space M_g is an algebraic curve C , then C is called Teichmüller curve.

A surface (X, q) , with a Riemann surface X and a holomorphic quadratic differential q such that the Teichmüller disk $\Delta = \Delta_q$ defined by q projects to a Teichmüller curve is called Veech surface.

How can one decide whether a surface (X, q) induces a Teichmüller curve or not? An answer to this question is given by the Veech group, a subgroup of $\mathrm{SL}_2(\mathbb{R})$ associated to (X, q) . This is explained in the following two subsections.

2.4.1 Veech groups

Let X be a Riemann surface and q a holomorphic quadratic differential on X . Let μ be the flat structure on X defined by q . One obtains a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ as follows: Let $\mathrm{Aff}^+(X, \mu)$ be the group of orientation preserving diffeomorphisms which are affine with respect to the flat structure μ , i.e. diffeomorphisms which are in terms of a local chart z of μ given by

$$z \mapsto A \cdot z + t, \quad \text{for some } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), t \in \mathbb{C}.$$

As above we identify the complex plane \mathbb{C} with \mathbb{R}^2 . Furthermore, we denote for $z = x + iy$: $A \cdot z = ax + by + i(cx + dy)$.

Since μ is a flat structure, up to change of sign the matrix A does not depend on the charts. Thus one has a group homomorphism:

$$D : \mathrm{Aff}^+(X, \mu) \rightarrow \mathrm{PSL}_2(\mathbb{R}), \quad f \mapsto [A].$$

For simplicity we will denote the image $[A]$ of the matrix A in $\mathrm{PSL}_2(\mathbb{R})$ often also just by A .

Definition 2.17. *The image $\bar{\Gamma}(X, \mu) = D(\mathrm{Aff}^+(X, \mu))$ of D is called the projective Veech group of (X, μ) .*

We will denote the projective Veech group also by $\bar{\Gamma}(X, q)$ and $\bar{\Gamma}_\iota$, where $\iota : \mathbb{D} \hookrightarrow T_g$ or $\iota : \mathbb{H} \hookrightarrow T_g$ is the Teichmüller embedding defined by q as described in 2.3. $\bar{\Gamma}(X, \mu)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, see [34, Prop. 2.7].

2.4.2 The action of the Veech group on the Teichmüller disk

Recall that the projection $T_g \rightarrow M_g$ from the Teichmüller space to the moduli space is given by the quotient for the action of the mapping class group

$$\Gamma_g = \mathrm{Diffeo}^+(X) / \mathrm{Diffeo}_0(X),$$

cf. (1) in the introduction. The action of $\mathrm{Diffeo}^+(X)$ on T_g is given by

$$\begin{aligned} \rho : \mathrm{Diffeo}^+(X) &\rightarrow \mathrm{Aut}(T_g) \cong \Gamma_g, \quad \varphi \mapsto \rho_\varphi \\ \text{with } \rho_\varphi : T_g &\rightarrow T_g, \quad (X_1, h) \mapsto (X_1, h \circ \varphi^{-1}). \end{aligned}$$

The affine group $\mathrm{Aff}^+(X, \mu)$ acts as subgroup of $\mathrm{Diffeo}^+(X)$ on T_g . The following remark (cf. [11, Theorem 1]) determines this action restricted to the Teichmüller disk

$$\Delta = \Delta_q = \{P_B = [(X, \mu_B), \mathrm{id}] \in T_g \mid B \in \mathrm{SL}_2(\mathbb{R})\}.$$

Remark 2.18. $\text{Aff}^+(X, \mu)$ stabilizes Δ . Its action on Δ is given by:

$$\varphi \in \text{Aff}^+(X, \mu), B \in \text{SL}_2(\mathbb{R}) \Rightarrow \rho_\varphi(P_B) = P_{BA^{-1}}$$

with $A \in \text{SL}_2(\mathbb{R})$ a preimage of $D(\varphi) = [A]$.

Proof. Let $\varphi \in \text{Aff}^+(X)$, $B \in \text{SL}_2(\mathbb{R})$ and $A \in \text{SL}_2(\mathbb{R})$ be a preimage of $D(\varphi) = [A] \in \text{PSL}_2(\mathbb{R})$. In the following commutative diagram

$$\begin{array}{ccccc} (X, \mu) & \xrightarrow{\varphi^{-1}} & (X, \mu) & \xrightarrow{\text{id}} & (X, \mu_B) \\ & & & \searrow \text{id} & \uparrow \\ & & & & (X, \mu_{BA^{-1}}) \end{array}$$

the map $(X, \mu_{BA^{-1}}) \rightarrow (X, \mu_B)$ is, as a composition of affine maps, itself affine. Its derivative is $D(\text{id} \circ \varphi^{-1} \circ \text{id}^{-1}) = BA^{-1}(BA^{-1})^{-1} = I$. Thus it is biholomorphic and $\rho_\varphi([(X, \mu_B), \text{id}]) = [(X, \mu_{BA^{-1}}), \text{id}]$. \square

It follows from Remark 2.18 that $\text{Aff}^+(X, \mu)$ is mapped by ρ to $\text{Stab}(\Delta)$, the global stabilizer of Δ in Γ_g . Furthermore $\rho : \text{Aff}^+(X, \mu) \rightarrow \text{Stab}(\Delta) \subseteq \Gamma_g$ is in fact an isomorphism: It is injective, see [11, Lemma 5.2] and surjective, see [11, Theorem 1]. Thus we have $\text{Aff}^+(X, \mu) \cong \text{Stab}(\Delta)$.

From Remark 2.18 it also becomes clear that the action of $\varphi \in \text{Aff}^+(X, \mu)$ depends only on $D(\varphi)$. Thus one obtains in fact an action of the projective Veech group $\bar{\Gamma}(X, \mu)$ on Δ .

Corollary 2.19. $\bar{\Gamma}(X, \mu) \subseteq \text{PSL}_2(\mathbb{R})$ acts on $\Delta = \{P_B \in T_g \mid B \in \text{SL}_2(\mathbb{R})\}$ by:

$$\rho_{[A]}(P_B) = P_{BA^{-1}} \quad \text{where } A \text{ is a preimage in } \text{SL}_2(\mathbb{R}) \text{ of } [A]. \quad (15)$$

Finally one may use the Teichmüller embedding $\iota_2 : \mathbb{H} \rightarrow T_g$ defined by q (cf. 2.7) in order to compare the action of $\bar{\Gamma}(X, \mu)$ on $\Delta = \Delta_\iota = \iota(\mathbb{H})$ with its action on \mathbb{H} via Möbius transformations. One obtains the diagram in the following remark (cf. [27, Proposition 3.2]).

Remark 2.20. Let $A \in \text{PSL}_2(\mathbb{R})$. Denote by $A : \mathbb{H} \rightarrow \mathbb{H}$ its action as Möbius transformation on \mathbb{H} . The following diagram is commutative:

$$\begin{array}{ccccc} \mathbb{H} & \xrightarrow{t \mapsto -\bar{t}} & \mathbb{H} & \xrightarrow{\iota} & \Delta \\ A \downarrow & & \downarrow RAR^{-1} & & \downarrow \rho_A \\ \mathbb{H} & \xrightarrow{t \mapsto -\bar{t}} & \mathbb{H} & \xrightarrow{\iota} & \Delta \end{array}$$

Figure 2

Here $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, thus R acts on $\mathbb{P}^1(\mathbb{C})$ by $z \mapsto -z$.

Proof. Let $t \in \mathbb{H}$. Choose some $B \in \mathrm{SL}_2(\mathbb{R})$ with $-\overline{B^{-1}(i)} = -\bar{t}$, thus $\iota(-\bar{t}) = P_B = [(X, \mu_B), \mathrm{id}]$ and using (15) we obtain the diagram:

$$\begin{array}{ccc} t & \xrightarrow{t \mapsto -\bar{t}} & -\bar{t} \xrightarrow{\iota} P_B = [(X, \mu_B), \mathrm{id}] \\ A \downarrow & & \downarrow \rho_A \\ A(t) & \xrightarrow{t \mapsto -\bar{t}} & -\overline{A(t)} \quad P_{BA^{-1}} = [(X, \mu_{BA^{-1}}), \mathrm{id}] \end{array}$$

The commutativity of the diagram in Figure 2 then follows from

$$RAR^{-1}(-\bar{t}) = -A(\bar{t}) = -\overline{A(t)} \text{ and} \\ -\overline{(BA^{-1})^{-1}(i)} = -A(\overline{B^{-1}(i)}) = -A(\bar{t}) = -\overline{A(t)}, \text{ thus } \iota(-\overline{A(t)}) = P_{BA^{-1}}.$$

□

2.4.3 Veech groups and Teichmüller curves

In Remark 2.18 we saw that the affine group $\mathrm{Aff}^+(X, \mu)$ maps isomorphically to the global stabilizer of the Teichmüller disk Δ in Γ_g . Denote by $\mathrm{proj} : T_g \rightarrow M_g$ the canonical projection. It then follows from Remark 2.20 that the map

$$\mathrm{proj} \circ \iota : \mathbb{H} \rightarrow \mathrm{proj}(\Delta) \subseteq M_g$$

factors through $\mathbb{H}/R\bar{\Gamma}(X, \mu)R^{-1}$. We call

$$\bar{\Gamma}^*(X, \mu) = R\bar{\Gamma}(X, \mu)R^{-1}$$

the *mirror projective Veech group*, since $\mathbb{H}/\bar{\Gamma}^*(X, \mu)$ is a mirror image of $\mathbb{H}/\bar{\Gamma}(X, \mu)$, and refer to it also as $\bar{\Gamma}^*(X, q)$ or $\bar{\Gamma}_\iota^*$.

$\mathbb{H}/\bar{\Gamma}^*(X, \mu)$ is a surface of finite type and hence an algebraic curve if and only if $\bar{\Gamma}^*(X, \mu)$ is a lattice in $\mathrm{PSL}_2(\mathbb{R})$. Altogether one obtains the following situation (cf. [27, Corollary 3.3]).

Corollary 2.21. *(X, q) induces a Teichmüller curve C if and only if $\bar{\Gamma}(X, \mu)$ is a lattice in $\mathrm{PSL}_2(\mathbb{R})$. In this case the following diagram holds:*

$$\begin{array}{ccccc} \mathbb{H} & \xrightarrow{t \mapsto -\bar{t}} & \mathbb{H} & \xrightarrow{\iota} & \Delta = \Delta_\iota \subseteq T_g \\ \downarrow & & \downarrow & & \downarrow \mathrm{proj} \\ \mathbb{H}/\bar{\Gamma}(X, \mu) & \xrightarrow{\text{antihol.}} & \mathbb{H}/\bar{\Gamma}^*(X, \mu) & \xrightarrow{\text{birat.}} & C \subseteq M_g \end{array}$$

In particular if $\bar{\Gamma}(X, \mu)$ is a lattice, then

- $\mathbb{H}/\bar{\Gamma}^*(X, \mu)$ is the normalization of the Teichmüller curve C ,
- $\mathbb{H}/\bar{\Gamma}(X, \mu)$ is antiholomorphic to $\mathbb{H}/\bar{\Gamma}^*(X, \mu)$.

3 Braungardt's construction of $\overline{T}_{g,n}$

Before we continue our study of Teichmüller disks and pass to the boundary, we want to explain the partial compactification $\overline{T}_{g,n}$ of the Teichmüller space $T_{g,n}$ that we shall use in the subsequent chapters. As mentioned in the introduction, $\overline{T}_{g,n}$ will be a locally ringed space which, as a topological space, coincides with Abikoff's augmented Teichmüller space $\hat{T}_{g,n}$ (see the discussion following Proposition 3.9). The points of this space can be considered as marked stable Riemann surfaces (X, f) , where $f : X_{\text{ref}} \rightarrow X$ is a deformation map. The forgetful map $(X, f) \mapsto X$ defines a natural map from $\overline{T}_{g,n}$ to the moduli space $\overline{M}_{g,n}$ of stable n -pointed Riemann surfaces of genus g . This map extends the projection $T_{g,n} \rightarrow M_{g,n}$ and is in fact also the quotient map for the natural action of the mapping class group $\Gamma_{g,n}$. But the stabilizers of the boundary points are infinite, and at the boundary the topology of $\overline{T}_{g,n}$ is quite far from that of a manifold.

In his thesis [8], V. Braungardt gave a construction of $\overline{T}_{g,n}$ which uses only the complex structure of $\overline{M}_{g,n}$ and the boundary divisor $\partial M_{g,n}$. Moreover his construction endows $\overline{T}_{g,n}$ with the structure of a locally ringed space and he shows that it is a fine moduli space for "marked" stable Riemann surfaces. In this chapter we give a brief account of his approach.

3.1 Coverings with cusps

The basic idea of Braungardt's construction is to study, for a complex manifold S , quotient maps $W \rightarrow W/G = S$ that have "cusps" over a divisor D in S . This concept, which will be explained in this section, generalizes the familiar ramified coverings. The key result is that, in the appropriate category of such quotient maps, there exists a universal object $p : \tilde{W} \rightarrow S$ with cusps over D .

In general \tilde{W} cannot be a complex manifold or even a complex space. Therefore we have to work in the larger category of *locally complex ringed spaces*, i.e. topological spaces W endowed with a sheaf \mathcal{O}_W of \mathbb{C} -algebras (called the *structure sheaf*) such that at each point $x \in W$ the stalk $\mathcal{O}_{W,x}$ is a local \mathbb{C} -algebra. The basic properties of such spaces can be found e.g. in [14, Ch. 1, § 1] (where they are called \mathbb{C} -ringed spaces).

In our situation Braungardt constructs a normal locally complex ringed space \tilde{W} such that the subspace $\tilde{W}_0 = \tilde{W} - p^{-1}(D)$ is a complex manifold and the restriction $p|_{\tilde{W}_0} : \tilde{W}_0 \rightarrow S_0 = S - D$ is the usual universal covering.

Example 3.1. *The simplest example is well known and quite typical: Take S to be the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $D = \{0\}$. The universal covering of $S - D$ is, of course, $\exp : \mathbb{H} \rightarrow \mathbb{D} - \{0\}$, $z \mapsto e^{2\pi iz}$. It turns out that the universal covering in Braungardt's sense is $\mathbb{H} = \mathbb{H} \cup \{i\infty\}$ with the horocycle topology, i.e. the sets $\mathbb{H}_R = \{z \in \mathbb{C} : \text{Im } z > R\} \cup \{i\infty\}$ for $R > 0$ form a basis of neighbourhoods of the point $i\infty$. Note that this topology is not the one induced from the Euclidean topology if $\mathbb{H} \cup \{\infty\}$ is considered as a subset of*

the Riemann sphere $\hat{\mathbb{C}}$.

$\hat{\mathbb{H}}$ is given the structure of a normal complex ringed space by taking $\mathcal{O}(U)$ to be the holomorphic functions on U for open subsets U of \mathbb{H} , and by defining $\mathcal{O}(\mathbb{H}_R)$ to be the set of holomorphic functions on $\{z \in \mathbb{C} : \text{Im} z > R\}$ that have a continuous extension to $i\infty$. Clearly $\mathcal{O}(\mathbb{H}_R)$ contains all functions of the form $z \mapsto e^{2\pi iz/n}$ for all $n \geq 1$.

