Origamis with non congruence Veech groups

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In this article we give an introduction to origamis (often also called square-tiled surfaces) and their Veech groups. As main theorem we prove that in each genus there exist origamis, whose Veech groups are non congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

The basic idea of an *origami* is to obtain a topological surface from a few combinatorial data by gluing finitely many Euclidean unit squares according to specified rules. These surfaces come with a natural *translation structure*. One assigns in general to a translation surface a subgroup of $\mathrm{GL}_2(\mathbb{R})$ called *the Veech group*. In the case of surfaces defined by origamis, the Veech groups are finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. These groups are the objects we study in this article.

One motivation to be interested in Veech groups is their relation to *Teichmüller disks* and *Teichmüller curves*, see e.g. the article [H 06] of F. Herrlich in this volume: A translation surface of genus g defines in a geometric way an embedding of the upper half plane into the Teichmüller space T_g of closed Riemann surfaces of genus g. The image is called Teichmüller disk. Its projection to the moduli space M_g is sometimes a complex algebraic curve, called *Teichmüller curve*. More precisely this happens, if and only if the Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case the algebraic curve can be determined from the Veech group up to birationality.

It is hard to determine the Veech group for a general translation surface. However, if the translation surface comes from an origami there is a special approach to this problem. It is based on the idea of describing origamis by finite index subgroups of F_2, the free group in two generators. This leads to a characterization of origami Veech groups as the images in $\mathrm{SL}_2(\mathbb{Z})$ of certain subgroups of $\mathrm{Aut}(F_2)$, the automorphism group of F_2.

Using this approach we will calculate Veech groups of two origamis explicitly. They turn out to be non congruence groups. Starting from these examples we obtain infinite sequences of origamis all of whose Veech groups are non congruence groups. This leads to the following theorem.

Theorem 1. *Each moduli space M_g ($g \geq 2$) contains an origami curve whose Veech group is a non congruence group.*

In Section 1 we introduce origamis and present different equivalent ways to describe them. In Section 2 we give a glance on the mathematical context. We describe, how an origami defines a family of translation surfaces and explain roughly , how one obtains a Teichmüller curve in moduli space starting from an origami. We introduce Veech groups and shortly point out their relation to Teichmüller curves. In Section 3 we turn to Veech groups of origamis and present a characterization of them in terms of automorphisms of the free group F_2 in two generators. We use this characterization to calculate two examples explicitly. Finally, in Section 4 we show that these two examples produce Veech groups that are non congruence groups and give a method to construct out of them infinite sequences of Veech groups that are again non congruence groups.

The first part (Section 1 - Section 3) of this article is meant to give a handy introduction to origamis and an overview on some of our results about their Veech groups. In the second part we state and prove Theorem 1 based on the results in the PhD thesis [S 05] of the author.

For a broader introduction and overview on origamis and Teichmüller curves as well as for references to the larger context, we refer the the reader e.g. to [HeSc 06], [S 04] and [S 05].

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1 Origamis

There are several ways to define origamis. We start with the somehow playful description that we have learned from [Lo 05], where also the name *origami* was introduced: An *origami* is obtained by gluing the edges of finitely many copies $Q_1, \ldots, Q_d$ of the Euclidean square Q via translations according to the following rules:

- Each left edge shall be identified to a right edge and vice versa.
- Similarly, each upper edge shall be identified to a lower one.
- The arising closed surface X shall be connected.

We only study what is called oriented origamis in [Lo 05] and call them just *origamis.*

Example 1.1.

a) *The simplest example is the origami that is made from only one square. There is precisely one possibility to glue its edges according to the rules. One obtains a torus E. We call this origami the* trivial origami O_0.

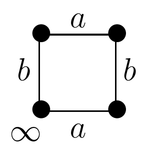

Figure 1: The trivial origami. Opposite edges are glued.

Observe that the four vertices of the square are all identified and become one point on the closed surface E. We call this point ∞.

b) *We now consider an origami made from four squares, see Figure 2. Some identifications of the edges are already done in the picture. For all other edges those having same labels are glued. The origami is called $L(2,3)$ for obvious reasons.*

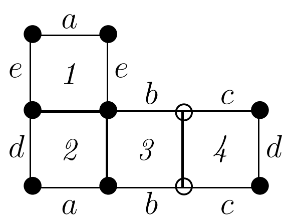

Figure 2: The origami $L(2,3)$. Opposite edges are glued.

Observe that in this case the vertices labeled with $\bullet$ and the vertices labeled with $\circ$ are respectively identified and become two points on the closed surface X. By calculating the Euler characteristic one obtains, that the genus of the surface X is 2.

c) *Finally, we consider an example with five squares, see Figure 3. Here, edges with same labels are identified. For the unlabeled edges, those which are opposite to each other are glued. We call the origami D.*

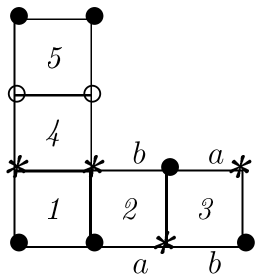

Figure 3: The origami D. Edges with the same label and unlabeled edges that are opposite are glued.

In this case, we obtain the three identification classes $\circ$, $\star$ and $\bullet$ for the vertices. The genus of the closed surface X is again 2.

Origamis as coverings of a torus

Observe, that the trivial origami O_0 from Example 1.1 a) is universal in the following sense: If X is the closed surface that arises from an arbitrary origami O and E the torus that arises from O_0, then we have a natural map $X \to E$ by mapping each of the unit squares of the origami O that form the surface X to the one unit square of O_0 that forms the torus E. This map is a covering that is unramified except over the one point $\infty \in E$. Conversely, given a closed surface X together with such a covering $p : X \to E$, we obtain a decomposition of X into squares by cutting X along the preimages of the edges of the one square of O_0 that forms E. This motivates the following definition of *origamis*.

Definition 1.2. *An* origami O *of* genus g *and* degree d *is a covering* $p : X \to E$ *of degree d from a closed, oriented (topological) surface X of genus g to the torus E that is ramified over at most one marked point $\infty \in E$.*

Remark that we have fixed here one torus E and one point $\infty \in E$. In particular we may furthermore fix a point $M \neq \infty$ on E and a set of standard generators of the fundamental group $\pi_1(E, M)$ that do not pass through ∞. That way we obtain a fixed isomorphism

$$\pi_1(E^*) \cong F_2, \tag{1}$$

where $E^* = E - \infty$ and $F_2 = F_2(x, y)$ is the free group in two generators x and y. Describing E by gluing the edges of the unit square via translations, we choose M to be the midpoint of the unit square and the standard generators to be the horizontal and the vertical simply closed curve through M, see Figure 4.

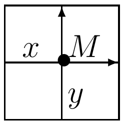

Figure 4: Generators of $\pi_1(E^*)$.

