Formalizing Cartan Geometry in Modal Homotopy Type Theory

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1 Introduction

Homotopy Type Theory[1] is a formal language with the intriguing feature that theorems and constructions done inside it, may be transported to a wide variety of categories including $(\infty, 1)$-toposes [2]. An idea due to William Lawvere is that toposes with suitable extra properties provide a useful context for setting up the foundations of differential geometry ([Koc06]). More recently it has been observed that $(\infty, 1)$-toposes with suitable extra structure provide a useful context for setting up the foundations of „higher" or „derived" differential geometry ([Sch13a]). It is the aim of this thesis to use both, the formal language of Homotopy Type Theory and some part of this extra structure, to prove certain theorems with geometric content in a natural and concise way. This means, that the geometric objects treated here, include ∞-stacks and ∞-gerbes. Furthermore, proofs in Homotopy Type Theory can be written down in a formally correct way and checked for correctness with the help of a software called a proof assistant. This was done for all the main theorems of this thesis with the proof assistant Agda.
One of the most important differences for the goals in this thesis of Homotopy Type Theory to the usual way of doing mathematics, are the two notions of equality. While what is called definitional or judgemental equality is just an equality of symbols allowing expansion of definitions, the second, the propositional equality, behaves more like the notion of isomorphism like it is commonly used in mathematics. But propositional equalities still have some basic properties one expects from a concept of equality, for example, the Leibniz Rule holds:

If $x = y$ and we know $P(x)$ for any property P, then we also know $P(y)$.

As two groups might be isomorphic via different isomorphisms, the propositional equality in Homotopy Type Theory does not rule out, that things are equal in different ways. Here, despite its name, propositional equality is not just a proposition in the usual sense, it is a type and therefore, of the same kind of object as the types representing spaces. So if our types are to be interpreted as some sheaves representing spaces with a smooth structure, the type of equalities between two points in such a space, has again the structure of a smooth space.
Sometimes, while doing mathematics, it is helpful to identify objects of a fixed kind via some equivalence relation. A common way to get a representation of this identification is to take the set-quotient. But sometimes, the information in what different ways two objects are identified is important – and the set-quotient is lacking this information. Objects providing this information, would be a homotopy quotient or a quotient groupoid. In some sense, this is what the notion of a moduli stack is about – we give some explanation in 1.5 how problems in geometry, so called moduli problems, can naturally lead to viewing objects to be equal in multiple ways [3].
The ease of use of Homotopy Type Theory compared to working with quotient stacks directly was one of the motivations of the author to consider Homotopy Type Theory as a tool for pure mathematics.

Since plain Homotopy Type Theory applies to any $(\infty, 1)$-toposes we cannot expect to prove anything about structures present only in particular toposes without using a more specific formal language. We add axioms to Homotopy Type Theory that make it

[1] Like presented in [Uni13].
[2] See [Shu15b] or [KL16].
[3] This is certainly not a universally accepted way to put it.

apply to toposes with a certain endofunctor. Some examples of such endofunctors, called infinitesimal shape functors, de Rham stack functors or coreductions, are described in 2.3 and in 4.4 we show, how the notion of formally étale maps from Algebraic Geometry may be recovered internally to some degree.
We add axioms to Homotopy Type Theory, formalizing existence and some of the properties of the endofunctors of interest. This means nothing else than using the known axioms of a modality[4]. The formalization is used to establish the foundations for Cartan Geometry. The formalized concepts include V-manifolds, their frame bundles and torsion free G-structures on them. While reasoning with these concepts in Homotopy Type Theory even seems naive at some points, the theorems we prove apply to what in higher geometry are known as étale ∞-stacks, ∞-gerbes on these and higher G-structures on those, such as, for instance String-structures [5].

1.1 Summary

In chapter 2 we present some known concepts and known examples of categories to which the results from this thesis may be applied to. Chapter 3 briefly introduces basic notions from Homotopy Type Theory and lists some lemmas about pullback-squares we need and which are known at least in their most basic versions and not hard to generalize to what we need. Then, left invertible H-spaces are defined, which are H-spaces as defined in [Uni13] with an additional property already mentioned in [Uni13]. Some basic properties and a theorem 3.3.6 in the form of a pullback square appearing in a proof of the Mayer-Vietoris fiber sequence are proven. In the last section, a modality $\Im$ is introduced and some known properties of modalities from [Uni13] are given. The section concludes with a proof that left invertible H-spaces are preserved by modalities 3.4.10 and proof of a connection between the difference maps of a left invertible H-space and its image under a modality 3.4.11.
The main goals of chapter 4 are the triviality theorem for formal disk bundles on left invertible H-spaces, definition of spaces locally modeled on a left invertible H-space, the construction of the frame bundle and the definition of torsion free G-structures. Among other things, the spaces locally modeled on a left invertible H-space V, or *V-manifolds*, will have étale ∞-stacks locally modeled on an ∞-group as one possible model.
This line of work was suggested to the author by Urs Schreiber in late 2015 and Urs Schreiber also explained his topos-theoretic definitions and proofs, leading to those goals, to the author. The basic definitions in chapter 4 are given in different equivalent versions, some close to the original topos-theoretic definitions, some in a different style, making more use of the type theory. For the triviality theorem for formal disk bundles, a generalization of the tangent bundle, over a left invertible H-space, two proofs are given. One is along the original topos theoretic proof 4.2.10, the other 4.2.9 uses the type theoretic notions and is by far easier. The „construction“ of the frame bundle is done by constructing a classifying map from a trivializing cover 4.3.7. Formally étale maps are discussed in 4.4 and some basic properties are proven in a way similar to the topos theoretic version. Finally, V-manifolds are defined and the moduli spaces of G-structures 4.6.2 and torsion free G-structures on them. Here, the main work was the construction of trivial G-structures and the type theoretic definition, of what it means for those to agree on different formal disks.

[4] See [Uni13, Section 7.7].
[5] See [Sch13a] and [Sch16] for higher G-structures in general and [SSS12] for the examples of String- and Fivebrane-structures.

All theorems mentioned above have been formalized by the author in Agda [6]. This is an important part of this thesis and, in the opinion of the author, had also a very positive effect on the form the results took in the informal presentation.

1.2 Agda code

All results of this thesis[7] have been checked in Agda. A lot of the problems encountered are not visible in the informal text. There are, for example, a lot of statements in the code concerning substitution of equivalent types or homotopic maps and similar issues. The green[8] check marks (✓) sometimes appearing behind propositions in this thesis, link to the corresponding Agda code sections in a git hub repository (see https://github.com/felixwellen/DCHoTT-Agda). If there is no check mark at an intermediate result, this means either that it was not possible to point to specific location in the code due to differences in the formulation, or that the result really was not proven formally and is not essential for the goals of the thesis.
It is important to note, that the links in the check marks point to code sections in specific commits [9], so there may always be newer versions than what is seen in the browser after clicking on a green check mark.
Even if one does not know how to read Agda, a look at the code might give some clue how things are proven, since there are a lot of comments with diagrams like they would appear in a mathematical exposition.
The code does not use any external library for Homotopy Type Theory. Yet the author copied some tricks from other projects, for example equational reasoning and the Licata-Trick for Higher Inductive Types, which appear only in the form of the interval with the sole purpose of constructing functional extensionality. Also not visible in the informal presentation, is the use of a convenient Agda-feature called „records", which is used to define a lot of basic types, which are then shown to be equivalent to the versions defined with basic type constructors.