We now give the precise definitions. We begin with the class of spaces that we need (cf. [8, 3.1.3]):

Definition 3.2. Let (W, \mathcal{O}_W) be a locally complex ringed space whose structure sheaf \mathcal{O}_W is a subsheaf of the sheaf $\mathcal{C}^\infty(W, \mathbb{C})$ of continuous complex valued functions on W .

a) A subset $B \subset W$ is called analytic if there is an open covering $(U_i)_{i \in I}$ of W and for each $i \in I$ there are finitely many elements $f_{i,1}, \dots, f_{i,n_i} \in \mathcal{O}_W(U_i)$ such that $B \cap U_i$ is the zero set of $\{f_{i,1}, \dots, f_{i,n_i}\}$.

b) We call (W, \mathcal{O}_W) an R -space if, for every open $U \subseteq W$ and every proper closed analytic subset $B \subset U$, a continuous function $f : U \rightarrow \mathbb{C}$ is in $\mathcal{O}_W(U)$ if and only if its restriction to $U - B$ is in $\mathcal{O}_W(U - B)$.

Note that all complex spaces are R -spaces: The required property is just Riemann's extension theorem, see [14, Chapter 7].

Definition 3.3. Let S be a complex manifold and $D \subset S$ a proper closed analytic subset. Then a surjective morphism $p : W \rightarrow S$ from an R -space (W, \mathcal{O}_W) to S is called a covering with cusps over D if there is a group G of automorphisms of W (as locally complex ringed space) such that

- (i) p is the quotient map $W \rightarrow W/G = S$,
- (ii) $W_0 = p^{-1}(S - D)$ is a complex manifold and $p|_{W_0} : W_0 \rightarrow S_0 = S - D$ is an unramified covering,
- (iii) for any $x \in W$ there is a basis of neighbourhoods U_x that are precisely invariant under the stabilizer G_x of x in G (i. e. $G_x(U_x) = U_x$ and $g(U_x) \cap U_x = \emptyset$ for each $g \in G - G_x$).

Note that, in particular, any ramified normal covering of complex manifolds is a covering in the sense of this definition (with cusps over the branch locus). As mentioned before, the basic result is (see [8, Satz 3.1.9])

Theorem 3.4. (i) For any complex manifold S and any proper closed analytic subset $D \subset S$ there exists an initial object $p : (\tilde{W}, \mathcal{O}_{\tilde{W}}) \rightarrow S$ in the category of coverings of S with cusps over D ; it is called the universal covering with cusps over D . The restriction of p to $\tilde{W}_0 = p^{-1}(S_0)$ is the universal covering of S_0 , and the group $G = \text{Aut}(\tilde{W}/S)$ is the fundamental group $\pi_1(S_0)$.

(ii) If S' is an open submanifold of S and \tilde{W}' the universal covering of S' with cusps over $D' = D \cap S'$, then \tilde{W}'/H' embeds as an open subspace into \tilde{W} , where H' is the kernel of the homomorphism $\pi_1(S' - D') \rightarrow \pi_1(S - D) = G$.

Proof. We only sketch the construction of the space $(\tilde{W}, \mathcal{O}_{\tilde{W}})$. The details that it satisfies all the required properties are worked out in [7]. For the proof of (ii) we refer to [8].

Let $S_0 = S - D$, $G = \pi_1(S_0)$ and $p_0 : W_0 \rightarrow S_0$ the universal covering. \tilde{W} is obtained from W_0 by “filling in the holes above D ” in such a way that the G -action extends from W_0 to \tilde{W} . More formally, the fibre \tilde{W}_s of \tilde{W} over any point $s \in S$ is constructed as follows: let $\mathfrak{U}(s)$ be the set of open connected neighbourhoods of s in S ; for any $U \in \mathfrak{U}(s)$ denote by $X(U)$ the set of connected components of $p_0^{-1}(U)$. Then

$$\tilde{W}_s = \{(x_U)_{U \in \mathfrak{U}(s)} : x_U \in X(U), x_U \cap x_{U'} \neq \emptyset \text{ for all } U, U' \in \mathfrak{U}(s)\}.$$

Clearly $\tilde{W}_s = p_0^{-1}(s)$ for $s \in S_0$. Note that by definition, G acts transitively on each \tilde{W}_s , thus $\tilde{W}/G = S$. For any $x = (x_U) \in \tilde{W}$ define the sets $x_U \cup \{x\}$, $U \in \mathfrak{U}(s)$, to be open neighbourhoods of x . Finally define the structure sheaf by

$$\mathcal{O}_{\tilde{W}}(U) = \{f : U \rightarrow \mathbb{C} \text{ continuous} : f \text{ holomorphic on } U \cap \tilde{W}_0\} \quad (16)$$

for any open subset U of \tilde{W} . □

A key point in Braungardt’s proof of Theorem 3.4 is the existence of neighbourhoods U for any point $a \in D$ such that the natural homomorphism $\pi_1(U - D) \rightarrow \varinjlim_{U' \in \mathfrak{U}(a)} \pi_1(U' - D)$ is an isomorphism. He calls such neighbourhoods *decent*. The importance of this notion is that if U is a decent neighbourhood of a point $a \in D$ and \bar{x}_U a connected component of $p^{-1}(U)$, then \bar{x}_U is precisely invariant under the stabilizer G_x in G of the unique point $x \in \bar{x}_U \cap p^{-1}(a)$.

Decent neighbourhoods in the above sense do not exist in general for singular complex spaces. For example, if S is a stable Riemann surface and $s \in S$ a node, $U - \{s\}$ is not even connected for small neighbourhoods U of s . Nevertheless the construction can be generalized to this case, and the proof of the theorem carries over to this case as Braungardt explains in [8, Anm. 3.1.4]; we therefore have:

Corollary 3.5. *Any stable Riemann surface has a universal covering with cusps over the nodes.*

Near the inverse image of a node, the universal covering of a stable Riemann surface looks like two copies of $\hat{\mathbb{H}}$ glued together in the cusps. If such a neighbourhood is embedded into the complex plane or $\mathbb{P}^1(\mathbb{C})$ it is called a *doubly cusped region*, cf. [25, VI.A.8].

3.2 The cusped universal covering of $\overline{M}_{g,n}$

Let us now fix nonnegative integers g, n such that $3g - 3 + n > 0$. We want to construct the space $\overline{T}_{g,n}$ as the universal covering of $\overline{M}_{g,n}$ with cusps

over the compactification (or boundary) divisor $\partial M_{g,n}$. But we cannot apply Theorem 3.4 directly to $\overline{M}_{g,n}$ since it is not a manifold, but only an orbifold (or smooth stack). Braungardt circumvents this difficulty by

Definition 3.6. *A morphism $p : Y \rightarrow \overline{M}_{g,n}$ of locally complex ringed spaces is called a covering with cusps over $D = \partial M_{g,n}$ if there is an open covering $(U_i)_{i \in I}$ of $\overline{M}_{g,n}$ and for each $i \in I$ a covering $q_i : U'_i \rightarrow U_i$ by a complex manifold U'_i such that $p|_{p^{-1}(U_i)}$ factors as $p^{-1}(U_i) \xrightarrow{p'_i} U'_i \xrightarrow{q_i} U_i$, where p'_i is a covering with cusps over $q_i^{-1}(D)$ (in the sense of Definition 3.3).*

Then one can use Theorem 3.4 to prove

Proposition 3.7. *There is a universal covering $\overline{T}_{g,n} \rightarrow \overline{M}_{g,n}$ with cusps over $\partial M_{g,n}$.*

Proof. We first construct local universal coverings and then glue them together. For any $s \in \overline{M}_{g,n}$ choose an open neighbourhood U and a covering $q' : U' \rightarrow U$ with a manifold U' . Let \tilde{W}' be the universal covering of U' with cusps over $D' = q'^{-1}(D)$. Let H' be the kernel of the homomorphism from $\pi_1(U' - D')$ to $\Gamma_{g,n}$. Theorem 3.4 (ii) suggests that the quotient \tilde{W}'/H' should be an open part of the universal covering of $\overline{M}_{g,n}$. All that remains to show is that the \tilde{W}'/H' glue together to a covering with cusps over D . This is done in [8, 3.2.1] \square

Locally $\overline{T}_{g,n}$ looks like a product of a ball with some copies of the universal covering $\hat{\mathbb{H}}$ of \mathbb{D} with cusps over $\{0\}$ which was explained in Section 3.1:

Corollary 3.8. *Let $x \in \overline{T}_{g,n}$ correspond to a stable Riemann surface X with k nodes. Then x has a neighbourhood that is isomorphic to*

$$\hat{\mathbb{H}}^k \times \mathbb{D}^{3g-3+n-k}.$$

Proof. Let $s \in \overline{M}_{g,n}$ be the image point of x . The deformation theory of stable Riemann surfaces gives us a map from \mathbb{D}^{3g-3+n} onto a neighbourhood of s such that the inverse image of $D = \partial M_{g,n}$ is the union of axes $D' = \{(z_1, \dots, z_{3g-3+n}) : z_1 \cdot \dots \cdot z_k = 0\}$, see [16, Sect. 3B]. The fundamental group of $\mathbb{D}^{3g-3+n} - D'$ is a free abelian group on k generators; they correspond to Dehn twists about the loops that are contracted in X . Thus the homomorphism $\pi_1(\mathbb{D}^{3g-3+n} - D') \rightarrow \Gamma_{g,n}$ is injective. By Proposition 3.7 and its proof the universal covering \tilde{W} of \mathbb{D}^{3g-3+n} with cusps over D' is therefore a neighbourhood of x . It is not hard to see that \tilde{W} is of the given form. \square

Our next goal is to compare $\overline{T}_{g,n}$ to the *augmented* Teichmüller space $\hat{T}_{g,n}$ introduced by Abikoff [1].

Proposition 3.9 (cf. [8], Satz 3.4.2). *$\overline{T}_{g,n}$ is homeomorphic to the augmented Teichmüller space $\hat{T}_{g,n}$.*

Before proving the proposition we summarize the definition and some properties of $\hat{T}_{g,n}$: As a point set,

$$\hat{T}_{g,n} = \{(X, f) : X \text{ a stable Riemann surface of type } (g, n), \\ f : X_{\text{ref}} \rightarrow X \text{ a deformation}\} / \sim \quad (17)$$

As mentioned in the introduction, a deformation is a map that contracts some disjoint loops on X_{ref} to points (the nodes of X) and is a homeomorphism otherwise. The equivalence relation is the same as for $T_{g,n}$: $(X, f) \sim (X', f')$ if and only if there is a biholomorphic map $h : X \rightarrow X'$ such that f' is homotopic to $h \circ f$.

Abikoff puts a topology on $\hat{T}_{g,n}$ by defining neighbourhoods $U_{V,\varepsilon}$ of a point (X, f) for a compact neighbourhood V of the set of nodes in X and $\varepsilon > 0$:

$$U_{V,\varepsilon} = \{(X', f') : \exists \text{ deformation } h : X' \rightarrow X, \text{ } (1 + \varepsilon)\text{-quasiconformal} \\ \text{on } h^{-1}(X - V), \text{ such that } f \text{ is homotopic to } h \circ f'\} / \sim \quad (18)$$

The action of the mapping class group $\Gamma_{g,n}$ extends continuously to $\hat{T}_{g,n}$ ([1, Thm. 4]), and the orbit space $\hat{T}_{g,n}/\Gamma_{g,n}$ is $\overline{M}_{g,n}$ (as a topological space).

Proof of Proposition 3.9. Braungardt shows (see [8, Hilfssatz 3.4.4]) that the stabilizer of a point $(X, f) \in \hat{T}_{g,n}$ in $\Gamma_{g,n}$ is an extension of the free abelian group generated by the Dehn twists about the contracted loops by the holomorphic automorphism group $\text{Aut}(X)$ of X . For any V and ε , $\bigcap_{\sigma \in \text{Aut}(X)} \sigma(U_{V,\varepsilon})$ is invariant under the stabilizer of (X, f) , and for sufficiently small V and ε , it is precisely invariant. Therefore the quotient map $\hat{T}_{g,n} \rightarrow \overline{M}_{g,n}$ is a covering with cusps over $\partial M_{g,n}$ in the sense of Definition 3.6, except that so far no structure sheaf has been defined on $\hat{T}_{g,n}$. But this can be done in the same way as in (16). The universal property of $\overline{T}_{g,n}$ then yields a map $p : \overline{T}_{g,n} \rightarrow \hat{T}_{g,n}$ compatible with the action of $\Gamma_{g,n}$ on both sides. To show that this map is an isomorphism we compare the stabilizers in $\Gamma_{g,n}$ for the points in both spaces. For a point in $\hat{T}_{g,n}$ we just described this stabilizer, and the proof of Corollary 3.8 shows that for a corresponding point in $\overline{T}_{g,n}$ it is also an extension of \mathbb{Z}^k by $\text{Aut}(X)$. \square

3.3 Teichmüller structures

In this section we explain how Braungardt extends the universal family of marked Riemann surfaces that is well known to exist over $T_{g,n}$ to a family over $\overline{T}_{g,n}$ which still is universal for the appropriate notion of marking or *Teichmüller structure*.

As above we fix a reference Riemann surface X_{ref} of type (g, n) ; let Q_1, \dots, Q_n be the marked points and $X_{\text{ref}}^0 = X_{\text{ref}} - \{Q_1, \dots, Q_n\}$. Let us also fix a universal covering $U_{\text{ref}} \rightarrow X_{\text{ref}}^0$ and identify $\pi_{g,n} = \pi_1(X_{\text{ref}}^0)$ with the group $\text{Aut}(U_{\text{ref}}/X_{\text{ref}}^0)$ of deck transformations.

A classical construction of the family $\mathcal{C}_{g,n}$ over $T_{g,n}$ goes as follows (cf. [5]): For every point $x = (X, P_1, \dots, P_n, f) \in T_{g,n}$ take a universal covering of $X^0 = X - \{P_1, \dots, P_n\}$ and arrange them so that they form an \mathbb{H} -bundle Ω^+ over $T_{g,n}$. Then $\mathcal{C}_{g,n}$ is obtained as the quotient of Ω^+ by the natural action of $\pi_{g,n}$. More precisely, Ω^+ is defined as follows: to $x \in T_{g,n}$ there corresponds the quasifuchsian group $G_x = w^\mu G (w^\mu)^{-1}$, where $G = \text{Aut}(U_{\text{ref}}/X_{\text{ref}}^0) \cong \pi_{g,n}$ and w^μ is the quasiconformal automorphism of $\mathbb{P}^1(\mathbb{C})$ associated to x , see e. g. [21, 6.1.1]. The domain of discontinuity of G_x consists of two connected components $\Omega^-(x) = w^\mu(L)$ (where L is the lower half plane) and $\Omega^+(x) = w^\mu(\mathbb{H})$. Then $\Omega^+(x)/G_x = X^0$, whereas $\Omega^-(x)/G_x = X_{\text{ref}}^{0,*}$, the mirror image of X_{ref}^0 .