Example 1.3. *In Example 1.1, in a) the covering is the identity $id : E \to E$. In b) we have a covering $p : X \to E$ of degree 4 that is ramified in the two points labeled by $\bullet$ and $\circ$. Recall that the genus of X is 2. In c) we have a covering $p : X \to E$ of degree 5 ramified in the two points labeled by $\bullet$ and $\star$. Observe that though the point on X labeled by $\circ$ is a preimage of ∞, the covering is not ramified in this point. The genus of X is again 2.*

Definition 1.4. *We say that two origamis $O_1 = (p_1 : X_1 \to E)$ and $O_2 = (p_2 : X_2 \to E)$ are* equivalent*, if there is a homeomorphism $\varphi : X_1 \to X_2$ with $p_1 = p_2 \circ \varphi$.*

Description by a pair of permutations

An origami $O = p : (X \to E)$ of degree d defines (up to conjugation in S_d)

- a homomorphism $m : F_2 = F_2(x, y) \to S_d$ or equivalently
- a pair of permutations (σ_a, σ_b) in S_d

as follows:
Let $M_1, \ldots, M_d$ be the preimages of the point M (defined as above) under p. Furthermore, let

$$m : \pi_1(E^*, M) \to \mathrm{Sym}(M_1, \ldots, M_d)$$

be the *monodromy map* defined by p, i.e. for the closed path $c \in \pi_1(E^*, M)$ the point M_i is mapped to M_j by $m(c)$ if and only if the lift of the curve c to X via p, that starts in M_i, ends in M_j.
Choosing an isomorphism $\mathrm{Sym}(M_1, \ldots, M_d) \cong S_d$ and using the isomorphism $\pi_1(E^*) \cong F_2$ fixed in (1) makes m into a homomorphism from F_2 to S_d. We set $\sigma_a = m(x)$ and $\sigma_b = m(y)$.
Observe that this homomorphism depends on the chosen isomorphism to S_d and on the choice of the origami in its equivalence class only up to conjugation in S_d. Therefore we consider two homomorphisms $m_1 : F_2 \to S_d$ and $m_2 : F_2 \to S_d$ to be *equivalent*, if they are conjugated by an element in S_d. Similarly we call two pairs (σ_a, σ_b) and (σ'_a, σ'_b) in S_d *equivalent*, if they are simultaneously conjugated, i.e. there is some $s \in S_d$ such that $\sigma_a = s\sigma'_a s^{-1}$ and $\sigma_b = s\sigma'_b s^{-1}$.

Example 1.5. *In Example 1.1 we obtain for the origami $L(2,3)$ in b) the monodromy homomorphism*

$$m : F_2 \to S_4, \quad x \mapsto (2\ 3\ 4) \ \ \textit{and} \ \ y \mapsto (2\ 1),$$

and thus $\sigma_a = (2\ 3\ 4)$ and $\sigma_b = (2\ 1)$.
For the origami D in c) we similarly obtain the permutations

$$\sigma_a = (1\ 2\ 3) \quad \textit{and} \quad \sigma_b = (1\ 4\ 5)(2\ 3).$$

Description as finite index subgroups of F_2

Origamis can be equivalently described as finite index subgroups of F_2, the free group in two generators, as stated in the following remark. The characterization of the Veech groups of origamis is mainly based on this observation.

Remark 1.6. *We have a one-to-one correspondence:*

origamis up to equivalence $\leftrightarrow$ finite index subgroups of F_2 up to conjugacy.

More precisely, this correspondence is given as follows:
Let $O = (p : X \to E)$ be an origami. Define $E^* = E - \{\infty\}$ and $X^* = X - p^{-1}(\infty)$. Thus we may restrict p to the unramified covering $p : X^* \to E^*$. This defines an embedding of the corresponding fundamental groups:

$$U = \pi_1(X^*) \ \subseteq \ \pi_1(E^*) \cong F_2$$

Again we use the fixed isomorphism in (1), see also Figure 4. Changing the origami in its equivalence class leads to a conjugation of U with an element in F_2. The index of the subgroup of F_2 is the degree d of the covering p.
Conversely, given a finite index subgroup U of F_2 we retrieve the origami in the following way: Let $v : \tilde{E^*} \to E^*$ be a universal covering of E^*. By the theorem of the universal covering, $\pi_1(E^*)$ is isomorphic to $\mathrm{Deck}(\tilde{E^*}/E^*)$, the group of deck transformations of $\tilde{E^*}/E^*$. Furthermore, the finite index subgroup U of $\mathrm{Deck}(\tilde{E^*}/E^*)$ corresponds to an unramified covering $p : X^* \to E^*$ of finite degree. This can be extended to a covering $X \to E$, where X is a closed surface.

Example 1.7. *In Example 1.1, we obtain the following subgroups of F_2:*
In a), X^* is the once punctured torus itself and $U = F_2$.
In b), X^* is a genus 2 surface with 2 punctures. Thus $U = \pi_1(X^*)$ is a free group of rank 5. Keeping in mind that we use the identification $\pi_1(E^*) \cong F_2 = F_2(x,y)$ described in Figure 4, one can read off from the picture in Figure 2 that

$$U \ = \ < x^3,\ xyx^{-1},\ x^2yx^{-2},\ yxy^{-1},\ y^2 >$$

In c), X^* is a genus 2 surface with three punctures. Thus U is a free group of rank 6. More precisely, we read off the picture in Figure 3, that

$$U \ = \ < x^3,\ xyx^{-2},\ x^2yx^{-1},\ yxy^{-1},\ y^2xy^{-2},\ y^3 >$$

Description as a finite graph

Finitely, sometimes it is convenient to describe an origami $O = (p : X \to E)$ as a finite, oriented labeled graph: Namely, let U be the finite index subgroup of F_2 (unique up to conjugation) that corresponds to O as described in the last paragraph. Then we represent the origami by the Cayley-Graph of $U \subseteq F_2$: The vertices of the graph are the coset representatives. They are labeled with a representative of the coset. The edges are labeled with x and y. For each vertex (with label $w \in F_2$) there is an x-edge from it to the vertex that belongs to the coset of wx. And similarly there is a y-edge to the vertex that belongs to the coset wy.

Example 1.8. *The following figure shows the Cayley-graph for the origami* $L(2,3)$ *from Example 1.1:*

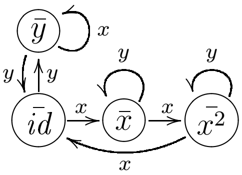

Figure 5: Graph for $O = L(2,3)$.