1.3 Acknowledgments

There are a lot of people without whom it is doubtful if I could ever have written a thesis with which I am as content as I am now. My supervisors, Frank Herrlich and Urs Schreiber, offered great support in very different ways. Urs Schreiber proposed an interesting task to solve and was always available to give very helpful answers to my numerous questions, via email and at our meetings in Bonn. Frank Herrlich had the courage to let me work on a topic as esoteric as Homotopy Type Theory and supported me througout the last years. Especially in the last weeks of writing the thesis, he and others protected me from distraction. Among those „others" is certainly one of the two groups at the department, I have been a member of for the last four and a half year, the group of didactics of mathematics, where there was never any hesitation, when I asked for short leaves to visit conferences and workshops.
Without the countless discussions with my office mate, Tobias Columbus, I would have known far less about a lot of topics important for the thesis and working on it would

[6] See [Nor07] for basic algorithms of this proof assitant and a discussion of the language itself or http://wiki.portal.chalmers.se/agda/pmwiki.php for information about Agda.
[7] All of them are internal statements in type theory.
[8] They should be black in the print version.
[9] In this instance, „commits" are snapshots in time of the agda code, managed by the version control system git (see https://git-scm.com/).

certainly have been less fun. Stefan Kühnlein had some crucial influence about ten years ago, when he convinced me with his topology course, that I am, contrary to my doubts at the time, studying the right subject. And since then, I have learnt a lot from former and current members of our working group.

There are more people with some direct positive influence on the thesis:

Ulrik Buchholtz and Thomas Streicher had a lot of helpful comments that certainly led to improvements of the thesis. One remark from Ulrik led me to use H-spaces in place of loop spaces, which was a major improvement. At the workshops in Bristol and Nantes, I had an opportunity to present my results leading to interesting discussions with lots of people from the Homotopy Type Theory community. Tobias Columbus and Fabian Januszewski helped me to calculate some examples from Algebraic Geometry. Konrad Völkel organised the workshop in Freiburg, which motivated me to learn how to prove things in Agda, since I promised to give a talk on it. In the summer of 2014, I gave a series of talks about Homotopy Type Theory to a very active audience, which I learnt a lot from.

There was also a lot of non-mathematical support. My flat mates made me a lot of tasty food, especially in the last weeks of writing my thesis and provided some entertaining company. My family was always very encouraging and showed sympathy when I was working a lot. At this point, I also like to mention some influence from my parents, that is certainly not unrelated to my interest in pure mathematics. My father taught me how to tinker with things and admire beautiful technology from a very early age on and my mother showed me by example how one can think with passion about details in nature and culture, that few people would ever notice. Then, finally, this would be unbearably incomplete without mentioning the one person I spend the most of the last five years with and learnt a lot from, the important things certainly far away from mathematics, yet certainly of positive influence on the way I did the things I have done in the last years.

1.4 Warnings

One of the theorems proved in this thesis, the triviality of the formal disk bundle over some class of spaces, is proved twice in very different ways. The first proof[10] is a lot shorter and more intuitive. The second proof is close to the topos version of the theorem by Urs Schreiber. Furthermore, the second proof needs more prerequisites, the main examples are pullback-pasting and the second-halves of both section 3.3 and 3.4. Pullback-pasting is needed in later chapters anyway and the other statements have some interest of their own. The statements in 3.3 could be used to transfer a classical proof of the homotopy-pullback version of the Mayer-Vietoris sequence to Homotopy Type Theory. Despite their formal uselessness for the main goals of the thesis, those statements are hence still included and the second proof of the triviality theorem for the formal disk bundle can be used to compare the style of reasoning: The second proof is basically done by pullback-pasting, where the first uses dependent sums and products to a great advantage which the author took a lot of time to realize despite Thomas Streicher's efforts to make him aware of the advantages of element based reasoning in Type Theory over an internal diagrammatic approach.

Here is another, completely different warning already hinted at above: The notion of homotopies we will discuss in this thesis do not arise as the well known continuous homotopies between paths in topological spaces. In fact, manifolds as they appear in Differential Geometry will appear as special sets in our intended way of transporting our type theoretic statements to the categories of interest. Homotopy types with homotopic information above level 0 might be obtained by taking the homotopy quotient of a manifold by some group action with non-trivial stabilizers. If spaces are modeled as sheaves on some site, such a homotopy quotient would be called a quotient stack or a higher sheaf in the classical world. Stacks also appear as solutions to *moduli problems*. There have been examples where things get easier once geometric notions and constructions are defined and performed directly with stacks. One first, famous example in Algebraic Geometry is certainly the proof of the irreducibility of moduli spaces of algebraic curves, proven in [DM69] with the stacky version of the moduli spaces. The result for stacks also yielded the desired result in the usual scheme-theoretic world.
In the following, we will first give a basic example how stacks may naturally appear in answering a basic geometric problem. We will proceed by explaining how stacks fit naturally into Homotopy Type Theory.

1.5 Moduli stacks and Univalence

We will introduce the concept of moduli stacks in the following. Apart from the discussion of classifying maps which will appear again in chapter 4, the following explanation is of little help for understanding everything else that is done in this thesis and was added for the sole reason of giving a good example from pure mathematics where homotopy theory appears like it does in the intended applications of this thesis.

A moduli space is supposed to be a space – for example a manifold – such that its points correspond to equivalence classes of geometric objects of a fixed kind. An easy well known example where everything works out nicely without using moduli *stacks*

[10] By the order they appear in this text – it was developed later.

are straight lines embedded in $\mathbb{R}^2$ which pass through the origin. These geometric objects have the 1-dimensional real projective space as their moduli space. This space is commonly defined as the following set topologized as a quotient of $\mathbb{R}^2 - \{(0,0)\}$:

$$\mathbb{RP}^1 := (\mathbb{R}^2 - \{(0,0)\})_{/\sim} \text{ where } v \sim v' \text{ means there is a } \lambda \in \mathbb{R} \text{ such that } v = \lambda \cdot v'$$