To extend this family we identify $\overline{T}_{g,n}$ with $\hat{T}_{g,n}$ by Corollary 3.9. As explained in [1], any point $x = (X, P_1, \dots, P_n, f) \in \hat{T}_{g,n} - T_{g,n}$ corresponds to a *regular B-group* G_x . This means that G_x is a Kleinian group isomorphic to $\pi_{g,n}$ whose domain of discontinuity $\Omega(G_x)$ has a unique simply connected invariant component $\Omega^-(G_x)$ such that $\Omega^-(G_x)/G_x$ is isomorphic to $X_{\text{ref}}^{0,*}$. For the union $\Omega^+(G_x) = \Omega^+(x)$ of the other components of $\Omega(G_x)$ it holds that $\Omega^+(G_x)/G_x \cong X^0 - \{\text{nodes}\}$. To every node in X there corresponds a conjugacy class of parabolic elements in G_x , each of which is accidental (i. e. it becomes hyperbolic in the Fuchsian group $hG_x h^{-1}$, where $h : \Omega^-(G_x) \rightarrow \mathbb{H}$ is a conformal map). Near a fixed point of such a parabolic element, $\Omega^+(G_x)$ is a doubly cusped region, cf. the remark at the end of Section 3.1. If we denote by $\hat{\Omega}^+(x)$ the union of $\Omega^+(G_x)$ with the fixed points of the parabolic elements in G_x (accidental or not), then $\hat{\Omega}^+(x) \rightarrow X$ is the universal covering of X with cusps over the nodes (cf. Corollary 3.5).

Definition 3.10. *Let*

$$\hat{\Omega}_{g,n}^+ = \{(x, z) \in \overline{T}_{g,n} \times \mathbb{P}^1(\mathbb{C}) : z \in \hat{\Omega}^+(x)\}.$$

On $\hat{\Omega}_{g,n}^+$, $\pi_{g,n}$ acts in such a way that for fixed $x \in \overline{T}_{g,n}$ the action on $\Omega^+(x)$ is that of G_x . $\overline{\mathcal{C}}_{g,n} = \hat{\Omega}_{g,n}^+/\pi_{g,n}$ is called the *universal family over $\overline{T}_{g,n}$* .

Braungardt shows ([8, Hilfssatz 4.2.1]) that $\Omega_{g,n}^+ = \{(x, z) \in \hat{\Omega}_{g,n}^+ : x \in T_{g,n}, z \in \Omega^+(x)\}$ is an open subset of $\overline{T}_{g,n} \times \mathbb{P}^1(\mathbb{C})$ and hence has a well defined structure of a complex ringed space. One can extend this structure sheaf to all of $\hat{\Omega}_{g,n}^+$ in the same way as in (16). Then clearly $\overline{\mathcal{C}}_{g,n}$ is also a complex ringed space, and the fibre over x is isomorphic to the stable Riemann surface X represented by x .

To justify the name “universal” family for $\overline{\mathcal{C}}_{g,n}$ we introduce the notion of a Teichmüller structure: For a single smooth Riemann surface (X, P_1, \dots, P_n) of type (g, n) , a Teichmüller structure is just a marking: so far in this article we used markings as classes of mappings $X_{\text{ref}} \rightarrow X$; equivalently a marking can be given as an isomorphism $\pi_{g,n} \rightarrow \pi_1(X - \{P_1, \dots, P_n\})$ inducing an isomorphism $\pi_g = \pi_1(X_{\text{ref}}) \rightarrow \pi_1(X)$ and respecting the orientation and the conjugacy classes of the loops around the Q_i resp. P_i . Yet another equivalent way to give a marking is as a universal covering $U \rightarrow X^0$ together with an

isomorphism $\pi_{g,n} \rightarrow \text{Aut}(U/X^0)$. This last characterization also works for a stable Riemann surface if we take for U a universal covering with cusps over the nodes. Before we extend this definition to the relative situation we recall the notion of a family of stable Riemann surfaces:

Definition 3.11. *Let S be a complex ringed space. A family of stable Riemann surfaces of type (g, n) over S is a complex ringed space \mathcal{C} together with a proper flat map $\pi : \mathcal{C} \rightarrow S$ such that the fibres $X_s = \pi^{-1}(s)$, $s \in S$, are stable Riemann surfaces of genus g . In addition we are given n disjoint sections $P_i : S \rightarrow \mathcal{C}$, $i = 1, \dots, n$, of π such that $P_i(s)$ is not a node on X_s . We denote by $\mathcal{C}^0 = \mathcal{C} - \bigcup_{i=1}^n P_i(S)$ the complement of the marked sections.*

Definition 3.12. *Let \mathcal{C}/S be a family of stable Riemann surfaces of type (g, n) over a complex ringed space S . A Teichmüller structure on \mathcal{C} is a complex ringed space \mathcal{U} together with a morphism $\mathcal{U} \rightarrow \mathcal{C}$ such that for every $s \in S$ the (restriction of the) fibre $U_s^0 \rightarrow X_s^0$ is a universal covering with cusps over the nodes, together with an isomorphism $\pi_{g,n} \rightarrow \text{Aut}(\mathcal{U}/\mathcal{C}^0)$.*

Putting everything together we obtain

Theorem 3.13. *$\overline{T}_{g,n}$ is a fine moduli space for stable Riemann surfaces with Teichmüller structure. $\overline{\mathcal{C}}_{g,n} \rightarrow \overline{T}_{g,n}$ is the universal family and $\hat{\Omega}_{g,n}^+ \rightarrow \overline{\mathcal{C}}_{g,n} = \hat{\Omega}_{g,n}^+/\pi_{g,n}$ is the universal Teichmüller structure.*

Finally Braungardt gives a very elegant and conceptual description of $\overline{\mathcal{C}}_{g,n}$ which extends a classical result of Bers ([5, Thm. 9]) to the boundary:

Proposition 3.14. *$\overline{T}_{g,n+1}/\pi_{g,n}$ is in a natural way isomorphic to $\overline{\mathcal{C}}_{g,n}$.*

Proof. The kernel of the obvious homomorphism $\Gamma_{g,n+1} \rightarrow \Gamma_{g,n}$ can be identified with $\pi_{g,n}$, which gives the action on $\overline{T}_{g,n+1}$. The holomorphic map $T_{g,n+1} \rightarrow T_{g,n}$ which forgets the last marked point extends to a map $\overline{T}_{g,n+1} \rightarrow \overline{T}_{g,n}$ by a general property of universal coverings with cusps. The difficult step in Braungardt's proof is to show that the induced map $\overline{T}_{g,n+1}/\pi_{g,n} \rightarrow \overline{T}_{g,n}$ has the right fibres. For this purpose he constructs a map $\hat{\Omega}_{g,n}^+ \rightarrow \overline{T}_{g,n+1}$ and shows that it is bijective and induces isomorphisms on the fibres over $\overline{T}_{g,n}$. \square

4 Boundary points of Teichmüller curves

The aim of this chapter is to study the boundary points of the Teichmüller disks and Teichmüller curves introduced in Chapter 2 in \overline{T}_g and \overline{M}_g , respectively. Here and later, whenever we speak about \overline{T}_g and its boundary, we mean the bordification of the Teichmüller space described in Chapter 3.

In particular we will derive, in Section 4.2, the following description of the boundary points of Teichmüller curves (see Proposition 4.14 and Corollary 4.15 for a more precise formulation):

Theorem 4.1. *One obtains the boundary points of a Teichmüller curve by contracting the centers of all cylinders in Strebel directions. They are determined by the parabolic elements in the associated mirror Veech group.*

This statement seems to be well known to the experts although we are not aware of a published proof.

In Section 4.1 we prepare for the proof of Theorem 4.1 by introducing Strebel rays. They are special geodesic rays in Teichmüller space which always converge to a point on the boundary. Following Masur [26], we describe this boundary point quite explicitly using the affine structure of the quadratic differential q that defines the Strebel ray.

In Section 4.2 we turn to the boundary points of Strebel rays that are contained in a Teichmüller disk. In particular if the Teichmüller disk descends to a Teichmüller curve in the moduli space, all its boundary points can be determined explicitly with the aid of the projective Veech group. One obtains Theorem 4.1 as a conclusion.

4.1 Hitting the boundary via a Strebel ray

In this section, we introduce Strebel rays and describe their end point on the boundary of \overline{T}_g . As before, everything might be done as well for punctured surfaces and the moduli space $T_{g,n}$ with $3g-3+n > 0$, but for ease of notation, we restrict to the case $n = 0$.

Let X be a Riemann surface of genus $g \geq 2$, q a holomorphic quadratic differential on X . Recall from Section 2.1 that with q we have chosen a natural flat structure μ on the surface $X^* = X - \{\text{critical points of } q\}$ whose charts were given in (2). The maximal real curves in X^* which are locally mapped by these charts to horizontal (resp. vertical) line segments are called *horizontal* (resp. *vertical*) *trajectories*. A trajectory is *critical* if it ends in a critical point. Otherwise it is *regular*.

Definition 4.2. *We say that a holomorphic quadratic differential q is Strebel, if all regular horizontal trajectories are closed.*

Strebel differentials play an exceptional role in the following sense. Recall from Section 2.2 that each holomorphic quadratic differential defines a geodesic ray. If q is Strebel, then the geodesic ray defined by its negative $-q$ converges in \overline{T}_g to an end point on the boundary. This is described more

precisely in the following proposition which was proven by Masur in [26]. We give a version of his proof with parts of the notation and arguments adapted to the context of our article.

Recall also from 2.2 that we obtain the geodesic ray to $-q$ as the image of the isometric embedding

$$\gamma = \gamma_{-q} : \begin{cases} [0, \infty) & \rightarrow T_g \\ t & \mapsto (X_K, f_K) = [(X, \mu_{-q}) \circ \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}, \text{id}] \quad \text{with } K = e^t. \end{cases} \quad (19)$$

Note that here (X_K, f_K) is the Teichmüller deformation of X of dilatation K with respect to $-q$. Furthermore, μ_{-q} is the translation structure on X^* defined by $-q$.

Proposition 4.3. *Suppose $q \neq 0$ is a Strebel differential. For the geodesic ray defined by γ_{-q} in T_g , one has:*

- a) *The ray converges towards a unique point (X_∞, f_∞) on the boundary of the Teichmüller space T_g .*
- b) *One obtains this point by contracting the central lines of the horizontal cylinders defined by q as is described in 4.1.4.*

Definition 4.4. *In the previous Proposition, the geodesic ray defined by $-q$, i. e. the image of γ_{-q} in T_g , is called a Strebel ray.*

For the proof of Proposition 4.3 one may use two slightly different perspectives of the Strebel ray. They are described in 4.1.1, 4.1.2 and 4.1.3, 4.1.4. In 4.1.5 we describe the boundary point (X_∞, f_∞) . In 4.1.6 we show that the Strebel ray in fact converges towards this point.

Throughout Section 4.1, we assume that the differential q is Strebel.

4.1.1 X as patchwork of rectangles

One may regard X as a patchwork of rectangles in the complex plane, as is described in the following.

Since q is Strebel, the surface X , with the critical points and critical horizontal trajectories removed, is swept out by closed horizontal trajectories. More precisely, it follows from the work of Strebel (cf. [33], also see [26, Theorem B] which contains a list of the results we use here) that the surface X , except for the critical points and critical horizontal trajectories, is covered by a finite number of maximal *horizontal cylinders* Z_1, \dots, Z_p , i. e. annuli that are swept out by closed horizontal trajectories. For each Z_i one may choose a vertical trajectory β_i joining opposite boundary components of Z_i . If we remove β_i from Z_i , the remainder is mapped, by the natural chart w_i defined

by μ (see (2)), to an open rectangle R_i in the complex plane. The horizontal and vertical edges have lengths

$$a_i = \int_{\alpha_i} |q(z)|^{\frac{1}{2}} dz \quad \text{and} \quad b_i = \int_{\beta_i} |q(z)|^{\frac{1}{2}} dz,$$

where α_i is any closed horizontal trajectory in the cylinder Z_i .

One may extend w_i^{-1} uniquely to a map from the closure \bar{R}_i of R_i to the closure of the annulus Z_i . Then the two horizontal edges of \bar{R}_i are mapped to the two horizontal boundary components of Z_i and the two vertical edges are both mapped to β_i . The critical points of q that lie on the boundary of Z_i define by their preimage marked points on the horizontal edges of \bar{R}_i and decompose them into segments.

For each such segment s on a horizontal edge of \bar{R}_i its image on X joins the annulus Z_i to an annulus Z_j possibly with $i = j$. Thus the map $w_j \circ w_i^{-1}$ (w_i^{-1} is the extended map, w_j is locally the inverse map of the extended map w_j^{-1}) is an *identification map* between s and a segment on a horizontal edge of \bar{R}_j . (Images of critical points have to be excluded.) These identification maps are of the form $z \mapsto \pm z + c$ with a constant $c \in \mathbb{C}$.

Conversely, given the closed rectangles $\bar{R}_1, \dots, \bar{R}_p$, the marked points on their horizontal edges and these identification maps, we may recover the surface X as follows: for each i glue the two vertical edges of \bar{R}_i by a translation and the horizontal edges (with the marked points removed) by the identification maps. In this way, one obtains a surface X^* with the flat structure on it inherited from the euclidean plane \mathbb{C} . By filling in the punctures at vertices, we obtain the original compact Riemann surface X .

In this sense one may consider X as a patchwork of the rectangles $\bar{R}_1, \dots, \bar{R}_p$. This description depends of course on the chosen holomorphic quadratic Strebel differential q .

Example 4.5. Two Riemann surfaces X given as a patchwork of rectangles:

In the two examples in Figure 3 and Figure 4, the two vertical edges of each rectangle are glued by a translation, respectively. Horizontal segments with the same name are glued. The direction of the arrow indicates whether the identification is a translation or a rotation by 180° . In the example in Figure 3 one only has translations, in the example in Figure 4 only rotations.

In the first example the surface X is of genus 2 and all marked points are identified and thus give only one point on X . In the second example one obtains a surface of genus 0 with four marked points indicated by the four symbols \bullet , \star , \circ and \square .

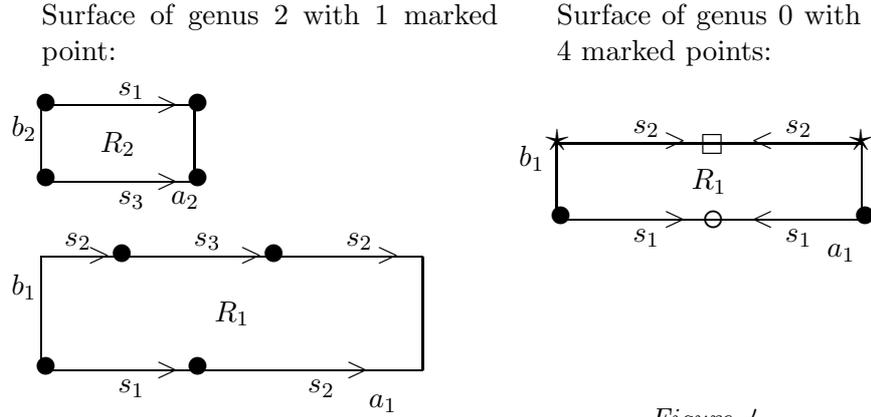


Figure 3

Figure 4

4.1.2 Stretching the cylinders

We will now redescribe the Strebel ray defined by $-q$ by stretching the rectangles in the 'patchwork' from 4.1.1 in the vertical direction.