2 Translation structures and Veech groups

Translation structures

Recall that an atlas on a surface is called *translation atlas*, if all transition maps are translations. An origami $O = (p : X \to E)$ naturally defines an $\mathrm{SL}_2(\mathbb{R})$-family of translation structures μ_A ($A \in \mathrm{SL}_2(\mathbb{R})$) on $X^* = X - p^{-1}(\infty)$ as follows:

- As first step, observe that each $A \in \mathrm{SL}_2(\mathbb{R})$ naturally defines a translation structure η_A on the torus E itself by identifying it with $\mathbb{C}/\Lambda_A$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \Lambda_A \text{ is the lattice } < \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} > \text{ in } \mathbb{C} \qquad (2)$$

- Then define the translation structure μ_A on X^* by lifting η_A via p, i.e.

$$\mu_A = p^* \eta_A.$$

Using the first description of an origami that we gave by gluing squares, we obtain the translation structure μ_I (where I is the identity matrix), if we identify the squares with the Euclidean unit square in $\mathbb{C}$. We obtain μ_A for a general matrix

$A \in \mathrm{SL}_2(\mathbb{R})$ from this by identifying the squares with the parallelogram spanned by the two vectors

$$\begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix}.$$

Thus the $\mathrm{SL}_2(\mathbb{R})$-variations of the translation structure μ_I can be thought of as affine shearing of the unit squares, see Figure 6.

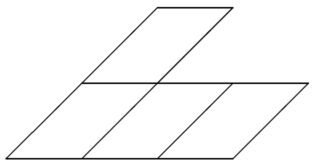

Figure 6: Sheared translation structure for the origami $L(2,3)$.

From an origami to a Teichmüller curve in the moduli space

By the $\mathrm{SL}_2(\mathbb{R})$-family of translation structures, the origami $O = (p : X \to E)$ defines a specific complex algebraic curve called *Teichmüller curve* in the moduli space M_g of closed Riemann surfaces of genus g. We state this construction here only briefly as motivation and refer e.g. to the overview article [HeSc 06] for a detailed description and links to references. A particular nice configuration of such Teichmüller curves is described in [H 06] in this volume.

The Teichmüller curve in M_g is obtained from the origami in the following way:

- The translation structure μ_A described in the previous paragraph is in particular a complex structure on the surface X^* which can be extended to the closed surface X. The Riemann surface (X, μ_A) together with the identity map $\mathrm{id} : X \to X$ as marking then defines a point in the Teichmüller space T_g. Thus we obtain the map: $\hat{\iota} : \mathrm{SL}_2(\mathbb{R}) \to T_g,\ A \mapsto [(X, \mu_A), \mathrm{id}]$.
- If $A \in \mathrm{SO}_2(\mathbb{R})$, then the affine map $z \mapsto A \cdot z$ is holomorphic. Thus the map $\hat{\iota}$ factors through $\mathrm{SO}_2(\mathbb{R})$. Furthermore using that $\mathrm{SL}_2(\mathbb{R})$ modulo $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the upper half plane $\mathbb{H}$, one obtains a map

  $$\iota : \mathbb{H} \cong \mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R}) \quad \to \quad T_g$$

  In fact, this map is an embedding that is in the same time holomorphic and isometric. A map with this property is called *Teichmüller embedding* and its image Δ in Teichmüller space is called a *Teichmüller disk* or a *geodesic disk*.

- Finally, one may compose the map ι with the projection to the moduli space M_g. The image of Δ in M_g is a complex algebraic curve. A curve in M_g that arises like this as the image of a Teichmüller disk is called *Teichmüller curve.*

Note: More generally, one obtains a Teichmüller disk Δ in a similar way starting from an arbitrary translation surface (or a bit more general: from a flat surface). However, the image of such a disk Δ in moduli space is not always a complex algebraic curve; in fact its Zariski closure tends to be of higher dimension. It is an interesting question how to decide whether a translation surface leads to a Teichmüller curve. One possible answer to this is given by the Veech group which we introduce in the following paragraph.

Veech groups

Let X^* be a connected surface and μ a translation structure on it. One assigns to it a subgroup of $\mathrm{GL}_2(\mathbb{R})$ called *Veech group* as described in the following: We consider the group $\mathrm{Aff}^+(X^*, \mu)$ of all orientation preserving *affine diffeomorphisms*, i.e. orientation preserving diffeomorphisms that are locally affine maps of the plane $\mathbb{C}$, see Figure 7. Here – and throughout the whole article – we identify $\mathbb{C}$ with $\mathbb{R}^2$ by the map $z \mapsto (\mathrm{Re}(z)\,,\,\mathrm{Im}(z))^t$. Thus an affine diffeomorphism f can be written in terms of local charts as

$$f:\ z = (\mathrm{Re}(z), \mathrm{Im}(z))^t \mapsto A \cdot (\mathrm{Re}(z), \mathrm{Im}(z))^t + z_0 \quad \text{with } A \in \mathrm{GL}_2(\mathbb{R}) \text{ and } z_0 \in \mathbb{C}. \tag{3}$$

Observe that A does not depend on the chart, since μ is a translation structure. Thus one obtains a well defined map

$$D:\ \mathrm{Aff}^+(X^*, \mu) \to \mathrm{GL}_2(\mathbb{R}), \quad f \mapsto A$$

called *Derivative map.*

Definition 2.1. *The* Veech group $\Gamma(X^*, \mu)$ *of the translation surface* (X^*, μ) *is the image of the derivative map* D*:*

$$\Gamma(X^*, \mu) = D(\mathrm{Aff}^+(X^*, \mu))$$

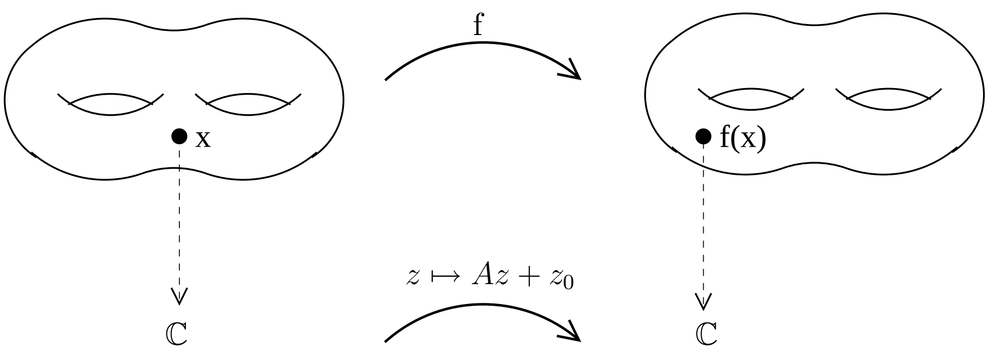

Figure 7: An affine diffeomorphism of a translation surface

Example 2.2. *Let (X^*, μ) be $\mathbb{C}/\Lambda_I$ with the natural translation structure. Here I is the identity matrix and Λ_I is the corresponding lattice as defined in (2).*
An affine diffeomorphisms of $\mathbb{C}/\Lambda_I$ lifts to an affine diffeomorphism of $\mathbb{C}$ respecting the lattice. Conversely, each such diffeomorphism descends to $\mathbb{C}/\Lambda_I$. Thus, we have in this case

$$\Gamma(X^*, \mu) = \mathrm{SL}_2(\mathbb{Z}).$$

Veech groups and Teichmüller curves

As indicated in the paragraph about Teichmüller curves, the Veech group "knows" whether a translation surface defines a Teichmüller curve in moduli space or not. More precisely, one has the following statement:

Fact: Let X be a surface of genus g and $X^* = X - \{P_1, \ldots, P_n\}$ for finitely many points $P_1, \ldots, P_n$ on X. Furthermore let μ be a translation structure on X^*. Then (X^*, μ) defines a Teichmüller curve C if and only if the Veech group $\Gamma(X^*, \mu)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case, the curve C is (antiholomorphic) birational to $\mathbb{H}/\Gamma(X^*, \mu)$.