Note that the relation $\sim$ identifies all points lying on the same line through the origin. And for each straight line through the origin, there is exactly one point in $\mathbb{RP}^1$. We will come back to this example later, when we have arrived at a general definition.
Despite the general nature of the notion of geometric objects which might form moduli spaces, we will – following a tradition[11] – restrict our investigation to the moduli space of plane triangles to convey the basic idea of this concept and to see how stacks naturally appear.
If $u, v, w \in \mathbb{R}^2$ are not contained in a single line, we denote the convex hull of these points by $\Delta((u, v, w))$ and call all subsets of $\mathbb{R}^2$ obtained in this way *plane triangles*. Two plane triangles Δ and Δ' are congruent, if there is an isometry $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(\Delta) = \Delta'$. In general, a moduli space for some kind of geometric object is supposed to have one point for each class of equivalent geometric objects. We consider two plane triangles to be equivalent, if they are congruent. Thus, if M_Δ denotes the moduli space of plane triangles, the points of M_Δ should bijectively correspond to congruence classes of plane triangles.
Since we want to have a moduli *space*, we should fix a notion of space and define what it means to be a moduli space for this notion of space. In general, there is virtually no limit to the notion of space that might be used. We will use real manifolds in this example. So our moduli space M_Δ – if it exists – should have the structure of a real manifold, capturing our expectations of a space of triangles – for example, similar triangles should be close to each other. The latter might be rephrased without a concept of distance: continuous variation of points in the moduli space should correspond to continuous variation of the triangles represented by those points.
On an open ball $U \subseteq \mathbb{R}^n$ a continuous variation of triangles with parameters from U, may be modeled by a map $t\colon U \to (\mathbb{R}^2)^3$, mapping a parameter $x \in U$ continuously to the vertices of a triangle. We say in this case, that we have a family of triangles over the base U and represent this family by the map $p\colon E \to U$, with fiber $p^{-1}(x) := \Delta(t(x))$ over each $x \in U$, where E might be constructed as a subset of $\mathbb{R}^2 \times U$. For bases not homeomorphic to an open ball, we require the above locally to get the notion of a continuously varying family of plane triangles:
A *family of plane triangles* (or just *family*) over a manifold B is a map $\varphi'\colon E' \to B$ isomorphic (in the category of manifolds over B) to a continuous map $\varphi\colon E \to B$, such that:

(i) $E \subseteq \mathbb{R}^2 \times B$,

(ii) φ is the projection onto the second factor restricted to E,

(iii) all fibers of φ are plane triangles

(iv) and there is some covering $(U_i)_{i\in I}$ of B, such that the vertices of the triangles $\varphi^{-1}(x)$, $x \in U_i$ are given by maps continuous on U_i.

As we will see later, the appropriate notion of morphism of families are commutative squares

[11]In [Beh+07] the authors attribute this idea to Michael Artin and follow this approach as well.

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "\varphi_D"'] & E \arrow[d, "\varphi_E"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

such that f' restricted to $\varphi_D^{-1}(b)$ is an isometry of plane triangles for all $b \in B$. Note that these squares are pullbacks. Let us look at some examples for families before we make any use of morphisms.

Example 1.5.1

(a) Let $I = [0, 1]$ and $\Delta \subseteq \mathbb{R}^2$ an equilateral triangle. Then $T := \Delta \times I$ is a family of plane triangles varying over I. Of course, this construction may be done with arbitrary manifolds and triangles.

(b) Let $S^1 \subseteq \mathbb{C}$ be the unit circle and $\Delta_e \subseteq \mathbb{R}^2$ an equilateral triangle centered at the origin. Then there is a family $R\colon \mathbb{R}^2 \times S^1 \to S^1$, such that

$$R^{-1}(e^{it}) = e^{i \cdot \frac{t}{3}} \cdot \Delta_e.$$

This means, if you „travel“ along S^1 the triangle in the fiber will rotate by 120° and finally, after one turn around S^1, the triangles will match up, due to the symmetry of Δ_e. One may check, that this family is not isomorophic to the projection $\Delta_e \times S^1 \to S^1$.

The notion of a family captures precisely, what it means for a family of triangles to vary continuously. This may be used to define a moduli space adhering to an analogous notion of continuous variation of triangles. But in the moduli space, the variation happens directly at the level of points, not fibers. These two notions of variation are connected by mapping a point $b \in B$ under a fiber of a family $F\colon E \to B$ to the point in M_Δ corresponding to the triangle $F^{-1}(b)$. This yields a map $\phi_F\colon B \to M_\Delta$ which we require to be continuous for all families and call the *classifying map* [12] of F.
Since families are preserved by pullback, we can rephrase this as follows:
Let P_Δ be the contravariant functor from manifolds to sets with

$$P_\Delta(B) := \{ \text{ Isomorphism classes of families over } B \ \}$$

and mapping a continuous $f\colon B' \to B$ to the map $f^*\colon P_\Delta(B) \to P_\Delta(B')$ pulling back along f, then M_Δ is required to represent P_Δ.
If we define a general *moduli problem* to be a functor $P\colon \mathcal{S} \to \mathrm{Set}$ for some meaningful category $\mathcal{S}$ of spaces, where P is obtained analogously to P_Δ, i.e. by giving a notion of family over a base $B \in \mathcal{S}$ and defining morphisms of families to be fiberwise equivalences of the geometric objects to classify, then a moduli space for P is, again just a representing object.
In our example, this implies $P_\Delta(M_\Delta) \cong \mathrm{Hom}(M_\Delta, M_\Delta)$ and we have a family $\chi\colon \tilde{M_\Delta} \to M_\Delta$, the preimage of id_{M_Δ}. The family χ is called the *universal family*, a name justified by the following property provable from the definition of P_Δ and the assumption that M_Δ represents P_Δ:
If M_Δ is a moduli space, all families F over any base B are pullbacks of χ along a unique map $\phi_F\colon B \to M_\Delta$.

[12] In 4.3 classifying maps will appear again. And in the beginning of 4.6 we will discuss classifying maps and provide some pictures.

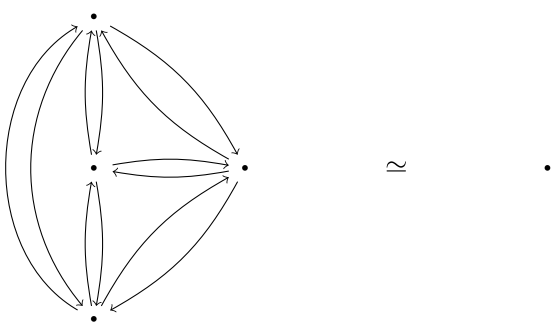
$$\begin{tikzcd} E \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\text{(pb)}"] & \tilde{M}_\Delta \arrow[d, "\chi"] \\ B \arrow[r, "\phi_F"'] & M_\Delta \end{tikzcd}$$