The flat structure defined by $-q = e^{\pi i} \cdot q$ is obtained from the flat structure μ defined by q by composing each chart with a rotation by $\frac{\pi}{2}$. Thus the deformation (X_K, f_K) of dilatation K with respect to $-q$ is equal to the affine deformation

$$\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ (X, \mu) = \begin{pmatrix} 0 & -K \\ 1 & 0 \end{pmatrix} \circ (X, \mu).$$

This defines by (10) the same point in T_g as the affine deformation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & -K \\ 1 & 0 \end{pmatrix} \circ (X, \mu) = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \circ (X, \mu).$$

Thus the isometric embedding $\gamma = \gamma_{-q}$ in (19) is equivalently given by

$$\gamma_{-q} : \begin{cases} [0, \infty) & \rightarrow T_g \\ t & \mapsto (X_K, f_K) = \left[\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \circ (X, \mu), \text{id} \right], \quad K = e^t \end{cases} \quad (20)$$

Recall again that here $(X_K, f_K) = (X_K^{-q}, f_K^{-q})$ is the Teichmüller deformation with respect to the differential $-q$.

Hence we obtain the point $\gamma_{-q}(t)$ as follows: Each chart of μ is composed with the map $x + iy \mapsto x + iKy$, $(x, y \in \mathbb{R})$ with $K = e^t$, and the marking is topologically the identity. Now, let X be given as a patchwork of the rectangles $\bar{R}_1, \dots, \bar{R}_p$ as in 4.1.1. Then we obtain the surface $X_K = X_K^{-q}$ in the following way: We stretch each rectangle \bar{R}_i , which has horizontal and vertical edges of lengths a_i and b_i , into a rectangle $\bar{R}_i(K)$ with horizontal and vertical edges of

lengths a_i and $K \cdot b_i$. The identification maps of the horizontal segments are again translations or rotations identifying the same segments as before. The surface $X_K = X_K^{-q}$ then is the patchwork obtained from $\bar{R}_1(K), \dots, \bar{R}_p(K)$ as described in 4.1.1.

On \bar{R}_i , the diffeomorphism $f_K = f_K^{-q}$ has image $\bar{R}_i(K)$ and is given by

$$x + iy \mapsto x + iKy.$$

This glues to a well defined diffeomorphism on X^* , which can be uniquely extended to X .

Example 4.6. K -stretched surfaces:

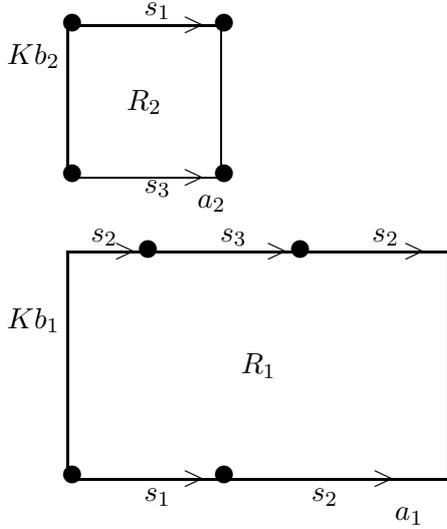


Figure 5

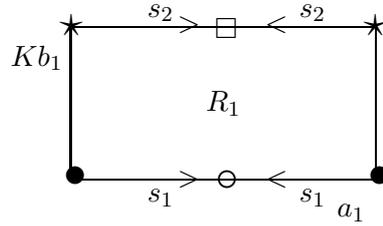


Figure 6

One obtains the surface $X_K = X_K^{-q}$ from the surface X in Example 4.5 as the patchwork of the stretched rectangles in Figure 5 and Figure 6, respectively.

4.1.3 S as patchwork of double annuli

Recall that, in 4.1.1, we used μ to identify the horizontal cylinder Z_i on X with the euclidean cylinder defined by the rectangle R_i in \mathbb{C} ; we did so by adding the vertical boundary edges and identifying them by a translation. It turns out to be easier to describe the end point of the Strebel ray, if we identify the Z_i with so called double annuli A_i .

Definition 4.7. A cylinder Z of length a and height b defines a double annulus A as follows:

- Take two disjoint open annuli A^1 and A^2 given as

$$A^1 = A^2 = \{z \in \mathbb{C} \mid r \leq |z| < 1\} \quad \text{with } r = e^{-\pi \frac{b}{a}}.$$

- Glue their inner boundary lines $\{|z| = r\}$ by the map $z \mapsto \frac{1}{z} \cdot r^2$.

- We call the resulting surface A the double annulus of Z .

Remark 4.8. A is biholomorphic to Z .

The identification is given explicitly as follows:

- Z is biholomorphic to the Euclidean cylinder defined by the rectangle

$$\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq a, 0 < \operatorname{Im}(z) < b\}.$$

- Decompose the rectangle into two halves of height $\frac{b}{2}$,
a lower half $R^1 = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq a, 0 < \operatorname{Im}(z) \leq \frac{b}{2}\}$
and an upper half $R^2 = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq a, \frac{b}{2} \leq \operatorname{Im}(z) < b\}$.
- The cylinder defined by R^1 is mapped to A^1 by $z \mapsto e^{2\pi i \frac{z}{a}}$.
The cylinder defined by R^2 is mapped to A^2 by $z \mapsto e^{2\pi i \frac{a+bi-z}{a}}$.

These maps respect the identifications and define a biholomorphic map from Z to A , as shown in Figure 7.

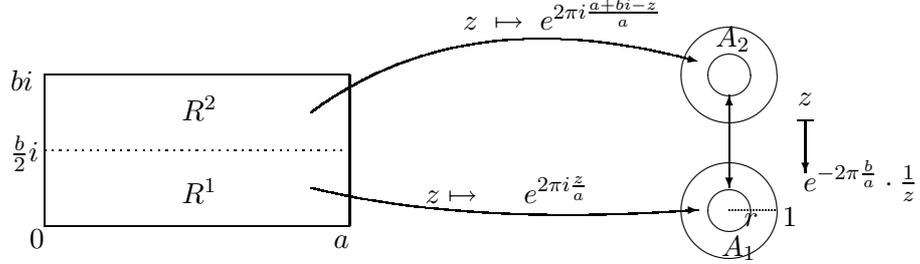


Figure 7

Consider the double annuli A_1, \dots, A_p defined by the cylinders Z_1, \dots, Z_p . The biholomorphic map $Z_i \rightarrow A_i$ extends to a continuous map from the closure of Z_i to the closure \bar{A}_i of A_i . The zeroes of q on the boundary of Z_i define marked points on the boundary of A_i and decompose it into segments. The surface X can now be described as a *patchwork of the closed double cylinders* $\bar{A}_1, \dots, \bar{A}_p$. The identification maps between the segments on the boundary of the A_i are essentially the same as in 4.1.1.

4.1.4 Contracting the central lines

Suppose that X is given as a patchwork of double annuli $\bar{A}_1, \dots, \bar{A}_p$ as in 4.1.3. We may describe the points $(X_K, f_K) = (X_K^{-q}, f_K^{-q})$ on the Strebel ray also as a patchwork of double annuli:

Let $A_i(K) = A_i^1(K) \cup A_i^2(K)$ ($i \in \{1, \dots, p\}$) be the double annulus from Definition 4.7 with $r = r_i(K) = r_i^K$ and define $X_K = X_K^{-q}$ to be the surface obtained by gluing the closures $\bar{A}_1(K), \dots, \bar{A}_p(K)$ with the same maps as $\bar{A}_1, \dots, \bar{A}_p$. Furthermore, define the diffeomorphism $f_K = f_K^{-q}$ on A_i by

$$f_K^{-q} : A_i^1 \rightarrow A_i^1(K) \text{ and } A_i^2 \rightarrow A_i^2(K), \\ z = r \cdot e^{i\varphi} \mapsto r^K \cdot e^{i\varphi} \text{ on both parts.}$$

Then the following diagram is commutative:

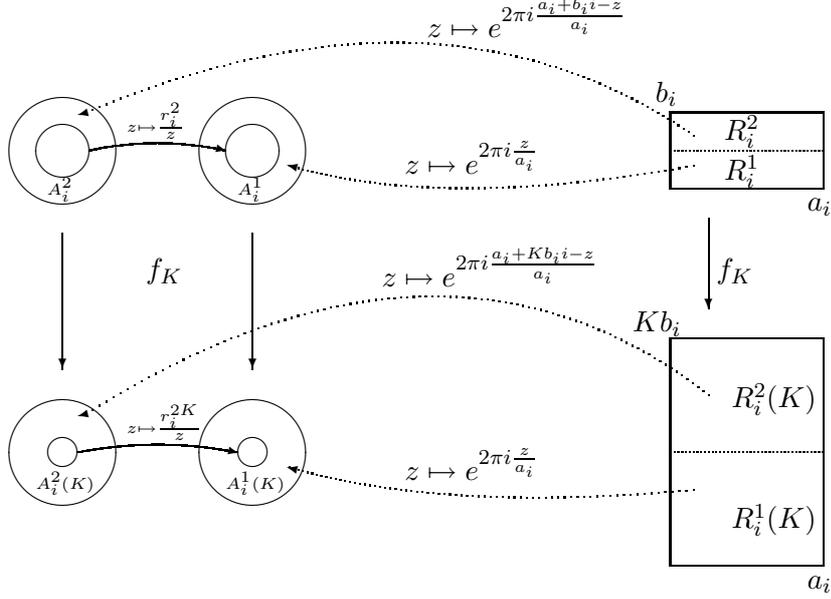


Figure 8

where $f_K = (re^{i\varphi} \mapsto r^K e^{i\varphi})$ on the left side and $f_K = (x + iy \mapsto x + Kiy)$ on the right side of the diagram. Thus, in particular, we have defined here with $(X_K, f_K) = (X_K^{-q}, f_K^{-q})$ the same surface (up to isomorphism) and the same diffeomorphism as in 4.1.2.

4.1.5 The end point of the Strebel ray

We use the description of the Strebel ray in 4.1.4 to obtain its end point $(X_\infty, f_\infty) \in \overline{T}_g$. Recall from 3.2 that a point in \overline{T}_g consists of a stable Riemann surface X_∞ and a deformation $f_\infty : X \rightarrow X_\infty$.

If $K \rightarrow \infty$ in 4.1.4, the interior radius $r_i(K) = r_i^K$ of the two annuli $A_i^1(K)$ and $A_i^2(K)$ that form the double annulus $A_i(K)$ tends to 0 ($i \in \{1, \dots, p\}$). $A_i(K)$ tends to a double cone $A_i(\infty)$ and the whole surface X_K to a stable Riemann surface X_∞ . More precisely, we define $A_i(\infty)$ and X_∞ as complex spaces in the following way.

Definition 4.9. Let $A_i^1(\infty)$ and $A_i^2(\infty)$ both be the punctured disk

$$\{z \in \mathbb{C} \mid 0 < |z| < 1\},$$

and let pt be an arbitrary point. The disjoint union

$$A_i(\infty) = A_i^1(\infty) \cup A_i^2(\infty) \cup \{pt\}$$

becomes a complex cone by the following chart:

$$\begin{aligned} \varphi : A_i(\infty) &\rightarrow \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \cdot z_2 = 0, |z_1|, |z_2| < 1\} \\ \varphi|_{A_i^1(\infty)} : z &\mapsto (0, z), \quad \varphi|_{A_i^2(\infty)} : z \mapsto (z, 0), \quad \varphi(pt) = (0, 0) \end{aligned}$$

The closures of the double cones $\bar{A}_1(\infty), \dots, \bar{A}_p(\infty)$ are glued to each other by the same identification maps as in the 'finite' case in 4.1.4. We call the resulting stable Riemann surface X_∞ . Topologically, X_∞ is obtained from the surface X by a contraction f_∞ of the middle curves of the cylinders.

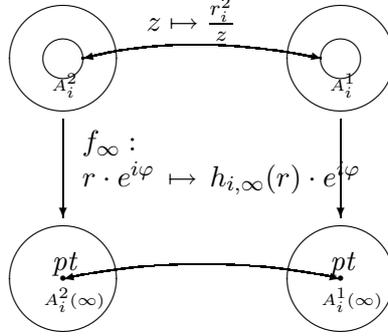


Figure 9

We now define the contraction f_∞ as the following map: Let A_i^1 and A_i^2 be the two annuli in Definition 4.7 that form the double annulus A_i ($i \in \{1, \dots, p\}$). Then f_∞ is given by

$$f_\infty : \begin{array}{l} A_i^j \\ z = r \cdot e^{i\varphi} \end{array} \rightarrow \begin{array}{l} A_i^j(\infty) \\ f_\infty(z) = h_{i,\infty}(r) \cdot e^{i\varphi} \end{array} \quad \text{for } j \in \{1, 2\}$$

with an arbitrary monotonously increasing diffeomorphism $h_{i,\infty} : [r_i, 1) \rightarrow [0, 1)$. The isotopy class of f_∞ is independent of the choices of $h_{i,\infty}$.

4.1.6 Convergence

We now show that, in the above notation, the Strebel ray γ_{-q} converges to the point (X_∞, f_∞) on the boundary of T_g . Recall from (18) in Chapter 3 that a base of open neighbourhoods of (X_∞, f_∞) is given by the open sets

$$U_{V,\varepsilon}(X_\infty, f_\infty) = \{(X', f') \mid \exists \varphi : X' \rightarrow X_\infty \text{ s.t. } \varphi \text{ is deformation,} \\ \varphi \circ f' \text{ is isotopic to } f_\infty \text{ and} \\ \varphi|_{X' \setminus \varphi^{-1}(V)} \text{ has dilatation } < 1 + \varepsilon\},$$

for all compact neighbourhoods V of the singular points of X_∞ and for all $\varepsilon > 0$. We may restrict to open neighbourhoods V of the form

$$V = V(\kappa) = V_1 \cup \dots \cup V_p, \quad \kappa = (\kappa_1, \dots, \kappa_p), \quad 0 < \kappa_i < 1$$

where V_i is a double cone defined by

$$V_i = V_i^1 \cup V_i^2 \cup \{pt\} \quad \text{with } V_i^j = \{0 < |z| \leq \kappa_i\} \subseteq A_i^j(\infty) \quad (j \in \{1, 2\}).$$

Lemma 4.10. *For each such $V = V(\kappa)$ and each $\varepsilon > 0$, there is some $K_0 \in \mathbb{R}_{>0}$ such that all points $(X_K, f_K) = (X_K^{-q}, f_K^{-q})$ with $K > K_0$ are in $U_{V,\varepsilon}(X_\infty, f_\infty)$.*

Proof. Choose K_0 such that $r_i^{K_0} < \kappa_i$ for all $i \in \{1, \dots, p\}$ and suppose that $K > K_0$. Define the diffeomorphism $\varphi : X_K \rightarrow X_\infty$ on $A_i^j(K)$ by

$$\varphi : z = r \cdot e^{i\theta} \mapsto \begin{cases} z \in A_i^j(\infty), & \text{if } 1 > |z| \geq \kappa_i \\ h_K^i(r) \cdot e^{i\theta} \in A_i^j(\infty), & \text{if } \kappa_i \geq |z| > r_i^K \\ pt \in A_i^j(\infty), & \text{if } |z| = r_i^K \end{cases}$$

with an arbitrary monotonously increasing diffeomorphism $h_K^i : (r_i^K, \kappa_i) \rightarrow (0, \kappa_i)$. Then $\varphi \circ f_K$ is isotopic to f_∞ and $\varphi|_{X_K \setminus \varphi^{-1}(V)}$ is holomorphic, hence its dilatation is 1. Thus (X_K, f_K) is in $U_{V,\varepsilon}(X_\infty, f_\infty)$.