We describe the relation to Teichmüller curves here just as motivation and in order to give a glance at the general frame. We have therefore resumed theorems contributed by several authors condensed in what is here called "fact". A good access to it can be found e.g. in [EG 97] or [Z 06]. A broader overview on Veech groups of translation surfaces is given e.g. in [HuSc 01] and in [Le 02]. Teichmüller disks, Teichmüller curves and Veech groups have intensively been studied by numerous authors, starting from Thurston [T 88] and Veech himself [V 89]. We refer to [S 04] and [HeSc 06] for more comprehensive overviews on references.

3 Veech groups of origamis

Let $O = p : (X \to E)$ be an origami. We have seen in Section 2 that O defines an $\mathrm{SL}_2(\mathbb{R})$-family of translation structures μ_A ($A \in \mathrm{SL}_2(\mathbb{R})$) on $X^* = X - p^{-1}(\infty)$. The corresponding Veech groups are not very different. In fact, they are all conjugated to each other. More precisely, we have:

$$\Gamma(X^*, \mu_A) = A\Gamma(X^*, \mu_I)A^{-1}.$$

Thus, we may restrict to the case where $A = I$ which justifies the following definition.

Definition 3.1. *The* Veech group $\Gamma(O)$ of the origami O *is defined to be* $\Gamma(X^*, \mu_I)$.

From Example 2.2 it follows that the Veech group of the trivial origami O_0 (defined in Example 1.1) is $\mathrm{SL}_2(\mathbb{Z})$. For a general origami one can show that $\Gamma(O)$

is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In fact, also the converse is true as it was shown by Gutkin and Judge in [GJ 00]: A Veech group is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ if and only if it comes from an origami.
From this it follows in particular by the Fact presented in Section 2 on page 10 that an origami always defines a Teichmüller curve in the moduli space.

Characterization of origami Veech groups

Recall from Section 1 that an origami O corresponds (up to equivalence) to a finite index subgroup U of $F_2 = F_2(x, y)$, the free group in two generators (up to conjugation). This description enables us to give a characterization of its Veech group entirely in terms of F_2 and its automorphisms.

For this we need the following two ingredients:

- Let $\hat{\beta} : \mathrm{Aut}(F_2) \to \mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$ be the natural projection. The fact that we only consider orientation preserving diffeomorphisms applies to only taking automorphisms of $\mathrm{Aut}(F_2)$ that are mapped to elements in $\mathrm{SL}_2(\mathbb{Z})$. We denote $\mathrm{Aut}^+(F_2) = \hat{\beta}^{-1}(\mathrm{SL}_2(\mathbb{Z}))$ and restrict to the map
$$\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z}).$$
- Let $\mathrm{Stab}(U) = \{\gamma \in \mathrm{Aut}^+(F_2) | \gamma(U) = U\}$

Using these ingredients, it was shown in [S 04] that Veech groups of origamis can be described as stated in the following theorem.

Theorem 2 (Proposition 1 in [S 04])**.** *For the Veech group $\Gamma(O)$ of the origami O holds:*
$$\Gamma(O) = \hat{\beta}(\mathrm{Stab}(U))$$

Let us make two comments on this description:
One consequence is, that one obtains an algorithm that can calculate the Veech group of an arbitrary origami explicitly. This algorithm is described in detail in [S 04].
As an other consequence, we have now a characterization of all origami Veech groups as stated in the following corollary.

Corollary 3.2. *A finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ occurs as origami Veech group if and only if it is the image of the stabilizing group $\mathrm{Stab}(U) \subseteq \mathrm{Aut}^+(F_2)$ for some finite index subgroup U in F_2.*

Thus the question, which finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are Veech groups becomes roughly speaking the same as the question which subgroups of $\mathrm{Aut}^+(F_2)$ are such stabilizing groups. So far, there is no general answer known.

In [S 05] it was shown that many congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are Veech groups. Recall that a *congruence group* of level n is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that is the full preimage of some subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ under the natural homomorphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ and n shall be minimal with this property. For prime level congruence groups the following statement is shown in [S 05, Theorem 4]

Theorem 3. *Let p be prime. All congruence groups Γ of level p are Veech groups except possibly $p \in \{2, 3, 5, 7, 11\}$ and Γ has index p in $\mathrm{SL}_2(\mathbb{Z})$.*

This result is generalized to a statement for arbitrary n in [S 05, Theorem 5]

Presenting the Veech group Γ and the quotient $\mathbb{H}/\Gamma$ for an origami

As mentioned above, using Theorem 2 the Veech group of an origami can be calculated explicitly. The Veech groups are described as subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by generators and coset representatives. We use for the notation that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T, with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Recall furthermore from the discussion on Veech groups and Teichmüller curves in Section 2 on page 10 that for a Veech group Γ we are in particular interested in the quotient $\mathbb{H}/\Gamma$, since this quotient is birational to the corresponding Teichmüller curve. Here Γ acts as Fuchsian group on the upper half plane $\mathbb{H}$, which is endowed with the Poincaré metric.

Since an origami Veech group Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the quotient $\mathbb{H}/\Gamma$ comes with a natural triangulation. More precisely, we choose the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ that is the geodesic pseudo-triangle Δ with vertices $P = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $Q = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $P_\infty = \infty$.

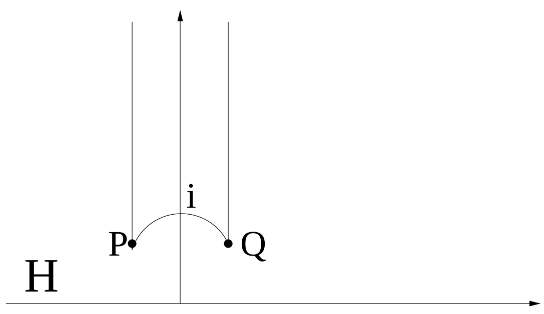

Figure 8: Fundamental domain of $SL_2(\mathbb{Z})$.

The surface $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ is obtained by identifying the vertical edges $P\infty$ and $Q\infty$ via T and the edge PQ with itself (with fixed point i) via S.

For an arbitrary subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index we obtain a fundamental domain as a union of translates of the triangle Δ: for each coset $\overline{A}$ we take the

triangle $A(\Delta)$, where A is a representative of the coset. The identification of the edges is given by the elements in Γ. Gluing the edges gives the quotient surface $\mathbb{H}/\Gamma$, filling in the cusps leads to a closed Riemann surface endowed with a triangulation. We draw stylized pictures of the fundamental domains that indicate the triangles (see Figure 9 and 10). The triangles are labeled with a coset representative, edges that are identified are labeled with the same letter and vertices that are identified with the same number. Vertices that come from cusps (i.e. points at ∞) are marked with $\bullet$.
In particular, one can read off from these stylized pictures the genus and the number of cusps of the quotient surface $\mathbb{H}/\Gamma$.