Or: For all B, there is a canonical bijection of isomorphism classes of families over B and morphisms $B \to M_\Delta$. We may now answer the question, if M_Δ really exists negatively: The second example above, R, has exactly the same map $\phi_R\colon S^1 \to M_\Delta$ as the family constantly Δ_e over S^1. Yet the two families are not isomorphic.
Let us look at the example mentioned in the beginning, $\mathbb{P}^1\mathbb{R}$, before we have a closer look at what went wrong with M_Δ and how we may fix it. The manifold $\mathbb{P}^1\mathbb{R}$ is supposed to be a moduli space for lines in the real plane passing through the origin. The appropriate notion of family may be defined like above, by replacing triangles with lines through the origin and requiring local continuity in the base of the lines in the fibers. But we could also use the usual definition of a vector bundle of dimension one, by defining a family to be a dimension one sub-bundle of the projection $\mathbb{R}^2 \times B$ for a base B. If we define morphisms of these bundles to be maps f of the bases $f\colon B \to B'$ such that for all $b \in B$ the fiber over b is equal to the fiber over $f(b)$, we can define the moduli problem functor like before and $\mathbb{P}^1\mathbb{R}$ would actually be a representing object and is therefore a moduli space for the problem of classifying lines in $\mathbb{R}^2$ passing through the origin. The important difference to the example with the triangles is not that we are talking about simpler objects, but the different notion of equivalence: We consider two lines equivalent, if they are the same line as a subset of $\mathbb{R}^2$. In the case of triangles, this would correspond to saying two triangles are equivalent, if they are the same subset of $\mathbb{R}^2$. With this notion of equivalence, the problem with the two non-isomorphic families above would disappear, since all triangles in the familiy with the rotating triangle would be different while the other family would still be constant.
On the other hand, we could alternate the moduli problem for lines through the origin in the plane by identifying isometric lines. On a first glance, this may look like making the whole moduli problem trivial, since there is only one line in the plane up to isometry. But now, a line can be equivalent to itself in more than one way allowing us to construct a family similar to the rotating triangle and therefore evidence, that we cannot construct a moduli space.
So the real problem we are observing here has little to do with the complexity of the geometric objects to classify – it is just about whether our notion of equivalence behaves like equality or more like isomorphy, since it is exactly this extra information given by non-trivial isomorphisms, which we lose by passing to equivalence classes.
So a solution to the problem above could also be a variation of the notion of equivalence of plane triangles, which is still in the spirit of the problem, but removes the possibility of triangles being equivalent in different ways. This can be done by labeling the edges of the triangles and requiring the isometries to respect the labels. Then, for two equivalent triangles in the old setting, labels may be chosen such that they are equivalent in the new setting. This means, in some sense, we do not change whether triangles are equivalent, but instead we change if we want to allow our objects to be equivalent in multiple different ways. But resorting to such a trick is not a very satisfying solution. It might be hard or impossible to come up with such a trick for more complicated objects and it

is still not a solution to the original problem.
Let us now, for this paragraph, look at something completely different. In homotopy type theory as introduced in [Uni13], we actually do have a solution to a moduli problem where the objects are allowed to be equivalent in different ways: the univalent universe. Here, the objects to classify are just small types, the notion of equivalence is equivalence of types and families are dependent types. The latter might seem strange, since dependent types in [Uni13] are already maps $P\colon A \to \mathcal{U}$ – this definition could be interpreted to represent families of types by their classifying map to the moduli space of types $\mathcal{U}$. To match the situation above more closely, we could also define a family over B to be a map $p\colon E \to B$. Then we would get the classifying map $\tilde{p}\colon B \to \mathcal{U}$ by mapping $b{:}B$ to the fiber $\sum_{e:E} p(e) = b$. And pulling back the universal family $\left(\sum_{A:\mathcal{U}} A\right) \to \mathcal{U}$ along $\tilde{p}$ yields a morphism equivalent to p over B. Hence $\mathcal{U}$ really is a moduli space of types in a natural translation of the definition to homotopy type theory.

Back to the triangles: One way to really fix the original problem is to replace sets of points by groupoids of points.

In the following, let $\mathcal{F}$ be a category of families over a fixed base together with their isomorphisms. We already noticed, that taking the set[13] quotient by isomorphy over a point as base cannot be a set of points of the moduli space M_Δ, since the information given by non trivial automorphisms is missing.
Usually, passing to quotients means building a set lacking some information irrelevant for some aspect of the problem while still maintaining the information given by the equivalence classes. This could alternatively be described as passing to the set of connected components, $\pi_0(\mathcal{F})$ of the groupoid $\mathcal{F}$.
To arrive at the notion of *quotient groupoid* partially solving our problem, we do *not* build a new groupoid taking the role of the quotient set. Instead, we change which objects possibly representing quotients we consider to be equal.
Using the notion of equivalence of categories restricted to groupoids turns out to be exactly the right notion of equality for our purpose. Let us first pretend, that all families in $\mathcal{F}$ are equivalent in precisely one way, i.e. for each pair of objects $x, y \in \mathcal{F}$ there is exactly one isomorphism $\varphi\colon x \to y$. This condition also defines a so called *codiscrete groupoid.* If we take a set quotient, $\mathcal{F}$ would collapse to a point. And, indeed, if we consider equivalent groupoids to be equal $\mathcal{F}$ really *is* a point, since any codiscrete groupoid is equivalent to the one object category. An example of a codiscrete groupoid is pictured below, where all drawn edges are supposed to be non identity isomorphisms.



For any set, we can construct a *discrete groupoid* by taking the set as the objects and introducing only identity morphisms. This extends to a fully faithful functor and we can

[13] We will ignore set theoretical problems.

identify sets with their discrete groupoids. If there are no non trivial automorphisms, the groupoid $\mathcal{F}$ is equivalent to its quotient set. Note that we can build a quotient groupoid with codiscrete components for any set X with an equivalence relation with set of objects X and an isomorphism for any equivalent pair of elements. Together with the π_0 functor mapping a groupoid G to the discrete groupoid of the set of the connected components of G, the concept of set quotients is included in the category of groupoids.
One way to arrive at a moduli space of plane triangles from this concept of quotient is by passing from manifolds to groupoid valued sheaves on a site. Sheaves with values in groupoids are called *stacks* and form a 2-category, where equivalences are induced by natural equivalences between the groupoids of points of two sheaves. We will not define those concepts here, a definition may be found in [Vis04]. Let us just remark, that presheaves are not strict functors but pseudofunctors and the more involved sheaf conditions for those functors, typically called *descent*, are more cumbersome to introduce and work with than the set valued versions.
We will now redefine the functor $P_\triangle$ describing our moduli problem as the functor mapping a base manifold B, to the groupoid of families over B with equivalences of families as morphisms. Let us take the category of manifolds with jointly surjective inclusions of open subsets as our site. Then with our definition of families, $P_\triangle$ turns out to be a sheaf. This is basically due to the fact, that our families are locally trivial bundles and those may be constructed by glueing local data.
With some appropriate version of the Yoneda-lemma, this entails that the problem functor $P_\triangle$ is the solution to the problem it describes. So, in some sense, we passed to a category of spaces, where a solution to the moduli problem trivially exists. And now, the interesting questions are, to which nice subcategories our moduli space belongs. For example, one could ask, if it admits some suitable generalization of the atlas on a manifold.
Since, according to the Yoneda-Lemma, morphisms from, e.g. $\mathrm{Hom}(\_, S^1)$ to $M_\triangle$ are equivalent to $M_\triangle(S^1)$, we know that the classifying map for the family over S^1 rotating an equilateral triangle by 120° degrees is now different from the classifying map for the constant family. The universal property of the moduli space and its universal family take a different form in this setting, which we will not discuss here.