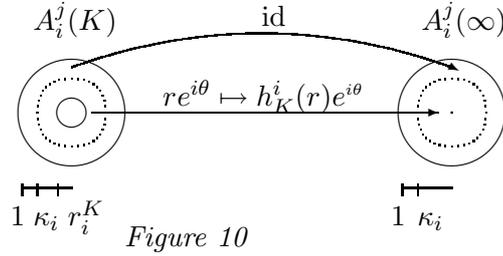


Figure 10

□

With Lemma 4.10 we have obtained the desired result and completed the proof of Proposition 4.3.

Corollary 4.11. *The Strebel ray defined by $-q$ converges to the point (X_∞, f_∞) on the boundary of T_g .*

4.2 Boundary points of Teichmüller disks

In this section we study the boundary points of a Teichmüller disk $\Delta = \Delta_\iota$ in the bordification \overline{T}_g of the Teichmüller space; in particular, we consider the case that Δ_ι projects to an affine curve in the moduli space M_g . For convenience, we use the upper half plane model and consider Teichmüller embeddings as maps from \mathbb{H} to T_g . We will obtain Theorem 4.1 as our final result. We proceed in two steps:

- In 4.2.1, we show that a Teichmüller embedding $\iota : \mathbb{H} \hookrightarrow T_g$ has a natural extension

$$\bar{\iota} : \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}_\iota^*\} \hookrightarrow \overline{T}_g,$$

- In 4.2.2, we show that the image of $\bar{\iota}$ is the whole closure of Δ_ι in \overline{T}_g , if the Teichmüller disk Δ_ι projects onto a Teichmüller curve in M_g . It will follow from this that one obtains the boundary points of Δ_ι precisely by contracting the central lines of the cylinders in “parabolic directions”. The parabolic directions correspond to the cusps of the projective mirror Veech group $\bar{\Gamma}_\iota^*$.

Throughout this section, we assume that $\iota : \mathbb{H} \hookrightarrow T_g$ is a Teichmüller embedding to a fixed holomorphic quadratic differential q on $X = X_{\text{ref}}$ and that μ is the translation structure defined by q as in Section 2.1. Recall from Section 2.4 that the associated projective Veech group $\bar{\Gamma}_\iota = \bar{\Gamma}(X, \mu)$ and its mirror image $\bar{\Gamma}_\iota^* = R\bar{\Gamma}_\iota R^{-1}$ (with R as in Remark 2.20) are both Fuchsian groups in $\text{PSL}_2(\mathbb{R})$.

4.2.1 Extending Teichmüller embeddings to the cusps of $\bar{\Gamma}^*$

Let $\tilde{s} \in \mathbb{R}^\infty = \mathbb{R} \cup \{\infty\}$ be a cusp of the Fuchsian group $\bar{\Gamma}_\iota^*$, i.e. \tilde{s} is a fixed point of some parabolic element \tilde{A} of $\bar{\Gamma}_\iota^*$. We associate to \tilde{s} a point $\bar{i}(\tilde{s}) = (X_\infty(\tilde{s}), f_\infty(\tilde{s}))$ on the boundary of T_g in the following way:

- In a natural way we associate to \tilde{s} a Strebel ray
- We show that this Strebel ray is the image in T_g of the hyperbolic ray in \mathbb{H} from i to \tilde{s} under ι .
- $\bar{i}(\tilde{s}) = (S_\infty(s), f_\infty(s))$ is defined to be the end point of the Strebel ray

The Strebel ray associated to \tilde{s} :

$A = R^{-1}\tilde{A}R$ is a parabolic element in the projective Veech group $\bar{\Gamma}_\iota$. Let v be its unit eigenvector.

By Proposition 2.4 in [34], the direction v is fixed by some affine diffeomorphism h of (X, μ) . The derivative of h is A and v is a Strebel direction. More precisely: The trajectories in the direction of v are preserved by h and each leaf is either closed or a saddle connection, i. e. connects two critical points.

As in 4.1.1, X decomposes into maximal cylinders of closed leaves parallel to v and the cylinders are bounded by saddle connections. The affine diffeomorphism h can be described nicely as follows: Passing to a power of h if necessary, one may assume that h fixes all critical points of q . Then h is the composition of Dehn twists along the core curves of the cylinders. Each trajectory is mapped by h to itself and the saddle connections are fixed pointwise.

Now, let us take the matrix

$$U = U_\theta \in \text{SO}_2(\mathbb{R}) \quad \text{such that} \quad U \cdot v = \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

with U_θ defined as in (13).

Consider the affine deformation $\text{id} : (X, \mu) \rightarrow (X, \mu_U) = (X, \mu) \circ U$ as in Definition 2.6. The vector v is mapped to \vec{e}_1 . Thus the same trajectories are now the horizontal ones.

Recall from 2.3.2 that the flat structure (X, μ_U) is defined by the quadratic differential $e^{2\theta i} \cdot q$. Thus $e^{2\theta i} \cdot q$ is Strebel. The ray is by (20) given as:

$$\begin{aligned} \gamma_{\tilde{s}} = \gamma_{-e^{2\theta i} \cdot q} : [0, \infty) &\rightarrow T_g \\ t &\mapsto \left[\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \circ (X, \mu_{U_\theta}), \text{id} \right] = [(X, \mu_{A_K}), \text{id}] \end{aligned}$$

with $K = e^t$ and $A_K = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix} \cdot U_\theta$

The Strebel ray $\gamma_{\tilde{s}}$ is the image of the geodesic ray in \mathbb{H} from i to the cusp \tilde{s} : From Remark 2.12 (see also Figure 1) one obtains that

$$\gamma_{\tilde{s}}(t) = [(X, \mu_{A_K}), \text{id}] = \hat{i}(A_K) = \iota(\overline{-A_K^{-1}(i)}).$$

Furthermore, we have

$$\overline{-A_K^{-1}(i)} = \overline{-U_\theta^{-1}(K \cdot i)} = -U_\theta^{-1}(-Ki) = RU_\theta^{-1}R^{-1}(Ki).$$

Thus the image of $\gamma_{\tilde{s}}$ is equal to the image of the ray $RU_\theta^{-1}R^{-1}(Ki)$ ($K \in [1, \infty)$) under ι . But the latter one is the geodesic ray in \mathbb{H} from i to $RU_\theta^{-1}R^{-1}(\infty) = -U^{-1}(\infty)$.

Observe finally that $-U^{-1}(\infty) = \tilde{s}$: Since $U \cdot v = \vec{e}_1$ for the eigenvector v of A , one has for the fixed point s of A that $U(s) = \infty$. Hence, one has for the fixed point \tilde{s} of $\tilde{A} = RAR^{-1}$ that $\tilde{s} = -s = -U^{-1}(\infty)$. Thus the Strebel ray defined by $\gamma_{\tilde{s}}$ is the image of the geodesic ray from i to \tilde{s} in \mathbb{H} under ι .

Finally we define $\bar{i}(\tilde{s}) = (X_\infty(\tilde{s}), f_\infty(\tilde{s})) \in \overline{T}_g$ to be the end point of the Strebel ray $\gamma_{\tilde{s}}$. We then define the map \bar{i} as follows.

Definition 4.12. \bar{i} is the extension of ι defined by

$$\begin{aligned} \bar{i}: \quad \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}_l^*\} &\rightarrow \overline{T}_g, \\ t &\mapsto \begin{cases} \iota(t), & \text{if } t \in \mathbb{H} \\ \bar{i}(t) = (X_\infty(\tilde{s}), f_\infty(\tilde{s})) & \text{if } t = \tilde{s} \text{ is a cusp of } \bar{\Gamma}_l^* \end{cases} \end{aligned}$$

We consider $\mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}_l^*\}$ as topological space endowed with the horocycle topology as in Example 3.1.

Proposition 4.13. \bar{i} is a continuous embedding.

Proof. \bar{i} is continuous:

Let s be a cusp of $\bar{\Gamma}_l^*$, i.e. s is a fixed point of some parabolic element $\tilde{A} \in \bar{\Gamma}_l^*$, and $c: [0, \infty) \rightarrow \mathbb{H}$ an arbitrary path in \mathbb{H} converging to s in the horocycle topology.

By Remark 2.20, the action of \tilde{A} on \mathbb{H} fits together with the action of $\rho(A) \in \Gamma_g$ on \overline{T}_g . Both actions may be extended continuously to $\mathbb{H}_s = \mathbb{H} \cup \{s\}$ (endowed with the horocycle topology) and to \overline{T}_g , respectively, and one obtains the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H}_s = \mathbb{H} \cup \{s\} & \xrightarrow{\bar{i}} & \overline{T}_g \\ p_{\tilde{A}} \downarrow & & \downarrow p \\ \overline{\mathbb{H}_s / \langle \tilde{A} \rangle} & \xrightarrow{i_{\tilde{A}}} & \overline{M}_g \end{array}$$

Here the map $i_{\bar{A}} : \mathbb{H}/\langle \bar{A} \rangle \rightarrow \overline{M}_g$ is the map induced by $\bar{\iota}$ and $\mathbb{H}/\langle \bar{A} \rangle$ is a disk with center $p_{\bar{A}}(s)$.

Let W be a neighbourhood of

$$\bar{P}_\infty = i_{\bar{A}}(p_{\bar{A}}(s)) = p(\bar{\iota}(s)).$$

For i in an index set I , let P_∞^i be the preimages of \bar{P}_∞ in \overline{T}_g under p . One of them is $\bar{\iota}(s)$, again by the commutativity of the diagram.

Since $\{P_\infty^i \mid i \in I\}$ is discrete we may choose the neighbourhood W in such a manner that its preimage under p is of the form:

$$V = p^{-1}(W) = \bigcup_{i \in I} V_i \subseteq \overline{T}_g$$

where the V_i are the connected components of V with $P_\infty^i \in V_i$ and V_i is invariant under the stabilizer of P_∞^i in the mapping class group Γ_g .

Furthermore, we may choose W such that the preimage of W under $i_{\bar{A}}$ is a simply connected neighbourhood of $p_{\bar{A}}(s)$. Then, again, the preimage

$$U = p_{\bar{A}}^{-1}(i_{\bar{A}}^{-1}(W))$$

is a neighbourhood of s in the horocycle topology.

Thus an end piece of the path c is completely contained in U , i.e. there is some $l \in \mathbb{R}_{>0}$ such that $c([l, \infty))$ is contained in U .

Since the above diagram is commutative and the V_i are disjoint, the image of U is one of the V_i . This V_i then contains $\bar{\iota}(c[l, \infty))$. In addition, V_i has to contain the end piece of the Strebel ray that leads to s used to define $\bar{\iota}(s)$. Hence, V_i is the component that contains $\bar{\iota}(s)$.

Making W arbitrarily small, the neighbourhood U of s becomes arbitrarily small. Thus $\iota \circ c$ converges to $\bar{\iota}(s)$.

$\bar{\iota}$ is injective:

Suppose there are two cusps s_1 and s_2 with $P_\infty = \bar{\iota}(s_1) = \bar{\iota}(s_2)$. Thus we have two Strebel rays defined by the negative of the Strebel differentials $q_1 = e^{i\theta_1} \cdot q$ and $q_2 = e^{i\theta_2} \cdot q$ with initial point $P_0 = \iota(i)$ and the same end point P_∞ in \overline{T}_g . Let (X_∞, f_∞) and (Y_∞, g_∞) be the two marked stable Riemann surfaces defined by the two Strebel rays, respectively. Since they define the same point in \overline{T}_g the following diagram is commutative up to homotopy with some biholomorphic h :

$$\begin{array}{ccc} & & X_\infty \\ & \nearrow f_\infty & \uparrow h \\ X_{\text{ref}} & & \\ & \searrow g_\infty & \downarrow \\ & & Y_\infty \end{array}$$

The core curves of the cylinders relative to the flat structure on X defined by q_1 are mapped by f_∞ to the singular points of X_∞ . Similarly the core curves

coming from q_2 are mapped to the singular points of Y_∞ . Since the diagram is commutative up to isotopy, the two systems of core curves are homotopic. Thus the two Strebel rays are similar by definition, using the terminology in [26, Section 5]. From Theorem 2 in [26] it follows that there is some constant $M < \infty$ such that for two points $Q \neq R$ lying on the two Strebel rays which are equidistant from the initial point P_0 , one has $d(Q, R) \leq M$. But then, since ι is an isometric embedding, M would have to be an upper bound for the distance of equidistant points on two different geodesic rays in \mathbb{H} starting from i . This cannot be true. \square

4.2.2 Boundary of Teichmüller disks that lead to Teichmüller curves

Let now $\iota : \mathbb{H} \hookrightarrow T_g$ be a Teichmüller embedding such that its image Δ_ι projects to a Teichmüller curve C in the moduli space M_g .

Proposition 4.14. *In this situation, the extended embedding from 4.2.1*

$$\bar{\iota} : \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}_\iota^*\} \hookrightarrow \bar{\Delta}_\iota \subseteq \bar{T}_g$$

is surjective onto the closure $\bar{\Delta}_\iota$ of Δ_ι in \bar{T}_g .

Proof. Recall from Corollary 2.21 that if ι leads to a Teichmüller curve then the projective Veech group $\bar{\Gamma} = \bar{\Gamma}_\iota$ is a lattice in $\text{PSL}_2(\mathbb{R})$, $\mathbb{H}/\bar{\Gamma}^*$ is a complex algebraic curve and $\mathbb{H}/\bar{\Gamma}^* \rightarrow C \subset M_g$ is the normalization of C . Thus it extends to a surjective morphism

$$\varphi : \overline{\mathbb{H}/\bar{\Gamma}^*} \rightarrow \bar{C} \subseteq \bar{M}_g,$$

where $\overline{\mathbb{H}/\bar{\Gamma}^*}$ and \bar{C} are the projective closure of $\mathbb{H}/\bar{\Gamma}^*$ and the closure of C in \bar{M}_g , respectively.

Furthermore, the map $\mathbb{H} \rightarrow \mathbb{H}/\bar{\Gamma}^*$ extends continuously to a surjective map $p_{\bar{\Gamma}} : \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}^*\} \rightarrow \overline{\mathbb{H}/\bar{\Gamma}^*}$, since $\bar{\Gamma}^*$ is a lattice in $\text{PSL}_2(\mathbb{R})$. Here we use the horocycle topology on $\mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}^*\}$.