Two examples: the origami L(2,3) and the origami D

The origami L(2,3):
In [S 04, Example 3.5] the Veech group is calculated as follows:

$$\Gamma(L(2,3)) = < \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}, \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} > .$$

More precisely, one obtains the generators presented as products of S and T as well as a list of coset representatives.

- List of generators:

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = T^3, \quad \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix} = TST^2ST^{-1}T^{-1}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = TSTST^{-1}S,$$

$$\begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} = T^2STST^{-1}S^{-1}T^{-2}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

- List of representatives:

$$I,\ T,\ S,\ T^2,\ TS,\ ST,\ T^2S,\ TST,\ T^2ST$$

Hence, $\Gamma(L(2,3))$ is a subgroup of index 9 in $\mathrm{SL}_2(\mathbb{Z})$.
The stylized picture of the quotient $\mathbb{H}/\Gamma(L(2,3))$ is determined in [S 04, Example 3.6] and is shown here in Figure 9.

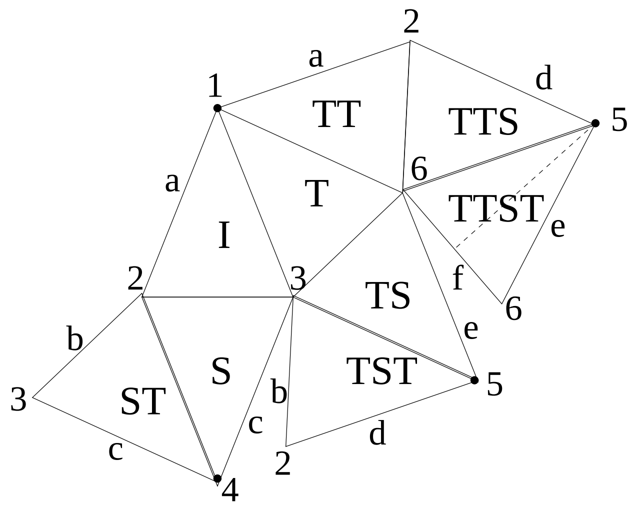

Figure 9: Fundamental domain of $\Gamma(L(2,3))$.

From this one can read off that the genus of the quotient $\mathbb{H}/\Gamma(L(2,3))$ is 0 and that it has 3 cusps, namely the vertices labeled by 1,4 and 5. It follows in particular that the corresponding Teichmüller curve has genus 0.

The origami D:
The Veech group of the origami D is calculated in [S 05, Section 7.3.2]. It has index 24 in $\mathrm{SL}_2(\mathbb{Z})$ and the following generators:

$$A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, \qquad A_1 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = T^3,$$

$$A_2 = \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix} = ST^6S^{-1}, \qquad A_3 = \begin{pmatrix} -7 & 16 \\ -4 & 9 \end{pmatrix} = (T^2S)T^4(T^2S)^{-1}.$$

$$A_4 = \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix} = (TS)T^4(TS)^{-1}, \quad A_5 = \begin{pmatrix} -9 & 5 \\ -20 & 11 \end{pmatrix} = (TST^2S)T^5(TST^2S)^{-1},$$

$$A_6 = \begin{pmatrix} 7 & 2 \\ -18 & -5 \end{pmatrix} = (ST^3S)T^2(ST^3S)^{-1},$$

The following is a system of cosets representatives:

I , T , S , T^2 , TS , ST , T^2S , TST , ST^2 , STS , T^2ST , TST^2 , ST^5 , ST^3 , T^2S , TST^3 , TST^2S , ST^4 , ST^3S , TST^2ST^{-1} , TST^2ST^{-2} , TST^2ST^{-3} ; TST^2ST^{-4} , ST^3ST

The corresponding origami curve $C(D)$ has genus 0. It is shown with its natural triangulation in Figure 10. It has six cusps, namely C_1, C_2, C_3, C_4, C_5 and C_6.

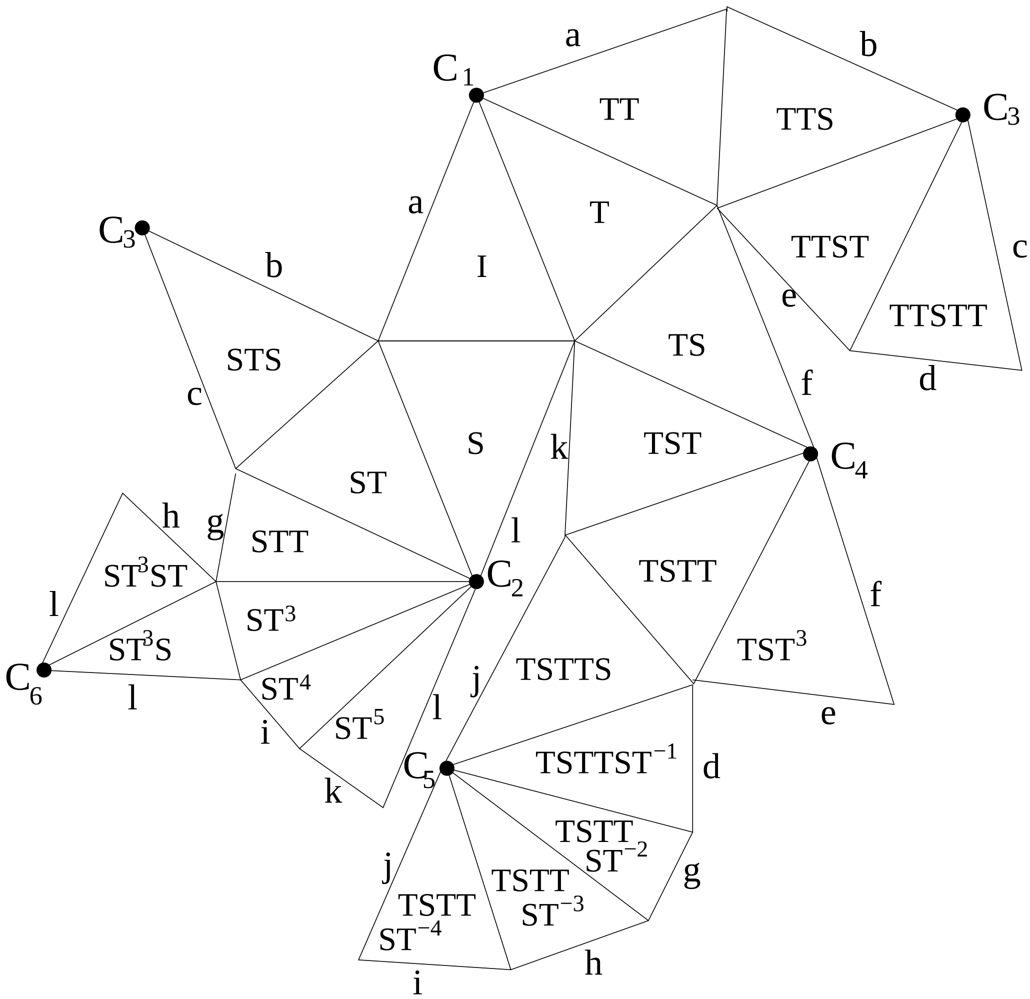

Figure 10: *The origami curve to D.*

4 Veech groups that are non congruence groups

Theorem 3 implies that there are many congruence groups which are Veech groups. How about non congruence groups? In this section we will see that the Veech groups for the two examples, the origami $L(2, 3)$ and the origami D, studied in the last paragraph are both non congruence groups. Furthermore, we give a construction that produces for both of them an infinite sequence of origamis whose Veech group is a non congruence group. We use this in order to prove our main theorem.