With a univalent universe, Homotopy Type Theory has a moduli space and a nice tool to construct other moduli spaces. One example is $\mathrm{BAut}(F)$, the moduli space of F-fiber bundles, which will be discussed in 4.3. And even if we do not assume existence of a univalent universe, type theoretic theorems may still be applied to stacks.

2 Models and differential cohesion

In the following, we will refer to the type theory used in this thesis, which is somewhat less than Homotopy Type Theory[14], as „the type theory“. In this short chapter we will indicate how the theorems proven in this thesis may be applied in contemporary pure mathematics. The word „interpretation“ is avoided deliberately in the following, since it has a technical meaning which is not provided by all things we want to call a model. More precisely, there is no proof yet, that theorems in the type theory can be interpreted in $(\infty, 1)$-toposes and one has to construct a representation by a certain kind of model

[14] We do not use any HITs, always assume function extensionality and sometimes assume univalence and the -1-truncation.

category to arrive at some category with extra structure that admits an interpretation of the type theory. And the resulting translation of constructions in the type theory to the original $(\infty, 1)$-topos is not yet proven to be independent of the choices along the way in a suitable sense. On the other hand, for 1-toposes, direct interpretation of the type theory without uses of univalence is known to be possible. We will not mention these differences later and use the word model as a non technical term, merely indicating that there is some meaningful way of translating theorems and constructions formulated in the type theory to theorems about objects in the model, typically for this thesis certain categories equipped with an endofunctor.
Furthermore, the examples of models given in the last part of this chapter are certainly useful for understanding what happens inside the type theory especially in the chapter about Cartan Geometry.

2.1 What can theorems in the type theory do for the working mathematician?

The purpose of this section is to briefly explain, in what kind of situations the theorems from this thesis may be translated to statements about objects appearing in contemporary pure mathematics. There are two classes of theorems in this thesis when it comes to applying their content to the classical mathematical world: The theorems that depend on univalence and those that do not but possibly use function extensionality. In the first case, it is conjectured that the models of the type theory with univalence include $(\infty, 1)$-toposes, but so far this is known to be true only for some special cases [15].
If we do not use the univalence axiom, all $(\infty, 1)$-toposes as well as all 1-toposes are known to be models – in the latter case we also know that there is some well defined process to interpret the type theoretic constructions in the topos.
Since we will later use what is called a modality to define geometric concepts, in addition to a topos we need an endofunctor I on it subject to the following conditions:

(i) there is a full subcategory with inclusion u and left adjoint l such that $I = u \circ l$

(ii) I is a left exact funtor, i.e. it preserves pullbacks.

In typical toposes arising in geometry, there are a lot of functors satisfying these conditions. For example, any Grothendieck Topology finer than the one used to create the topos gives rise to such an endofunctor, since sheafification is left exact. But so far, the author knows of just one case where there might be a meaningful application[16] of a theorem in this thesis to functors not of a very special kind. Those functors of interest are called *coreductions* and we will give examples in 2.3.1 and 2.3.2.

2.2 Differential Cohesion

We will give a very short introduction to differential cohesion to comment on its relation to this thesis. See [Sch13a, Chapter 4] for more on differential cohesive toposes.

In [Law86] William Lawvere introduces the concept of what is know called a *cohesive topos*[17].

[15] See [LKV14] for the first $(\infty, 1)$-topos model supporting univalence and [Shu15a], [Shu15b] for extensions to larger classes of $(\infty, 1)$-toposes.
[16] This will be mentioned at the end of 4.
[17] The word „cohesion“ appears later, e.g. in [Law07].

Definition 2.2.1
A topos E is cohesive over a base topos S, if there is a geometric morphism $(p^*, p_*)\colon E \to S$, such that

(a) there exist a further left adjoint $p_!$ to p^* and further right adjoint $p^!$ to p_*,

(b) p^* and $p^!$ are fully faithful

(c) and $p_!$ preserves finite products.

This quadruple may be „condensed“ to an adjoint triple on E:

$$\int \quad \dashv \quad \flat \quad \dashv \quad \sharp$$

by the compositions $\int = p^* \circ p_!$, $\flat = p^* \circ p_*$ and $\sharp = p^! \circ p_*$ called „shape“, „flat“ and „sharp“. Then $\int$ and $\sharp$ are reflections. The objects of the subcategory $\int$ reflects to, are called *discrete* and those of the other subcategory *codiscrete*. The remaining functor $\flat$ is a coreflection into the subcategory of discrete objects. Note that the base topos may be recovered as one of those equivalent subcategories.
The intuition is, that points in objects might „cohere“ in some way. For example, in a topological space, the topology fixes how points „cohere“. The discrete objects do not cohere in any way and the points of codiscrete objects cohere in every possible way. And with being able to compare objects of the topos with these extremes, it is possible to access the cohesive structure.
One of the simplest non trivial examples of a cohesive topos is the category of reflexive graphs, as a presheaf-topos over Set. If G is such a graph, $\int(G)$ is the graph of the connected components of G, $\flat(G)$ is the graph with the same vertices as G but just reflexive edges and $\sharp(G)$ is the complete graph on the vertices of G. So one could say, two vertices in a graph cohere, if there is an edge joining them.
There is an extension of the axioms of cohesion to what is called *differential cohesion*. In addition to the adjoint triple on a cohesive topos, a differential cohesive topos has a triple

$$\mathfrak{R} \quad \dashv \quad \mathfrak{I} \quad \dashv \quad \mathcal{E}t$$

where $\mathfrak{R}$ is a coreflection, $\mathfrak{I}$ a reflection and $\mathcal{E}t$ a coreflection, such that the subcategory $\mathfrak{I}$ reflects to, contains the discrete objects.
For an $(\infty, 1)$-topos, the definition is given as [Sch13a, Definition 4.2.1] and the adjoint triple is for example used in [ST].
In this thesis, only the middle functor of the triple above appears again. It is called *coreduction*, *de Rham stack* or *infinitesimal shape* and we will give axioms for its internal version in 3.4. For algebro-geometric versions of $\mathfrak{I}$, it might not be part of a differential cohesive structure, but still of an adjoint triple. Yet for the smooth models, we will discuss below, it is a part of a differential cohesive structure.
It is not known, if it is possible to access all the operations of a differential cohesive topos and their properties internally. In [Shu15c], the problems with cohesion and Homotopy Type Theory are discussed and a variation of Homotopy Type Theory is presented, that supports a full cohesion triple and relies on the Dedekind-reals for the construction of the shape functor.
So one has reason to hope, that it is possible to find a Type Theory supporting differential cohesion in a natural way, allowing a lot more Differential Geometry to be done in Type Theory than by the means used in this thesis.