Thus one has the following commutative diagram of continuous maps:

$$\begin{array}{ccc} \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}^*\} & \xrightarrow{\bar{\iota}} & \bar{\Delta}_\iota \subseteq \bar{T}_g \\ p_{\bar{\Gamma}} \downarrow & & p|_{\bar{\Delta}_\iota} \downarrow \quad \downarrow p \\ \overline{\mathbb{H}/\bar{\Gamma}^*} & \xrightarrow{\varphi} & \bar{C} \subseteq \bar{M}_g \end{array}$$

Let now P_∞ be a point on the boundary of Δ_ι . Similarly as in the proof of the continuity of $\bar{\iota}$ we may choose a neighbourhood W of $p(P_\infty)$ in \bar{C} such that all connected components V_i of the preimage $p^{-1}(W)$ contain only one preimage of $p(P_\infty)$. One of them, let's say V_0 , contains of course P_∞ itself.

We choose an arbitrary path $c_\iota : [0, \infty) \rightarrow W \setminus \{p(P_\infty)\} \subseteq C$ that converges to $p(P_\infty)$. Let $\hat{c}_\iota : [0, \infty) \rightarrow V_0$ be an arbitrary lift of c_ι via p in V_0 . Since we may choose W arbitrarily small, V_0 may become arbitrarily small and \hat{c}_ι

converges to P_∞ .

Now let $c : [0, \infty) \rightarrow \mathbb{H}$ be the preimage of \hat{c}_ι under ι , i. e. the path such that $\iota \circ c = \hat{c}_\iota$. We project it by $p_{\bar{\Gamma}}$ to $\mathbb{H}/\bar{\Gamma}^*$, i. e. we take the path $p_{\bar{\Gamma}} \circ c$. Its image under φ is $\varphi \circ p_{\bar{\Gamma}} \circ c = p \circ \hat{c}_\iota = c_\iota$ and converges to $p(P_\infty)$ in \bar{C} . Thus $p_{\bar{\Gamma}} \circ c$ converges in $\mathbb{H}/\bar{\Gamma}^*$, since φ is an open map.

Since also $p_{\bar{\Gamma}}$ is open, c converges to some $t_\infty \in \mathbb{H} \cup \{\text{cusps of } \bar{\Gamma}^*\}$. By continuity of $\bar{\iota}$ one has $\bar{\iota}(t_\infty) = P_\infty$. Thus $\bar{\iota}$ is surjective onto $\bar{\Delta}_\iota$. \square

One obtains immediately the following conclusions.

Corollary 4.15. *If $\iota : \mathbb{H} \hookrightarrow T_g$ leads to a Teichmüller curve C , then*

- a) *the boundary points of the Teichmüller disk Δ_ι are precisely the end points of the Strebel rays in Δ_ι with initial point $\iota(i)$.*
- b) *These boundary points correspond to the fixed points of parabolic elements in the projective Veech group.*
- c) *Each boundary point of the Teichmüller curve C is obtained by contracting the core curves of the cylinders in the direction of v , where v is the eigenvector of a parabolic element in the Veech group.*

This finishes the proof of Theorem 4.1.

5 Schottky spaces

In this chapter we first recall the construction of Schottky coverings for smooth and stable Riemann surfaces. We use them to define markings called Schottky structures. In the smooth case they are classified by the well known Schottky space S_g , a complex manifold of dimension $3g - 3$ (if $g \geq 2$). In [13] it was shown that also the Schottky structures on stable Riemann surfaces are parameterized by a complex manifold \overline{S}_g . Here we show how to obtain \overline{S}_g from Braungardt's extension \overline{T}_g of the Teichmüller space introduced in Chapter 3. In the last section of this chapter we study the image of a Teichmüller disk in the Schottky space.

5.1 Schottky coverings

We recall the basic definitions and properties of Schottky uniformization of Riemann surfaces. We introduce the Schottky space S_g and sketch, following [13], the construction of a universal family over it.

Definition 5.1. *A group $\Gamma \subset PSL_2(\mathbb{C})$ of Möbius transformations on $\mathbb{P}^1(\mathbb{C})$ is called a Schottky group if there are, for some $g \geq 1$, disjoint closed simply connected domains $D_1, D'_1, \dots, D_g, D'_g$ bounded by Jordan curves $C_i = \partial D_i$, $C'_i = \partial D'_i$, and generators $\gamma_1, \dots, \gamma_g$ of Γ such that $\gamma_i(C_i) = C'_i$ and $\gamma_i(D_i) = \mathbb{P}^1(\mathbb{C}) - \overline{D'_i}$ for $i = 1, \dots, g$. The generators $\gamma_1, \dots, \gamma_g$ are called a Schottky basis of Γ .*

In Schottky's original paper [32], the D_i in the definition were disks. With the same notation let

$$F = F(\Gamma) = \mathbb{P}^1(\mathbb{C}) - \cup_{i=1}^g (\overline{D}_i \cup \overline{D}'_i) \quad \text{and} \quad \Omega = \Omega(\Gamma) = \cup_{\gamma \in \Gamma} \gamma(F).$$

It is well known, see e. g. [25, X.H.] that Γ is a Kleinian group, free of rank g with free generators $\gamma_1, \dots, \gamma_g$, that Ω is the region of discontinuity of Γ , and that $X = \Omega/\Gamma$ is a closed Riemann surface of genus g . The quotient map $\Omega \rightarrow X$ is called a *Schottky covering*.

An important fact is the following uniformization theorem:

Proposition 5.2. *Every compact Riemann surface X of genus $g \geq 1$ admits a Schottky covering by a Schottky group of rank g .*

Proof. The proof is based on the following construction that we shall extend to stable Riemann surfaces in Section 5.4: choose disjoint simple loops c_1, \dots, c_g on X which are independent in homology, i. e. $F = X - \cup_{i=1}^g c_i$ is connected. Then F is conformally equivalent to a plane domain that is bounded by $2g$ closed Jordan curves. For $i = 1, \dots, g$ denote by C_i and C'_i the two boundary components of F that result from cutting along c_i . Now let Φ_g be a free group on generators $\varphi_1, \dots, \varphi_g$, and take a copy F_w of F for every element $w \in \Phi_g$. The F_w are glued according to the following rule: if w and w' are reduced words in $\varphi_1, \dots, \varphi_g$ and if $w = w'\varphi_i$ then the boundary component

C_i on $F_{w'}$ is glued to C'_i on F_w ; if w ends with φ_i^{-1} the roles of C_i and C'_i are interchanged. By this construction we obtain a plane domain Ω together with a holomorphic action of Φ_g on it: an element $\varphi \in \Phi_g$ maps the copy F_w to $F_{w\varphi}$. The crucial step in the proof now is to show that this action extends to all of $\mathbb{P}^1(\mathbb{C})$, i. e. Φ_g acts by Möbius transformations. For this we refer to [3, Ch. IV, Thm. 19F]. \square

Definition 5.3. Let \tilde{S}_g be the set of all $(\gamma_1, \dots, \gamma_g) \in \mathrm{PSL}_2(\mathbb{C})^g$ that generate a Schottky group Γ and form a Schottky basis for Γ . The set S_g of equivalence classes of g -tuples $(\gamma_1, \dots, \gamma_g) \in \tilde{S}_g$ under simultaneous conjugation is called the Schottky space of genus g .

For a point $s = (\gamma_1, \dots, \gamma_g) \in \tilde{S}_g$ let $\Gamma(s)$ be the Schottky group generated by $\gamma_1, \dots, \gamma_g$, $\Omega(s)$ the region of discontinuity of $\Gamma(s)$, and $X(s) = \Omega(s)/\Gamma(s)$ the associated Riemann surface. This leads to an alternative description of the Schottky space:

Remark 5.4. S_g is the set of equivalence classes of pairs (X, σ) , where X is a Riemann surface of genus g and $\sigma : \Phi_g \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is an injective homomorphism such that $\Gamma := \sigma(\Phi_g)$ is a Schottky group and $\Omega(\Gamma)/\Gamma \cong X$. (X, σ) and (X', σ') are equivalent if there is some $A \in \mathrm{PSL}_2(\mathbb{C})$ such that $\sigma'(\gamma) = A\sigma(\gamma)A^{-1}$ for all $\gamma \in \Phi_g$. Note that then X' is isomorphic to X .

To endow S_g with a complex structure we proceed as follows: Taking the fixed points and the multipliers of the γ_i we obtain an embedding of \tilde{S}_g as an open subdomain of $\mathbb{P}^1(\mathbb{C})^{3g}$. For $g = 1$ each equivalence class contains a unique Möbius transformation of the form $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$. If $g \geq 2$ we find in each equivalence class in \tilde{S}_g a unique representative $(\gamma_1, \dots, \gamma_g)$ such that γ_1 and γ_2 have attracting fixed points 0 and 1, respectively, and γ_1 has repelling fixed point ∞ . This defines a section to the projection $\tilde{S}_g \rightarrow S_g$ and embeds S_g as a closed subspace of \tilde{S}_g which, moreover, lies in $\{0\} \times \{\infty\} \times \{1\} \times \mathbb{C}^{3g-3} \subseteq \mathbb{P}^1(\mathbb{C})^{3g}$. Thus we have shown.

Proposition 5.5. a) S_1 is a punctured disk.

b) For $g \geq 2$, S_g carries a complex structure as an open subdomain of \mathbb{C}^{3g-3} .

Our next goal is to show that this complex structure on S_g is natural. The main step in this direction is

Proposition 5.6. The forgetful map $\mu : S_g \rightarrow M_g$, that sends $s = (X, \sigma)$ to the isomorphism class of X , is analytic and surjective.

Proof. The surjectivity of μ follows from Prop. 5.2. To show that μ is analytic we use the fact that M_g is a coarse moduli space for Riemann surfaces. Therefore it suffices to find a holomorphic family $\pi : \mathcal{C}_g \rightarrow S_g$ of Riemann surfaces over S_g which induces μ in the sense that for $s \in S_g$, $\mu(s)$ is the isomorphism class of the fibre $C_s = \pi^{-1}(s) \subset \mathcal{C}_g$.

The family \mathcal{C}_g is obtained as in Section 3.3: Let

$$\Omega_g = \{(s, z) \in S_g \times \mathbb{P}^1(\mathbb{C}) : z \in \Omega(s)\}.$$

Ω_g is a complex manifold on which the free group Φ_g acts holomorphically by $\varphi(s, z) = (s, \sigma(\varphi)(z))$ for $s = (X, \sigma) \in S_g$, $\varphi \in \Phi_g$ and $z \in \Omega(s)$.

The projection $\text{pr}_1 : \Omega_g \rightarrow S_g$ onto the first component factors through the orbit space $\mathcal{C}_g = \Omega_g/\Phi_g$, and the induced map $\pi : \mathcal{C}_g \rightarrow S_g$ is the family of Riemann surfaces we were looking for. \square

The family \mathcal{C}_g is in fact universal for Riemann surfaces with *Schottky structure*, a kind of marking that we now recall from [13, Section 1.3]:

Definition 5.7. a) *Let $\mathcal{U} \rightarrow S$ be an analytic map of complex manifolds and $\Gamma \subset \text{Aut}(\mathcal{U}/S)$ a properly discontinuous subgroup. Then the analytic quotient map $\mathcal{U} \rightarrow \mathcal{U}/\Gamma = \mathcal{C}$ is called a Schottky covering if the induced map $\mathcal{C} \rightarrow S$ is a family of Riemann surfaces and if for every $x \in S$ the restriction $U_x \rightarrow C_x$ of the quotient map to the fibres is a Schottky covering.*

b) *A Schottky structure is a Schottky covering $\mathcal{U} \rightarrow \mathcal{U}/\Gamma = \mathcal{C}$ together with an equivalence class of isomorphisms $\sigma : \Phi_g \rightarrow \Gamma$, where σ and σ' are considered equivalent if they differ only by an inner automorphism of Φ_g .*

Note that the construction in the proof of Proposition 5.6 endows the family \mathcal{C}_g/S_g with a Schottky structure.

A Schottky structure on a single Riemann surface X is given by a Schottky covering $\Omega \rightarrow \Omega/\Gamma = X$ and an isomorphism $\sigma : \Phi_g \rightarrow \Gamma$. Comparing the respective equivalence relations we find that the points (X, σ) in S_g correspond bijectively to the isomorphism classes of Riemann surfaces with Schottky structure. In fact a much stronger result holds:

Theorem 5.8. *S_g is a fine moduli space for Riemann surfaces with Schottky structure.*

Proof. Let \mathcal{C}/S be a family of Riemann surfaces and $(\mathcal{U} \rightarrow \mathcal{U}/\Gamma = \mathcal{C}, \sigma : \Phi_g \xrightarrow{\sim} \Gamma)$ a Schottky structure on \mathcal{C} . Then we have a map $f : S \rightarrow S_g$ which maps a point x to the isomorphism class of the Schottky covering $U_x \rightarrow C_x$. We have to show that f is analytic. Then the other properties of a fine moduli space follow easily from the definitions, namely that \mathcal{C} is the fibre product $\mathcal{C}_g \times_{S_g} S$ and that \mathcal{U} is isomorphic to $\Omega_g \times_{\mathcal{C}_g} \mathcal{C} = \Omega_g \times_{S_g} S$ such that the projection $\mathcal{U} \rightarrow \Omega_g$ onto the first factor is equivariant for the actions of Γ and Φ_g via the isomorphism σ .

The universal property of M_g as a coarse moduli space gives us, as above for μ , that the composition $\mu \circ f$ is analytic. Since μ has discrete fibres, it therefore suffices to show that f is continuous. This is quite subtle, see [13, § 3]. \square

5.2 Relation to Teichmüller space

In this section we explain that Schottky space can be obtained as a quotient space of the Teichmüller space which was introduced in Section 3.3. For this purpose we first endow the universal family $\mathcal{C}_{g,0}$ over the Teichmüller space $T_g = T_{g,0}$ with a Schottky structure as follows:

Let $a_1, b_1, \dots, a_g, b_g$ be a set of standard generators of π_g , the fundamental group of the reference surface X_{ref} ; this means that they satisfy the relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$. Then b_1, \dots, b_g are homologically independent, hence the construction in the proof of Prop. 5.2 provides us with a corresponding Schottky covering $\Omega_{\text{ref}} \rightarrow X_{\text{ref}}$. The group $\text{Aut}(\Omega_{\text{ref}}/X_{\text{ref}})$ of deck transformations is isomorphic to the free group on b_1, \dots, b_g . Denoting $U_{\text{ref}} \rightarrow X_{\text{ref}}$ the universal covering, there is a covering map $U_{\text{ref}} \rightarrow \Omega_{\text{ref}}$ over X_{ref} . The group $\text{Aut}(U_{\text{ref}}/\Omega_{\text{ref}})$ is the kernel N_α of the homomorphism $\alpha : \pi_g \rightarrow \Phi_g$ which maps b_i to φ_i and a_i to 1; in other words, N_α is the normal closure in π_g of the subgroup generated by a_1, \dots, a_g .