An other generalization of the example $L(2,3)$ was given by Hubert and Lelièvre in [HL 05], where they show for certain "L-shaped" origamis or square-tiled surfaces, how they are called there, that their Veech groups are non congruence groups. These surfaces are all of genus 2, hence it follows that there are infinitely many origamis of genus 2 whose Veech group is a non congruence group.

Recall that a group is a congruence group, whose level is a divisor of n, if and only if it contains the principal congruence group

$$\Gamma(n) \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n\} \;=\; \mathrm{kernel}(\mathrm{proj}: \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))$$

In [S 04, Proposition 3.8] it was shown using a proof of Stefan Kühnlein that the Veech group of $L(2,3)$ is a non congruence group. The basic tool for this is the general level that is defined for any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ as follows: For each cusp we define its *amplitude* to be the smallest natural number n such that there is an element of Γ conjugated in $\mathrm{SL}_2(\mathbb{Z})$ to the matrix

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

which fixes the cusp. Observe that this is equal to the number of triangles around the vertex that represents the cusp in our stylized picture of the quotient surface (see Figures 9 and 10). The *general level* of Γ is the least common multiple of the amplitudes of all its cusps. A theorem of Wohlfahrt [W 64, Theorem 2] states that the level and the general level of a congruence group coincide.
The amplitude of the three cusps of $\mathbb{H}/\Gamma(L(2,3))$ labeled with 1, 4 and 5 in Figure 9 is 3, 2 and 4 respectively. Hence, the general level of $\Gamma(L(2,3))$ is 12. Then it is shown in the proof that $\Gamma(L(2,3))$ does not contain $\Gamma(12)$ which gives the contradiction.

The same method can be used in order to show that $\Gamma(D)$ is a non congruence group. We here carry out the proof for it. Observe from Figure 10 that the six cusps $C_1, \ldots, C_6$ have the amplitude 3, 6, 4, 4, 5 and 2, respectively. Thus the general level is 60.

Proposition 4.1. *The Veech group $\Gamma(D)$ is a non congruence group.*

Proof. Suppose that $\Gamma = \Gamma(D)$ is a congruence group. Since the general level of Γ is 60, we have by the theorem of Wohlfahrt mentioned above, that $\Gamma(60)$ is a subgroup of Γ.

We will use the following facts, which can be checked e.g. in Figure 10:

$$A_1 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \in \Gamma, \quad A_6 = \begin{pmatrix} 7 & 2 \\ -18 & -5 \end{pmatrix} \in \Gamma \quad \text{and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \notin \Gamma$$

In order to verify this in Figure 10, use that

$$A_1 = T^3 \text{ and } A_6 = S^{-1}T^2S^{-1}T^{-1}S^{-1}TS^{-1}T^{-3}S^{-1}.$$

We will find an element in Γ whose projection to $\mathrm{SL}_2(\mathbb{Z}/60\mathbb{Z})$ is equal to that of T, which gives us the desired contradiction.
Recall that

$$\mathrm{SL}_2(\mathbb{Z}/60\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z}).$$

We identify in the following these two groups. Furthermore we denote by p_4, p_3, p_5 and p_{60} the projection from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}/60\mathbb{Z})$, respectively. Then $p_{60} = p_4 \times p_3 \times p_5$.
We have

$$\begin{aligned} p_{60}(A_1) &= (\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}) \quad \text{and} \\ p_{60}(A_6) &= (\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}) \end{aligned}$$

The order of $p_4(A_1)$ in $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is 4, the order of $p_3(A_1)$ in $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is 1 and the order of $p_5(A_1)$ in $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ is 5. We also say: *The order of* $p_{60}(A_1)$ *is* $(4, 1, 5)$. Since $7 \equiv 3 \mod 4$ and $7 \equiv 2 \mod 5$ we have

$$p_{60}(A_1^7) = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \tag{4}$$

Furthermore:

$$p_{60}(A_6^2) = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix})$$

and with the same notation as above $p_{60}(A_6^2)$ *has the order* $(1, 3, 5)$. Thus

$$p_{60}(A_6^{20}) = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}). \tag{5}$$

From (4) and (5) it follows that

$$p_{60}(A_6^{20} \cdot A_1^7) = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = p_{60}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = p_{60}(T)$$

But $A_6^{20} \cdot A_1^7 \in \Gamma$ and $T \notin \Gamma$, thus $\Gamma(60) = \ker(p_{60})$ cannot be contained in Γ. Therefore, Γ cannot be a congruence group of level 60. Contradiction! $\square$

Sequences of origamis with non congruence Veech groups

Starting from the origamis $L(2,3)$ and D we will define respectively a sequence O_n, such that for each $n \in \mathbb{N}$ the Veech group $\Gamma(O_n)$ again is a non congruence group. The basic idea is to "copy and paste": we will cut the origami along a segment, take n copies of it and glue them along the cuts.

In Figure 11 we show the origami O_n for $L(2,3)$:

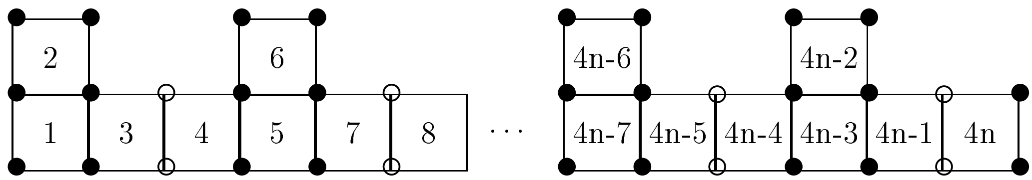

Figure 11: *n copies of $L(2,3)$. Opposite edges are glued.*

Using the description of an origami by a pair of permutations from Section 1, O_n is given as:

$\sigma_a = (1\ 3\ 4\ 5\ 7\ 8\ 9\ 11\ 12\ \dots\ 4n{-}3\ 4n{-}1\ 4n), \quad \sigma_b = (1\ 2)(5\ 6)\dots(4n{-}3\ 4n{-}2).$

Observe that the genus of O_n is $n+1$ and it has $2n$ cusps: n of order 3 (all n marked by $\bullet$ in Figure 11), and n of order 1 (all n marked by $\circ$ in Figure 11).