2.3 Some models

We will now look at some examples of models.
We start, with what we will call *smooth models*. The first topos similar to the following with a similar aim[18] was constructed in [Dub79]. Below, we repeat the definitions we need from [KS17, Section 2.1].
The models we will discuss, are called smooth because they admit a fully faithful inclusion of the category of smooth finite-dimensional manifolds. All manifolds mentioned in this thesis are supposed to be of the latter kind[19]. We denote the subcategory of the manifolds $\{\mathbb{R}^n \mid n \in \mathbb{N}\}$ by CartSp and call them *cartesian spaces*. The category CartSp is turned into a site with the locally finite open covers with contractible multiple intersections.
The sheaves on this site already form the subcategory of reduced objects of the topos we are really aiming at. What is still missing, are infinitesimal directions, i.e. objects which allow us to probe the smooth structure of our sheaves by mapping into them. For example, there should be what is called an *infinitesimal line segment* or *first order disk* $\mathbb{D}^1$ such that morphisms $\mathbb{D}^1 \to M$ to a sheaf M representing a manifold correspond to tangent vectors at points in M.
To get these infinitesimal directions, we have to pass to $\mathbb{R}$-algebras with the fully faithful[20] functor $\mathcal{C}^\infty \colon \mathrm{CartSp}^{\mathrm{op}} \to \mathbb{R}-\text{algebras}$. We will now define the ring of function on an infinitesimal line segment:

$$\mathcal{C}^\infty(\mathbb{D}^1) = \mathcal{C}^\infty(\mathbb{R})/(\mathrm{id}_{\mathbb{R}}^2).$$

This means we could add an object $\mathbb{D}^1$ formally to the category CartSp with opposites of all morphisms from and to images of objects in CartSp. But we will make a more systematic definition soon. Let us see, why $\mathbb{D}^1$ has the property mentioned above in some simple case. Morphisms $\mathbb{D}^1 \to \mathbb{R}$ should correspond to tangent vectors. Morphisms $\mathbb{D}^1 \to \mathbb{R}$ are the same as morphisms $\mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})/(\mathrm{id}^2)$ and from the fully faithfulness of $\mathcal{C}^\infty$, we know that any morphism $\varphi\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ is given by composition with a smooth map $f\colon \mathbb{R} \to \mathbb{R}$ which we get as $\varphi(\mathrm{id})$. Furthermore, composition with the projection is a surjective map from maps $\mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})/(\mathrm{id}^2)$. Therefore, any map $\psi\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})/(\mathrm{id}^2)$ is already determined by its value $f + (\mathrm{id}^2)$ on id.
Let us now see, when two such values, $f+(\mathrm{id}^2)$ and $g+(\mathrm{id}^2)$ are the same. By Hadamard's Lemma, we can rewrite f as

$$f = f(0) + x \cdot f'(0) + x^2 \cdot h(x)$$

with $h \in \mathcal{C}^\infty(\mathbb{R})$. Thus, $f+(\mathrm{id}^2) = g+(\mathrm{id}^2)$ if and only if, they take the same value at 0 and have the same derivative at 0. So we have a value in $\mathbb{R}$ and the value of a derivative and therefore enough data to specify a tangent to $\mathbb{R}$.
Note that our algebra of functions on $\mathbb{D}^1$ has a nilpotent element, $\mathrm{id}+(\mathrm{id}^2)$ and splits into $\mathbb{R}$ and a nilpotent summand of dimension one. We may turn this around and consider any $\mathbb{R}$-algebra of the form $\mathbb{R} \oplus V$, for a finite dimensional nilpotent algebra V, as the algebra of functions on a disk having infinitesimal directions. But it is also useful to stop at a specific order in some cases:

[18]Which was having a model for synthetic Differential Geometry.
[19]Excepting, of course, the internal notion of V-manifolds defined in 4.5.
[20]See [IS93, Corollary 35.10].

Definition 2.3.1

(a) Let $\mathrm{FormalCartSp}_k$ for $k \in \mathbb{N}$ be the category opposite to the full subcategory of $\mathbb{R}$-algebras spanned by algebras of the form

$$\mathcal{C}^{\infty}(\mathbb{R}) \otimes (\mathbb{R} \oplus V)$$

for V a finite dimensional $\mathbb{R}$-algebra such that $V^k = 0$. As a site, let $\mathrm{FormalCartSp}_k$ carry the topology given by covers U_i such that composing with the maps

$$\mathcal{C}^{\infty}(\mathbb{R}) \otimes (\mathbb{R} \oplus V) \to \mathcal{C}^{\infty}(\mathbb{R})$$

gives a cover in CartSp. We write FormalCartSp, if we require $V^k = 0$ just for some not fixed $k \in \mathbb{N}$.

(b) The sheaves on FormalCartSp are called *formal smooth sets* and form a model with the coreduction

$$\Im(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}) := \mathcal{F}(\mathbb{R}^n)$$

where $\mathbb{R}^n \times \mathbb{D}$ stands for the formal opposite of some $\mathcal{C}^{\infty}(\mathbb{R}) \otimes (\mathbb{R} \oplus V)$.
The category of sheaves on $\mathrm{FormalCartSp}_k$ together with the coreduction

$$\Im_k(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}) := \mathcal{F}(\mathbb{R}^n)$$

is refered to as the *smooth models* with *k-th order infinitesimals*.

In [Sch13a, Section 6.5] the formal smooth sets are lifted to an $(\infty, 1)$-topos setting, giving a model consisting of *formal smooth ∞-groupoids*.

Throughout chapter 4 we will calculate small examples in the Zariski-sheaves:

Definition 2.3.2
On $\mathrm{Rings}^{\mathrm{op}}$ we consider the *Zariski-topology* generated by covers (f_i) such that each f_i is opposite to a localization $R \to R_{r_i}$ and the r_i generate the unit ideal. Sheaves with respect to this topology are called *Zariski-sheaves*. The formal opposite of a ring R is denoted $\mathrm{Spec}(R)$ and the *coreduction* $\Im$ is given by

$$\Im(\mathcal{F})(\mathrm{Spec}(R)) = \mathcal{F}(\mathrm{Spec}(R_{\mathrm{red}}))$$

– where R_{red} denotes the *reduction* of the ring R, i.e. the quotient ring $R/\sqrt{0}$.
We merely want to give special emphasis to the Zariski-sheaves over $\mathrm{Spec}(\mathbb{Z})$ – there is no reason not to consider relative versions or different topologies. At least for an affine base scheme, $\Im$ may be defined in the same way and for the sake of this thesis, we do not have to care about if $\Im(\mathcal{F})$ is still a sheaf – we can just apply sheafification to it and still have a left exact functor.

A calculation similar to what was done above to see that tangent vectors can be detected with a line segment $\mathbb{D}^1$ is presented in the discussion following definition 4.2.3. And as already mentioned above, the Zariski-sheaves will be used to give external examples for some of the definitions in chapter 4. Here is a short overview of the examples:

(i) In 4.1, we calculate the formal neighbourhood of 0 in $\mathbb{A}^1_k$ in order to show how the coreduction functor may be used to define such concepts.

(ii) Following definition 4.4.1, we show for some special cases, to which external maps the internally defined *formally étale*, *formally unramified* and *formally smooth* maps correspond to.

(iii) In 4.4.10 we give an example of a Zariski-sheaf with a non formally smooth terminal morphism.