In Section 3.3 we described the family $\Omega_{g,0}^+ \rightarrow \mathcal{C}_{g,0}$ of universal coverings of the surfaces in the family $\mathcal{C}_{g,0}$; the fundamental group π_g and hence also N_α acts on the fibres of this covering, and we obtain:

Remark 5.9. *The induced map $\Omega_{g,0}^+/N_\alpha \rightarrow \mathcal{C}_{g,0}$ is a Schottky covering, and the universal Teichmüller structure $\tau : \pi_g \xrightarrow{\sim} \text{Aut}(\Omega_{g,0}^+/\mathcal{C}_{g,0})$ (cf. Theorem 3.13) descends via α to a Schottky structure $\sigma_\alpha : \Phi_g = \pi_g/N_\alpha \xrightarrow{\sim} \text{Aut}((\Omega_{g,0}^+/N_\alpha)/\mathcal{C}_{g,0})$ on $\mathcal{C}_{g,0}$.*

By Theorem 5.8 this Schottky structure induces an analytic map $s_\alpha : T_g \rightarrow S_g$. To describe s_α as the quotient map for a subgroup of the mapping class group Γ_g , we first identify Γ_g with the group $\text{Out}^+(\pi_g)$ of orientation preserving outer automorphisms of π_g ; then, to a diffeomorphism $f : X_{\text{ref}} \rightarrow X_{\text{ref}}$, we associate the induced automorphism $\varphi = f_* : \pi_g \rightarrow \pi_g$. It follows from the Dehn-Nielsen theorem that this gives an isomorphism $\Gamma_g \xrightarrow{\sim} \text{Out}^+(\pi_g)$. In this chapter, by $\varphi \in \Gamma_g$ we always mean an element of $\text{Out}^+(\pi_g)$.

Proposition 5.10. a) s_α is the quotient map for the subgroup

$$\Gamma_g(\alpha) = \{\varphi \in \Gamma_g : \alpha \circ \varphi \equiv \alpha \text{ mod } \text{Inn}(\pi_g)\}$$

of the mapping class group Γ_g (where $\text{Inn}(\pi_g)$ denotes the group of inner automorphisms).

b) $s_\alpha : T_g \rightarrow S_g$ is the universal covering of the Schottky space.

c) s_α lifts to maps \tilde{s}_α and ω_α that make the following diagram commutative:

$$\begin{array}{ccccc}
\Omega_{g,0}^+ & & & & \\
\downarrow / \pi_g & \searrow / N_\alpha & & & \\
& & \Omega_{g,0}^+ / N_\alpha & \xrightarrow{\omega_\alpha} & \Omega_g \\
& & \swarrow & & \downarrow / \Phi_g \\
\mathcal{C}_{g,0} & \xrightarrow{\tilde{s}_\alpha} & \mathcal{C}_g & & \\
\downarrow & & \downarrow & & \\
T_g & \xrightarrow{s_\alpha} & S_g & & \\
\swarrow / \Gamma_g & & \swarrow \mu & & \\
& & M_g & &
\end{array}$$

Proof. **a)** Let $x = (X, f) \in T_g$. Recall from Section 3.3 that the fibre over x in $\Omega_{g,0}^+$ is the component $\Omega^+(x)$ of the region of discontinuity of the quasifuchsian group G_x associated to x . The universal Teichmüller structure on $\mathcal{C}_{g,0}$ induces an isomorphism $\tau_x : \pi_g \rightarrow G_x = \text{Aut}(\Omega^+(x)/X)$. From Remark 5.9 we see that the point $s_\alpha(x) = (X, \sigma) \in S_g$ is given by the restriction $\sigma_{\alpha,x}$ of σ_α to the fibre over x ; explicitly,

$$\sigma = \sigma_{\alpha,x} : \Phi_g = \pi_g / N_\alpha \xrightarrow{\sim} \text{Aut}((\Omega^+(x) / \tau_x(N_\alpha)) / X) = G_x / \tau_x(N_\alpha).$$

For $\varphi \in \Gamma_g$ we have $s_\alpha(x) = s_\alpha(\varphi(x))$ if and only if $\sigma_{\alpha,x} = \sigma_{\alpha,\varphi(x)}$ up to an inner automorphism. Since $\tau_{\varphi(x)} = \tau_x \circ \varphi^{-1}$ this happens if and only if φ induces an inner automorphism on π_g / N_α , i. e. if and only if $\varphi \in \Gamma_g(\alpha)$.

b) This is clear from the fact that T_g is simply connected and $\Gamma_g(\alpha)$ is torsion free, hence s_α is unramified. Using the construction in a) one can give a direct proof which in turn provides an independent proof that T_g is simply connected, see [13, Prop. 6].

c) It follows from Remark 5.9 that $\Omega_{g,0}^+ / N_\alpha \rightarrow \mathcal{C}_{g,0}$ is a Schottky covering. Therefore, by the universal property of S_g (Theorem 5.8), $\mathcal{C}_{g,0}$ is the fibre product $T_g \times_{S_g} \mathcal{C}_g$, and \tilde{s}_α is the projection to \mathcal{C}_g . Moreover the Schottky covering $\Omega_{g,0}^+ / N_\alpha \rightarrow \mathcal{C}_{g,0}$ is a pullback of the universal Schottky covering $\Omega_g \rightarrow \mathcal{C}_g$, i. e. $\Omega_{g,0}^+ / N_\alpha = \mathcal{C}_{g,0} \times_{\mathcal{C}_g} \Omega_g$, and again ω_α is the projection to the second factor. \square

In fact, the action of $\Gamma_g(\alpha)$ on T_g extends to $\Omega_{g,0}^+$, $\Omega_{g,0}^+ / N_\alpha$ and $\mathcal{C}_{g,0}$; then \tilde{s}_α and ω_α are the quotient maps for these actions.

5.3 Schottky coverings of stable Riemann surfaces

In this and the following section we introduce a partial compactification \overline{S}_g of S_g that fits in between \overline{T}_g and \overline{M}_g . We have presented two different ways to define S_g , and we shall see that both are suited for extension to stable Riemann surfaces: The first way is to construct Schottky coverings for surfaces

with nodes, define Schottky structures and find parameters for them. This approach was pursued in [13] and will be sketched in this section. The other possibility is to extend the action of $\Gamma_g(\alpha)$ to (part of) the boundary of \overline{T}_g and show that the quotient exists and has the desired properties; this will be done in Section 5.4.

Definition 5.11. *Let X be a stable Riemann surface of genus g . A cut system on X is a collection of disjoint simple loops c_1, \dots, c_g on X , not passing through any of the nodes, such that $X - \cup_{i=1}^g c_i$ is connected.*

Proposition 5.12. *On any stable Riemann surface there exist cut systems.*

Proof. Let $f : X_{\text{ref}} \rightarrow X$ be a deformation; we must find disjoint and homologically independent loops $\tilde{c}_1, \dots, \tilde{c}_g$ on X_{ref} that are disjoint from the loops a_1, \dots, a_k that are contracted by f . For this we complete a_1, \dots, a_k to a maximal system a_1, \dots, a_{3g-3} of homotopically independent loops (such a system decomposes X_{ref} into pairs of pants). Among the a_i we find a_{i_1}, \dots, a_{i_g} that are homologically independent. If $i_\nu > k$ we take $\tilde{c}_\nu = a_{i_\nu}$, and for $i_\nu \leq k$ we replace a_{i_ν} by a loop \tilde{c}_ν that is homotopic to a_{i_ν} and disjoint from it. \square

Once we have found c_1, \dots, c_g as above, we proceed as in the proof of Proposition 5.2 to construct a Schottky covering of X : Let $F = X - \cup_{i=1}^g c_i$, take a copy F_w of F for each $w \in \Phi_g$, and glue these copies exactly as before to obtain a space Ω . Of course, neither F nor Ω is planar whenever X has nodes. In all cases, the complex structure on X lifts to a structure of a one-dimensional complex space on F . The group Φ_g acts on this space by holomorphic automorphisms. Precisely, there is an isomorphism $\Phi_g \rightarrow \Gamma = \text{Aut}(\Omega/X)$, and X is isomorphic to Ω/Γ as complex space.

Definition 5.13. *The covering $\Omega \rightarrow X$ constructed above for a cut system $c = (c_1, \dots, c_g)$ on a stable Riemann surface X is called the Schottky covering of X relative to c . A covering of X is called a Schottky covering if it is the Schottky covering relative to some cut system.*

The next goal is to define a space \overline{S}_g that classifies Schottky coverings in a way analogous to Definition 5.3. Since the covering space Ω is in general not a subspace of $\mathbb{P}^1(\mathbb{C})$ and thus the group of deck transformations not a subgroup of $\text{PSL}_2(\mathbb{C})$, we cannot directly extend 5.3.

A closer look at the construction of a Schottky covering $\Omega \rightarrow \Omega/\Gamma = X$ of a stable Riemann surface X shows the following:

Each irreducible component L of Ω is an open dense subset of a projective line; more precisely, the stabilizer of L in Γ is a Schottky group as in Definition 5.1, and L is its region of discontinuity. Moreover the intersection graph of the irreducible components of Ω is a tree (hence Ω is called a *tree of projective lines*).

Therefore, for each irreducible component L , there is a well defined projection $\pi_L : \Omega \rightarrow L$ which is the identity on L : For an arbitrary point $x \in \Omega$ there is a unique chain $L_0, L_1, \dots, L_n = L$ of mutually distinct components such that

$x \in L_0$ and L_i intersects L_{i+1} for $i = 0, \dots, n-1$; then define $\pi_L(x)$ to be the intersection point of L_{n-1} and L .

An *end* of Ω is an equivalence class of infinite chains L_0, L_1, L_2, \dots of irreducible components as above (i. e. $L_i \neq L_j$ for $i \neq j$ and $L_i \cap L_{i+1} \neq \emptyset$), where two chains are equivalent if they differ only by finitely many components. Let $\Omega^* = \Omega \cup \{\text{ends of } \Omega\}$. Clearly the projection π_L to a component L can be extended to Ω^* .

For any three different points or ends y_1, y_2, y_3 in Ω^* there is a unique component $L = L(y_1, y_2, y_3)$ (called the *median* of the three points) such that the points $\pi_L(y_1), \pi_L(y_2), \pi_L(y_3)$ are distinct. Now given any four distinct points or ends y_1, \dots, y_4 in Ω^* we can define a *cross ratio* $\lambda(y_1, \dots, y_4)$ by taking the usual cross ratio of $\pi_L(y_1), \dots, \pi_L(y_4)$ on the median component $L = L(y_1, y_2, y_3)$ of the first three of them; note that $\lambda(y_1, \dots, y_4)$ will be 0, 1 or ∞ if $\pi_L(y_4)$ coincides with $\pi_L(y_1)$, $\pi_L(y_2)$ or $\pi_L(y_3)$.

To obtain parameters for the group Γ observe that any $\gamma \in \Gamma$, $\gamma \neq 1$, has exactly two fixed points on the boundary of Ω , where boundary points of Ω are either points in the closure of a component, or ends of Ω ; one of the fixed points is attracting, the other repelling. For any four different (primitive) elements $\gamma_1, \dots, \gamma_4$ in Γ we define $\lambda(\gamma_1, \dots, \gamma_4)$ to be the cross ratio of their attracting fixed points. It is a remarkable fact that from these cross ratios both the space Ω and the group $\Gamma \subset \text{Aut}(\Omega)$ can be recovered. For any particular Schottky covering finitely many of them suffice, but for different Schottky coverings we must take, in general, the cross ratios of different elements of Φ_g . To parameterize all Schottky coverings we therefore have to use infinitely many of these cross ratios. We consider them as (projective) coordinates on an infinite product of projective lines $\mathbb{P}^1(\mathbb{C})$. The cross ratios satisfy a lot of algebraic relations, which define a closed subset B of this huge space. Every point of B represents a tree of projective lines Ω as above together with an action of Γ on it. \overline{S}_g is the open subset of B , where this action defines a Schottky covering. For details and in particular the technical complication caused by the presence of infinitely many variables and equations, see [13, §2] and [17]. In principle, one can proceed as in Section 5.1 to construct a family of stable Riemann surfaces over \overline{S}_g .

Given a family \mathcal{C}/S of stable Riemann surfaces over a complex manifold S , we can define the notion of a *Schottky covering* $\mathcal{U}/S \rightarrow \mathcal{C}/S$ and of a *Schottky structure* on \mathcal{U} exactly as in Definition 5.7, except that now \mathcal{U} is not assumed to be a manifold, but only a complex space. It is shown in [13, §3] that the family over \overline{S}_g carries a universal Schottky structure:

Theorem 5.14. *\overline{S}_g is a fine moduli space for stable Riemann surfaces with Schottky structure.*

5.4 \overline{S}_g as quotient of \overline{T}_g

It is not possible to extend the quotient map $s_\alpha : T_g \rightarrow S_g$ constructed in Section 5.2 to the whole boundary of T_g in \overline{T}_g . Instead we shall, for each

α , identify a part $\overline{T}_g(\alpha)$ of \overline{T}_g to which the action of $\Gamma_g(\alpha)$ and hence the morphism s_α can be extended. It will turn out that the quotient space is the extended Schottky space \overline{S}_g described in the previous section.