Finally, we want to present the origami O_n by the finite index subgroup $H_n = \pi_1(X^*)$ of F_2, that corresponds to O_n by Remark 1.6.
Recall from Example 1.7 that for $O_1 = L(2,3)$, we obtain the free group of rank 5:

$$U = H_1 = < g_1 = x^3,\ g_2 = xyx^{-1},\ g_3 = x^2yx^{-2},\ g_4 = yxy^{-1},\ g_5 = y^2 > = F_5.$$

The group H_n is obtained as as follows:

$$H_n = < g_1^n,\ g_1^i\, g_j\, g_1^{-i} \in F_5 \mid\ i \in \{0,\dots,n-1\} \text{ and } j \in \{2,\dots,5\} >$$

In Figure 12, we show the origami D_n:

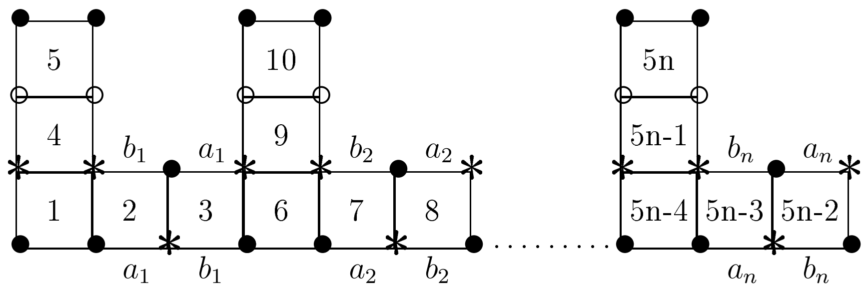

Figure 12: *n copies of D. Edges with the same label or unlabeled opposite edges are glued.*

The pair of permutations describing D_n is:

$$\begin{aligned}\sigma_a &= (1\,2\,3\ \ 6\,7\,8\ \ \ldots\ \ 5n-4\ 5n-3\ 5n-2),\\ \sigma_b &= (1\,4\,5)(6\,9\,10)\ldots(5n-4\ 5n-1\ 5n)(2\,3)(7\,8)\ldots(5n-3\ 5n-2)\end{aligned}$$

The genus of D_n is $2n$ and it has $n+2$ cusps: 2 of order $2n$ (marked as $\bullet$ and $\star$) and n of order 1 (all n marked by $\circ$).

Again, we present O_n by the corresponding finite index subgroup H_n of F_2. We have from Example 1.7 that $U = H_1 = F_6$, the free group of rank 6:

$$U\ = < g_1' = x^3,\ g_2' = xyx^{-2},\ g_3' = x^2yx^{-1},\ g_4' = yxy^{-1},\ g_5' = y^2xy^{-2},\ g_6' = y^3 >\ = F_6$$

And similarly as above, we obtain:

$$H_n\ = < g_1'^{\,n},\ \ g_1'^{\,i}\, g_j'\, g_1'^{\,-i}\ \in F_6\,|\ \ i \in \{0,\ldots,n-1\} \text{ and } j \in \{2,\ldots,6\} >$$

We will see in the following that for both sequences all Veech groups $\Gamma(O_n)$ are non congruence groups. More precisely, we will show:

Proposition 4.2. *For both sequences O_n the following holds:*

- $\Gamma(O_n) \subseteq \Gamma(O_1)$, *which is for both sequences a non congruence group.*
- *More generally one has:*
  n *divides* $m \ \Rightarrow\ \ \Gamma(O_m) \subseteq \Gamma(O_n)$.
- *Different origamis in one sequence have different Veech groups, i.e.:*
  $\Gamma(O_n) \neq \Gamma(O_m)$ *for* $n \neq m$.

To prove this, let us detect that we are in the following more general setting.

Setting A:

- Let U be a finite index subgroup of F_2. Then U is a free group of rank k for some $k \geq 2$, i.e.
  $$U\ = < g_1, \ldots, g_k > =\ F_k$$
- Let $\alpha : F_k \to \mathbb{Z}$ be the projection $w \mapsto \sharp_{g_1} w$
  where $\sharp_{g_1} w$ is the number of g_1 in the word $w = w(g_1, \ldots, g_k)$ with g_1^{-1} counted as -1.
- Let H_n be the kernel of $p_n \circ \alpha$, where $p_n : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the natural projection, i.e.

$$H_n\ = < g_1^n,\ \ g_1^i\, g_j\, g_1^{-i}\ \in F_k\,|\ \ i \in \{0,\ldots,n-1\} \text{ and } j \in \{2,\ldots,k\} > .$$

- Finally, let H_0 be the kernel of α, i.e.:

$$H_0 = \bigcap_{n\in\mathbb{N}} H_n = << g_2, \ldots, g_k >>_U,$$

is the normal subgroup in U generated by $g_2, \ldots, g_k$.

Observe that we are in this setting with
$U = \pi_1(X^*) = < x^3,\ xyx^{-1},\ x^2yx^{-2},\ yxy^{-1},\ y^2 >$ for the origami $L(2,3)$ and
$U = \pi_1(X^*) = < x^3,\ xyx^{-2},\ x^2yx^{-1},\ yxy^{-1},\ y^2xy^{-2},\ y^3 >$ for the origami D.

In order to prove the properties in Proposition 4.2, we will need that U fulfills the following a bit technical condition:

Property B: Let $U = < g_1, \ldots, g_k >$ $(k \geq 2)$ be as above a finite index subgroup of F_2 of rank k and $\{w_i\}_{i\in I}$ a system of coset representatives with $w_1 = \mathrm{id}$. Suppose that U has the following property:

$$\forall\, j \in I - \{1\}: \quad w_j << g_2, \ldots, g_k >>_U w_j^{-1} \not\subseteq U.$$

One can check by hand that for both origamis, $L(2,3)$ and D, this property is fulfilled. In this setting we obtain the following conclusions.

Proposition 4.3. *Let $n \in \mathbb{N} \cup \{0\}$. Let U be a finite index subgroup of F_2 fulfilling property B. With the notations from Setting A, we have:*

a) The normalizer of H_n in F_2 is equal to U: $\mathrm{Norm}_{F_2}(H_n) = U$

b) $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \overset{\mathrm{Def}}{=} \mathcal{G}$

c) Recall that $U = F_k$, the free group in k generators.
Let $\beta_n : \mathrm{Aut}(F_k) \to \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z})$ be the natural projection.
Then $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n)$ is equal to

$$\beta_n^{-1}(\{A = (a_{i,j})_{1\leq i,j\leq k} \in \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z}) |\ a_{1,2} = \ldots = a_{1,k} = 0\}) \cap \mathcal{G}.$$

Here we use the notation $\mathbb{Z}/(0\mathbb{Z}) = \mathbb{Z}$ thus β_0 is the natural projection $\mathrm{Aut}(F_k) \to \mathrm{GL}_k(\mathbb{Z})$.