Finally, algebro-geometric coreductions are also defined and used in $(\infty, 1)$-toposes and non-commutative settings. We give a few examples:

(i) Simpson and Teleman define the whole differential cohesion triple in [ST]. The base sites they consider include schemes of finite type over the complex numbers with étale and Zariski-topology.

(ii) Dennis Gaitsgory and Nick Rozenblyum use a functor called *de Rham prestack* analogous to the coreductions above in derived Algebraic Geometry. For example in [GR14], they have a friendly explanation how one can get to a definition of D-modules and crystals using their coreduction[21] and attribute it to Grothendieck, [Gro68].

(iii) Maxim Kontsevich and Alexander Rosenberg use Q°-categories in [KR] for non commutative geometry which are nothing else than categories together with a reflective subcategory. So if the categories are nice enough, they provide models [22]. In [KR] they also define formally étale, formally unramified and formally smooth maps relative to a reflection. This is essentially the central definition in 4.4.

3 Preliminaries from Type Theory

3.1 Basic constructions

The types in the type theory we use here represent spaces as well as the propositions we will make about those spaces. We will not give a precise definition what classical mathematical objects we call spaces – one nice advantage of the type theory is that due to its abstract nature, extra structure on our types is not ruled out. This is similar to working in an abstract Grothendieck-topos, where we can do a lot without specifying what kind of geometric structure our sheaves carry. A 1-topos has a lot of similarities to the category of sets and that is certainly a reason why they are used in Geometry: Despite the complicated geometric structures the sheaves might encode we can reason about them set-theoretically.

In the type theory, this is even more explicit since we have what is called a judgement of kind „$x:X$“ expressing that x is a term of the type X, which is used in ways similar to the membership relation „ϵ“ in set theory. We will prefer to say „x is an element of the type X“ or „x is a point in the type X“ to use of the more syntactic word „term“.

We will start by looking at how types may be constructed, or which basic *type constructors* we will use. All type constructors in the following, incomplete summary correspond to functors or objects defined by universal properties in models. The following table is a variant of [Shu17, p. 17] reduced to what we need here:

[21] And other concepts which might be inaccessible from inside the type theory.

[22] The presheaf categories on associative algebras they use certainly are nice enough.

Type constructor	Example notation	Universal property
unit type	1	terminal object
product type	$A \times B$	product
function type	$A \to B$	exponential
identity type	$x =_A y$	equalizer
dependent product	$\prod_{x:A} B(x)$	right adjoint to pullback
dependent sum	$\sum_{x:A} B(x)$	left adjoint to pullback

Note that all those universal problems have solutions in a locally cartesian closed category with a terminal object. A locally cartesian closed category is equivalently[23] a category C with pullbacks such that for all morphisms $f: A \to B$ the pullback functor $f_*: C/B \to C/A$ is left and right adjoint. The left adjoint always exists and is given by $f \circ \_ : C/A \to C/B$ and in spite of its seemingly primitive nature, the type constructor $\sum$ it corresponds to is far from trivial. The right adjoint, sometimes denoted $\prod_f$, could be described as a relative version of an exponential. As described in [MM12, p. 57], $\prod_f$ also corresponds to application of the quantifier „$\forall$" in some situations. Both of the above fit to $\prod$-usages in Type Theory.

The pullback functor f_* does not appear in the table above. In our type theory, types are allowed to depend on terms. If $P(x)$ is a type depending on $x:X$ and we have a map $f: Y \to X$, then $P(f(y))$, is a type depending on $y:Y$. In models, the types depending on $x:X$ are given as a subclass of the morphisms $P: E \to X$ and substituting $f(y)$ for x corresponds to pulling P back along $f: Y \to X$.

A thorough introduction on how these constructions work inside the type theory are in the first chapters of [Uni13]. An explanation connecting everything also to models is given in [Shu17]. We will continue by giving short explanations of the type constructors above and present some of their properties relevant to our goals. Note that this is not intended to be an introduction to those type constructors, we will merely highlight some things important for this thesis and fix some notation.

3.1.1 Dependent types

In [Uni13, p. 32] a hierarchy of type universes

$$\mathcal{U}_0 : \mathcal{U}_1 : \ldots$$

is postulated and the informal statement „A is a type" is given the meaning, that A is an element of some universe. We will join [Uni13] in dropping the universe levels, i.e. we write $A:\mathcal{U}$ and not $A:\mathcal{U}_i$ for some $i \in \mathbb{N}$. Most of the types representing the objects we are interested in, are in the lowest universe $\mathcal{U}_0$ – thus the only $\mathcal{U}_i$ appearing as a $\mathcal{U}$ apart from a few exceptions [24]. In the Agda code repository, the universe levels are explicitly mentioned most of the time.

It is not known for all models of interest, that we really have an universe which we can use as a type like this. But as long as we do not use our universe for other things apart from stating that something is a type or a dependent type, the problem may be ignored, since those two notions are supported by any model we want to consider. In the models supporting univalence, there is always a universe type, and we will sometimes use universes in more ways than mentioned above, if univalence is assumed.

[23] To the definition suggested by the name, i.e. that all slices are cartesian closed.

[24] One example where we will use $\mathcal{U}_1$ is when we define and reason about $\mathrm{BAut}(F)$.

For any types A and B, we may form the type of functions $A \to B$. An element $(x \mapsto b(x)) : A \to B$ may be constructed by demonstrating [25] that $b(x) : B$ under the assumption $x : A$. Map $f : A \to B$ may be used in the usual way – applying $f : A \to B$ to some point $a : A$ gives us the result $f(a) : B$. And finally, the two rules are related by the definitional equalities $(x \mapsto b(x))(a) \equiv b(a)$ we have for any $a : A$. See [Uni13, p. 29] for more on function types. Sometimes, we will write the type function with multiple arguments as e.g. $A \to (B \to C)$ and not $A \times B \to C$. The equivalences between the two types are called *currying* and uncurrying.
For convenience, we will write dependent types as morphisms to the universe $P \colon A \to \mathcal{U}$ – alternatively, to formally avoid universes, we could write something like „$x : A \vdash P(x)$ is a type“ to indicate that we have demonstrated, that $P(x)$ is a type under the assumption $x : A$. So a dependent type $P \colon A \to \mathcal{U}$ provides us with a type $P(a)$ for any $a : A$.

3.1.2 Dependent sums and products

The *dependent sum* of a dependent type $P \colon A \to \mathcal{U}$ is the type denoted

$$\sum_{x : A} P(x)$$

containing all dependent pairs (a, p_a), where $a : A$ and $p_a : P(a)$. This is similar to the proposition „$\exists x : A$ such that $P(x)$“, but with all the evidence, namely the pairs, collected. There is an operation called *propositional truncation* which turns $\sum_{x : A} P(x)$, and all other types, into an actual proposition. Some authors implicitly assume, that statements of the form „There exists $x : A$, such that ...“ when turned into types are not the dependent sum but the *truncated* sum. We will always assume we take just the sum and make all applications of the propositional truncation explicit – sometimes just by using the adverb *merely*.
Dependent sums may be used to construct total spaces of „fibrations“ given by dependent types. This will happen a lot in chapter 4.