We begin with the definition of the admissible group homomorphisms α and the associated parts $\overline{T}_g(\alpha)$ of \overline{T}_g :

Definition 5.15. a) *A surjective homomorphism $\alpha : \pi_g \rightarrow \Phi_g$ is called symplectic if there are standard generators $a_1, b_1, \dots, a_g, b_g$ of π_g (in the sense of Section 5.2) such that $\alpha(a_i) = 1$ for $i = 1, \dots, g$.*

b) *Recall from Chapter 3 that a point in \overline{T}_g can be described as an equivalence class of pairs (X, f) , where X is a stable Riemann surface and $f : X_{\text{ref}} \rightarrow X$ is a deformation (see Corollary 3.9).*

For a symplectic homomorphism $\alpha : \pi_g \rightarrow \Phi_g$ let

$$\overline{T}_g(\alpha) = \{(X, f) \in \overline{T}_g : \ker(f_*) \subseteq \ker(\alpha)\}.$$

Proposition 5.16. a) *$\overline{T}_g(\alpha)$ is an open subset of \overline{T}_g ; it contains T_g and is invariant under the group $\Gamma_g(\alpha)$ introduced in Prop. 5.10.*

b) *\overline{T}_g is the union of the $\overline{T}_g(\alpha)$, where α runs through the symplectic homomorphisms.*

c) *The restriction to $\overline{T}_g(\alpha)$ of the universal covering $p : \overline{T}_g \rightarrow \overline{M}_g$ is surjective for every symplectic α .*

Proof. a) Let (X, f) be a point in \overline{T}_g and c_1, \dots, c_k the loops on X_{ref} that are contracted under f . Then the kernel of $\pi_1(f) : \pi_g \rightarrow \pi_1(X)$ is the normal subgroup generated by c_1, \dots, c_k . The local description of \overline{T}_g in Corollary 3.8 shows that there is a neighbourhood U of (X, f) in \overline{T}_g such that for every $(X', f') \in U$ the map $f' : X_{\text{ref}} \rightarrow X'$ contracts a subset of $\{c_1, \dots, c_k\}$. Hence the kernel of $\pi_1(f')$ is contained in $\ker(\pi_1(f))$. Thus if $(X, f) \in \overline{T}_g(\alpha)$, also $U \subseteq \overline{T}_g(\alpha)$. The remaining assertions are clear.

b) Again let (X, f) be a point in \overline{T}_g and c_1, \dots, c_k the loops on X_{ref} contracted by f . By Proposition 5.12 we can find a cut system a_1, \dots, a_g on X and a corresponding Schottky covering. This covering induces a surjective homomorphism $\pi_1(X) \rightarrow \Phi_g$. Composing this homomorphism with $\pi_1(f)$ yields a homomorphism $\alpha : \pi_g \rightarrow \Phi_g$ which corresponds to a Schottky covering of X_{ref} (relative to the cut system $f^{-1}(a_1), \dots, f^{-1}(a_g)$) and hence is symplectic. By construction, c_1, \dots, c_k are in the kernel of α .

c) Let $\alpha : \pi_g \rightarrow \Phi_g$ be symplectic and $a_1, b_1, \dots, a_g, b_g$ standard generators of π_g such that $\alpha(a_i) = 1$ for all i . For an arbitrary stable Riemann surface X choose a deformation $f : X_{\text{ref}} \rightarrow X$ and let c_1, \dots, c_k be the loops that are contracted by f . As in the proof of b) we find standard generators $a'_1, b'_1, \dots, a'_g, b'_g$ such that the c_j are contained in the normal subgroup generated by the a'_i . The map $a_i \mapsto a'_i, b_i \mapsto b'_i$ defines an automorphism φ of π_g and thus an element of Γ_g . Then by construction $(X, f \circ \varphi)$ lies in $\overline{T}_g(\alpha)$ and $p(X, f \circ \varphi) = X$. \square

As a side remark we note that $\overline{T}_g(\alpha)$ is not only invariant under $\Gamma_g(\alpha)$, but also under the larger “handlebody” group

$$H_g(\alpha) = \{\varphi \in \Gamma_g : \varphi(N_\alpha) = N_\alpha\}$$

(where $N_\alpha = \ker(\alpha)$ as in Section 5.2). Note that $H_g(\alpha)$ is the normalizer of $\Gamma_g(\alpha)$ in Γ_g , and that we have an exact sequence

$$1 \rightarrow \Gamma_g(\alpha) \rightarrow H_g(\alpha) \rightarrow \text{Out}(\Phi_g) \rightarrow 1.$$

The quotient space $\hat{S}_g = T_g/H_g(\alpha) = S_g/\text{Out}(\Phi_g)$ is a parameter space for Schottky groups of rank g (without any marking).

Proposition 5.17. *For any symplectic homomorphism $\alpha : \pi_g \rightarrow \Phi_g$, the quotient space $\overline{T}_g(\alpha)/\Gamma_g(\alpha)$ is a complex manifold $\overline{S}_g(\alpha)$.*

Proof. This is a local statement which is clear for points $(X, f) \in T_g$ since $\Gamma_g(\alpha)$ is torsion free. For an arbitrary $x = (X, f) \in \overline{T}_g$ we saw in Section 3.2 that the Dehn twists τ_1, \dots, τ_k around the loops c_1, \dots, c_k that are contracted by f generate a finite index subgroup Γ_x^0 of the stabilizer Γ_x of x in Γ_g (the quotient being the finite group $\text{Aut}(X)$). Let α be a symplectic homomorphism with respect to standard generators $a_1, b_1, \dots, a_g, b_g$, and assume $(X, f) \in \overline{T}_g(\alpha)$. Since the c_i are in the normal subgroup generated by a_1, \dots, a_g , they do not intersect any of the a_j and thus $\tau_i(a_j) = a_j$ for all i and j . This shows $\Gamma_x \subseteq \Gamma_g(\alpha)$.

Now choose a neighbourhood U of $x = (X, f)$ in $\overline{T}_g(\alpha)$ which is precisely invariant under Γ_x . Then it follows, from Proposition 3.7 (and Definition 3.6), that U/Γ_x is a complex manifold. \square

For any two sets $a_1, b_1, \dots, a_g, b_g$ and $a'_1, b'_1, \dots, a'_g, b'_g$ of standard generators, $a_i \mapsto a'_i, b_i \mapsto b'_i$ defines an automorphism of π_g . Therefore for any two symplectic homomorphisms α and α' there is an automorphism $\psi \in \Gamma_g$ such that $\alpha = \alpha' \circ \psi$. Then clearly $N_\alpha = \psi(N_{\alpha'})$ and $\Gamma_g(\alpha') = \psi\Gamma_g(\alpha)\psi^{-1}$. This shows that, as an automorphism of \overline{T}_g , ψ maps $\overline{T}_g(\alpha)$ to $\overline{T}_g(\alpha')$ and descends to an isomorphism $\overline{\psi} : \overline{S}_g(\alpha) \rightarrow \overline{S}_g(\alpha')$. We have shown:

Remark 5.18. *The complex manifolds $\overline{S}_g(\alpha)$ are isomorphic for all symplectic homomorphisms α .*

It remains to show that the $\overline{S}_g(\alpha)$ coincide with the fine moduli space \overline{S}_g of Section 5.3. This is achieved by showing that $\overline{S}_g(\alpha)$ satisfies the same universal property as \overline{S}_g :

Proposition 5.19. *For any symplectic α , $\overline{S}_g(\alpha)$ is a fine moduli space for stable Riemann surfaces with Schottky structure and hence isomorphic to \overline{S}_g .*

Proof. The idea of the proof is to endow the universal family over $\overline{T}_g(\alpha)$ with a Schottky structure and to transfer this to a Schottky structure on the image family over $\overline{S}_g(\alpha)$.

Before explaining this for the whole family we consider a single stable Riemann

surface X . Let d_1, \dots, d_k be the nodes on X , $f : X_{\text{ref}} \rightarrow X$ a deformation and $\alpha : \pi_g \rightarrow \Phi_g$ a symplectic homomorphism such that $x = (X, f) \in \overline{T}_g(\alpha)$. In Section 3.3 we described the universal covering $\hat{\Omega}^+(x) \rightarrow \hat{\Omega}^+(x)/G_x = X$ of X with cusps over the nodes. Recall that $\hat{\Omega}^+(x)$ is the union of the plane region $\Omega^+(x)$ with the common boundary points of the doubly cusped regions lying over the nodes d_i , and that G_x is isomorphic to π_g .

Remark 5.20. *Using the above notation, let $\rho : \pi_g \rightarrow G_x$ be an isomorphism and $N_\alpha^{G_x} = \ker(\alpha \circ \rho^{-1}) \subseteq G_x$. Then $\Omega = \hat{\Omega}^+(x)/N_\alpha^{G_x}$ is a complex space, $G_x/N_\alpha^{G_x} \cong \Phi_g$ acts holomorphically on Ω , and $\Omega \rightarrow \Omega/\Phi_g = X$ is a Schottky covering.*

Proof. The key observation is that the stabilizer in G_x of a point $\tilde{d}_i \in \hat{\Omega}^+(x)$ lying over d_i is generated by an element γ_i corresponding under ρ to a conjugate of the loop $f^{-1}(d_i)$. Since we assumed $(X, f) \in \overline{T}_g(\alpha)$, we have $\gamma_i \in N_\alpha^{G_x}$. This shows that Ω is a complex space, more precisely: a Riemann surface with nodes. The other assertions then follow directly from the definitions. \square

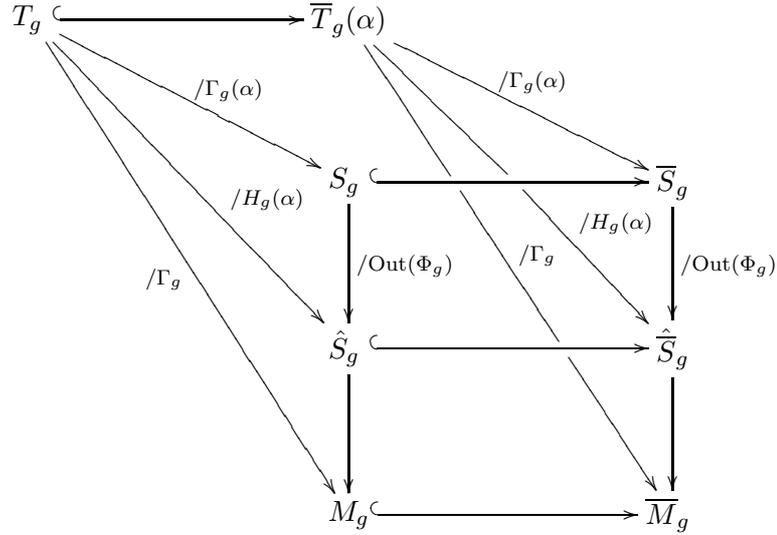
The above construction can be carried over to families in the following way: First consider the universal family $\overline{\mathcal{C}}_g$ over \overline{T}_g and the universal Teichmüller structure $\hat{\Omega}_g^+ \rightarrow \overline{\mathcal{C}}_g$ on it. Denote by $\overline{\mathcal{C}}_g(\alpha)$ resp. $\hat{\Omega}_g^+(\alpha)$ the restriction to $\overline{T}_g(\alpha)$. Then the quotient space $\hat{\Omega}_g^+(\alpha)/N_\alpha$ is a complex space on which $\Phi_g = \pi_g/N_\alpha$ acts. The quotient map $\hat{\Omega}_g^+(\alpha)/N_\alpha \rightarrow \overline{\mathcal{C}}_g(\alpha)$ is a Schottky covering and the identification of Φ_g with the group of deck transformations defines a Schottky structure.

The group $\Gamma_g(\alpha)$ acts not only on $\overline{\mathcal{C}}_g(\alpha)$, but also on $\hat{\Omega}_g^+(\alpha)$ as follows: for $\varphi \in \Gamma_g(\alpha)$ and $(x, z) \in \hat{\Omega}_g^+(\alpha)$ with $x \in \overline{\mathcal{C}}_g(\alpha)$ and $z \in \hat{\Omega}_g^+(\alpha)$ we set

$$\varphi(x, z) = (\varphi(x), z).$$

Note that the groups G_x and $G_{\varphi(x)}$ are the same (only the isomorphism with π_g has changed); therefore $\hat{\Omega}^+(x) = \hat{\Omega}^+(\varphi(x))$. This action, which is trivial on the fibres, descends to actions of $\Gamma_g(\alpha)$ on $\hat{\Omega}_g^+(\alpha)/N_\alpha$ and on $\overline{\mathcal{C}}_g(\alpha)$. The respective orbit spaces give a family $\overline{\mathcal{C}}_g = \overline{\mathcal{C}}_g(\alpha)/\Gamma_g(\alpha)$ over $\overline{S}_g(\alpha)$ and a Schottky structure on it. Using the universal property of the family over $\overline{T}_g(\alpha)$ (see Theorem 3.13) and the fact that Schottky structures are locally induced by Teichmüller structures, we find that the Schottky structure on $\overline{\mathcal{C}}_g$ is in fact universal. \square

The following diagram collects the relations between the spaces introduced and used in this section. The horizontal maps are open embeddings, the last two vertical maps are analytic with discrete fibres; all other maps in the diagram are quotient maps for the groups indicated (to be precise, the map from $\overline{T}_g(\alpha)$ to \overline{M}_g is the restriction of the orbit map for the action of Γ_g on \overline{T}_g).



5.5 Teichmüller disks in Schottky space

Let $\iota : \mathbb{H} \rightarrow T_g$ be a Teichmüller embedding as in Definition 2.4 and $\Delta = \iota(\mathbb{H})$ its image in T_g . Let $\text{Stab}(\Delta)$ be the stabilizer of Δ in Γ_g . We have seen in Section 2.4.3 that $\text{Stab}(\Delta)$ maps surjectively to the projective Veech group $\overline{\Gamma}_\iota$ of ι (see Definition 2.17); the kernel of this map is the pointwise stabilizer of Δ .

In this section we assume that $\overline{\Gamma}_\iota$ is a lattice in $\text{PSL}_2(\mathbb{R})$, or equivalently that the image C_ι of Δ in M_g is a Teichmüller curve (cf. Corollary 2.21). As mentioned in the introduction, Veech showed that C_ι is not a projective curve and thus cannot be closed in \overline{M}_g .

Proposition 5.21. *Let $\iota : \mathbb{H} \rightarrow T_g$ be a Teichmüller embedding such that $\overline{\Gamma}_\iota$ is a lattice in $\text{PSL}_2(\mathbb{R})$. Then there exists a symplectic homomorphism $\alpha : \pi_g \rightarrow \Phi_g$ such that*

$$\text{Stab}(\Delta) \cap \Gamma_g(\alpha) \neq \{1\}.$$

Since $\Gamma_g(\alpha)$ is torsion free, this implies that the intersection is infinite. As a consequence, the image of the Teichmüller disk Δ in the Schottky space S_g is the quotient by an infinite group and in particular not isomorphic to a disk.

Proof. Denote by $\overline{\Delta}$ and \overline{C}_ι the closures of Δ and C_ι in \overline{T}_g and \overline{M}_g , respectively. Since C_ι is not closed, we can find a point $z \in \overline{C}_\iota - C_\iota$; let $x \in \overline{\Delta}$ be a point above z . By Prop. 5.16 b) there is a symplectic homomorphism α such that $x \in \overline{T}_g(\alpha)$.

Let $\overline{s}_\alpha : \overline{T}_g(\alpha) \rightarrow \overline{S}_g$ be the quotient map for $\Gamma_g(\alpha)$ (see Prop. 5.17 and Prop. 5.19) and let $D(\iota) = s_\alpha(\Delta)$ be the image of Δ in S_g . Then the closure $\overline{D}(\iota)$ of $D(\iota)$ in \overline{S}_g contains $\overline{s}_\alpha(x)$, and we have $\overline{C}_\iota = \overline{\mu}(\overline{D}(\iota))$, cf. the diagram at the end of Section 5.4.

By our assumption, \overline{C}_ι is Zariski closed in \overline{M}_g . Therefore $\overline{\mu}^{-1}(\overline{C}_\iota)$ is an analytic subset of \overline{S}_g . $\overline{D}(\iota)$ is an irreducible component of $\overline{\mu}^{-1}(\overline{C}_\iota)$ and hence also an analytic subset.

Recall, from Corollary 2.21, that $\Delta/\text{Stab}(\Delta)$ is the normalization of C_ι . Furthermore, by Prop. 4.14, $\overline{\Delta}$ is isomorphic to $\mathbb{H} \cup \{\text{cusps of } \overline{\Gamma}_\iota^*\}$. Therefore $\overline{\Delta}/\text{Stab}(\Delta)$ is the normalization of \overline{C}_ι . The restriction of the quotient map $\overline{\Delta} \rightarrow \overline{\Delta}/\text{Stab}(\Delta)$ to the intersection $\overline{\Delta}_\alpha = \overline{\Delta} \cap \overline{T}_g(\alpha)$ factors through \overline{s}_α . If the intersection $\text{Stab}(\Delta) \cap \Gamma_g(\alpha)$ was trivial, this restriction would be an isomorphism. But then $\overline{\Delta}_\alpha$ would be isomorphic to an analytic subset of a complex manifold. This is impossible since $\overline{\Delta}_\alpha$ contains $x \in \overline{T}_g - T_g$ and hence is not a complex space. \square

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