Proof.
a)
By definition H_n is normal in U, i.e. $U \subseteq \mathrm{Norm}_{F_2}(H_n)$.
Let now w be an element of $F_2 \backslash U$. Hence, $w = w_j \cdot u$ for some $j \in I - \{1\}$, $u \in U$. By Property B, there exists some $h_0 \in << g_2, \ldots, g_k >>_U = H_0$, such that $w_j h_0 w_j^{-1} \notin U$. Therefore we have $w(u^{-1}h_0u)w^{-1} \notin U$. But $u^{-1}h_0u \in H_0 \subseteq H_n$, since H_0 is normal in U. This shows that $w \notin \mathrm{Norm}_{F_2}(H_n)$.

b)
This follows from a), since for a subgroup H of F_2 in general holds:
$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(\mathrm{Norm}_{F_2}(H))$, see e.g. [S 06, Remark 3.1].

c)
Define $M = \{A = (a_{i,j})_{1\le i,j\le k} \in \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z}) |\ a_{1,2} = \ldots = a_{1,k} = 0\}$.
Let $\gamma \in \mathcal{G}$. We have to show that $\gamma(H_n) = H_n$ if and only if $\beta_n(\gamma) \in M$.
Let furthermore $p_n^k : F_k \to (\mathbb{Z}/n\mathbb{Z})^k$ be the natural projection.
Consider the following commutative diagram:

$$\begin{array}{ccccccc}
H_n & \subseteq & F_k & \xrightarrow{\gamma} & F_k & \supseteq & H_n \\
\downarrow{\scriptstyle p_n^k} & & \downarrow{\scriptstyle p_n^k} & & \downarrow{\scriptstyle p_n^k} & & \downarrow{\scriptstyle p_n^k} \\
\overline{H}_n = p_n^k(H_n) & \subseteq & (\mathbb{Z}/n\mathbb{Z})^k & \xrightarrow{\beta_n(\gamma)} & (\mathbb{Z}/n\mathbb{Z})^k & \supseteq & p_n^k(H_n) = \overline{H}_n
\end{array}$$

Since p_n^k is surjective and H_n is the full preimage of $\overline{H}_n = p_n^k(H_n)$, it follows that $\gamma(H_n) = H_n$ if and only if $\beta_n(\gamma)(\overline{H}_n) = \overline{H}_n$.

Observe finally that:

$$\begin{aligned}
\overline{H}_n &= \{(0, x_2, \ldots, x_k) \in (\mathbb{Z}/n\mathbb{Z})^k\} \quad \text{and} \\
\mathrm{Stab}_{\mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z})}(\overline{H}_n) &= \{\ A = (a_{i,j})_{1\le i,j\le k} \in \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z}) | \\
&\qquad (y_1, \ldots, y_k) = A \cdot (0, x_2, \ldots, x_k) \ \Rightarrow\ y_1 = 0\ \} \\
&= \{A = (a_{i,j}) \in \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z}) |\ a_{1,2} = \ldots = a_{1,k} = 0\} \\
&= M.
\end{aligned}$$

□

Theorem 2 suggests the following notation.

Definition 4.4. *Let U be a subgroup of F_2.*
With $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ as in Theorem 2, we define

$$\Gamma(U) = \hat{\beta}(\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U))$$

and call $\Gamma(U)$ the Veech group *of U.*

We now obtain from Proposition 4.3 the following conclusions.

Corollary 4.5. *Suppose that we are in the same situation as in Proposition 4.3, in particular that U is a finite index subgroup of F_2 fulfilling property B. Then we have for all $n \in \mathbb{N}$:*

a) $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_0) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n)$ *and* $\Gamma(H_0) \subseteq \Gamma(H_n)$.

b) If $m \in \mathbb{N}$ with $n|m$, then:

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_m) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n) \quad \textit{and} \quad \Gamma(H_m) \subseteq \Gamma(H_n).$$

c)

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_0) = \bigcap_{n\in\mathbb{N}} \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n) \quad \textit{and} \quad \Gamma(H_0) = \bigcap_{n\in\mathbb{N}} \Gamma(H_n)$$

Proof.

a) and b):

Let $\gamma \in \mathcal{G}$. By Proposition 4.3 we have that

$$\begin{aligned} \forall n \in \mathbb{N}:\ \gamma \in \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n) \quad &\Leftrightarrow \quad \beta_n(\gamma) = A = (a_{i,j}) \\ &\qquad \text{with } a_{1,2} \equiv \ldots \equiv a_{1,k} \equiv 0 \mod n \\ \text{and} \quad \gamma \in \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_0) \quad &\Leftrightarrow \quad \beta_0(\gamma) = A = (a_{i,j}) \\ &\qquad \text{with } a_{1,2} = \ldots = a_{1,k} = 0. \end{aligned}$$

Thus we have for all $n \in \mathbb{N}$ and for all $m \in \mathbb{N}$ with $n|m$, that

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_0) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_m) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H_n).$$

We have in particular by the definition of the Veech group of a subgroup of F_2:

$$\Gamma(H_0) \subseteq \Gamma(H_m) \subseteq \Gamma(H_n).$$

c):

$\subseteq$ follows from a). $\supseteq$ follows from Remark [S 06, Remark 3.1].

□

We now return to the language of origamis: Let O be an origami, U the corresponding subgroup of F_2. Define for U the subgroups H_n $(n \in \mathbb{N})$ as in Setting A and let O_n be the origamis corresponding to the groups H_n.
By Corollary 4.5 and Theorem 2 we obtain immediately the following result.

Proposition 4.6. *If U has the Property B, then*

$$\forall n \in \mathbb{N}:\ \Gamma(O_n) \subseteq \Gamma(O) \quad \textit{and} \quad \forall n, m \in \mathbb{N}:\ n|m \ \Rightarrow\ \Gamma(O_m) \subseteq \Gamma(O_n).$$

In particular, if $\Gamma(O)$ is a non congruence group, each $\Gamma(O_n)$ is a non congruence group. Thus in this case, we obtain infinitely many origamis whose Veech group is a non congruence group.

In order to conclude Proposition 4.2, it is now just left to prove the last item. But this follows , since we have (see [S 05]) for both sequences O_n, the one coming from the origami $L(2,3)$ and the one coming from the origami D, that

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \Gamma(O_n) \quad \Leftrightarrow \quad 3n \text{ divides } s. \tag{6}$$

This finishes the proof of Proposition 4.2.
Furthermore, Theorem 1 follows from Proposition 4.2.

Remark: From Corollary 4.5 and (6) it follows that $\Gamma(H_0)$ has infinite index in $\mathrm{SL}_2(\mathbb{Z})$. Furthermore it is non trivial, since it contains

$$B_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ for } L(2,3) \quad \text{respectively} \quad B_3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \text{ for } D.$$

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