Let $P \colon A \to \mathcal{U}$ be a dependent type. The dependent product

$$\prod_{x : A} P(x)$$

is a generalization of the function type introduced above – the functions $s \colon \prod_{x : A} P(x)$ are allowed to have values $s(x)$ in a type $P(x)$ depending on the value $x : A$. Analogous to the sums, dependent products can be truncated to get „$\forall$“-quantified propositions. But if we say something like „For all $x : A$ we have a point in $P(x)$ by ...“, this will be assumed to be an element of the dependent product and not the truncated version. All truncations will be mentioned explicitly.

3.1.3 Identity types

In [AW09] it was noticed that Martin-Löf Type Theory with identity types admits models in Quillen model categories. The dependent identity type of a type A is modeled by the path object PA, obtained by factoring the diagonal $\Delta \colon A \to A \times A$ into a trivial cofibration and a fibration:

[25] That means we have to supply a proof, that $b(x) : B$ using the rules of type theory and the assumption $x : A$.

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\Delta"'] & PA \arrow[d] \\ & A \times A \end{tikzcd}$$

In a well behaved situation, where we have actual paths in spaces, the cofibration $A \to PA$ corresponds to the inclusion of constant paths into the space of all path and the fibration $PA \to A \times A$ maps a path to its endpoints. Keeping this in mind might help to understand the rules for identity types and what they mean for the intended applications of the results in this thesis.

For a type A and any points $x, y \in A$, we write $x =_A y$ or just $x = y$ for the *identity type* or type of *equalities* between x and y. We prefer to call the elements of $x = y$ equalities, to calling them paths, since that could lead to confusion if our types correspond to manifolds which do have non trivial paths in the classical sense, but no non trivial equalities in the internal type theoretic sense.

For any $x : A$, there is always an equality $\mathrm{refl} : x = x$, where refl stands for reflexivity. Now, if we want to define a function taking equalities $\gamma : x = y$ as input or prove some statement quantifying over equalities $\gamma : x = y$, we have to do it just for the case of $\mathrm{refl} : x = x$ and arbitrary x. That means, we can define a concatenation of equalities

$$\_ \bullet \_ : \prod_{x, y, z : A} (x = y) \to (y = z) \to (x = y)$$

just by the equations $\mathrm{refl} \bullet \mathrm{refl} = \mathrm{refl}$. This might seem a bit like cheating, but might be more plausible when looking at the above diagram, since reducing to the case refl just means we define something on A and are allowed to push it forward along the trivial cofibration $A \to PA$.

The composition $\_ \bullet \_$ extends to an ∞-groupoid structure on A, see [Uni13, Chapter 2]. This is an example where a structure with an infinite number of compatibility conditions may be used inside the type theory by making the compatibility relations explicit to the level they are needed.

Definition 3.1.1

Let A and B be types. For any two functions $f, g \colon A \to B$, we define the *type of homotopies* from A to B as

$$(f \Rightarrow g) :\equiv \left(\prod_{a : A} f(a) = g(a) \right).$$

It is possible to define composition, inversion and images under functions for equalities as well as for homotopies [26]. We will use the symbol $\bullet$ again to denote pointwise concatenation of homotopies. This is the same as composition with reversed argument order. The pointwise inversion will again be denoted by $^{-1}$.

We call triangles

[26] See [Uni13, Section 1.12] and [Uni13, Section 2.4].

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

homotopy commutative or just *commutative*, if there is a homotopy $g \circ f \Rightarrow h$. Of course, we will express commutativity of more general shapes in the same way. We will sometimes call homotopies *2-cells* and apply constructions form 2-category theory like whiskering to them:

Remark 3.1.2
Let $f, g\colon A \to B$ be maps and $H\colon f \Rightarrow g$ a homotopy. A function $h\colon B \to C$ may be *whiskered* to H, i.e. we can paste the following diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "H\Downarrow"] & B \arrow[r, "h"] & C \end{tikzcd}$$

to get a new 2-cell

$$\begin{tikzcd} A \arrow[r, bend left, "hf"] \arrow[r, bend right, "hg"'] \arrow[r, phantom, "hH\Downarrow"] & C \end{tikzcd}$$

by defining

$$hH :\equiv (a : A) \mapsto h(H(a)).$$

Of course, an analogous construction gives us a precomposition of maps with 2-cells.

It is possible to define a „homotopy level“ for types:

Definition 3.1.3

(a) A type A is called *contractible* if there is a point a, the center of the contraction, and a witness

$$\kappa\colon \prod_{x : A} a = x$$

that all points are joined to a by an equality.

(b) A -2-type is a contractible type and an „n-type“ is a type such that all its identity types are $(n-1)$-types. [27]

(c) -1-types are called *propositions* and may equivalently be defined as the types A with

$$\prod_{x, y : A} x = y.$$

[27]See [Uni13, Chapter 7] for details on how this definition works.

Sometimes, it is desirable to turn some type into a proposition. This might be done, with the *propositional truncation* $\|\_\|_{-1}$ which we will use in 4.3. This is an operation, that may be applied to a type A to get a proposotion $\|A\|_{-1}$. It comes with a constructor $|\_|\colon A \to \|\_\|_{-1}$ and is an example of what is called a *modality*. We will discuss axioms for and theorems about the modality $\Im$ in 3.4. Since $\Im$ is just a general modality, everything proven there about $\Im$ is also true of $\|\_\|_{-1}$. Of course, the propositional truncation is also discussed in [Uni13, pp. 146-148].

We will now turn to some important dependent function.

Definition 3.1.4
For all dependent types $P\colon A \to \mathcal{U}$ and points $x, y : A$, equal by $\gamma : x = y$, there is a function

$$\mathrm{transport}_P(\gamma)\colon P(x) \to P(y)$$

given by the equation $\mathrm{transport}_P(\mathrm{refl}) = \mathrm{id}$.

This also proves an important property of the propositional equality, the Leibniz Rule:

If $x = y$ and we know $P(x)$ for any property P, then we also know $P(y)$.

If we look at some dependent type P as a fibration and not a property, the transport is the fiber-transport.
As transports turn out to be equivalences, they can also be seen as a means to substitute equal elements of a type in a dependent type. This is certainly a reason why transports appear a lot. We will suppress their appearance often in the informal exposition. In [Uni13] some calculations are given, what the transports on some basic types turn out to be. This is usually easy to guess and not hard to prove. For example, on an identity type, the transport of the dependent type $y \mapsto x = y$ for some fixed x, is given by conjugation with the equality we are transporting along.

3.1.4 Equivalences

There is an internal notion of equivalence known to give weak equivalences in the models.

Definition 3.1.5
Let $f\colon A \to B$ be a map. We say that f is an equivalence, if one of the following equivalent conditions is true:

(i) All fibers of f are contractible.

(ii) There exist maps $g\colon B \to A$ and $h\colon B \to A$ and homotopies

$$g \circ f \Rightarrow \mathrm{id} \text{ and } f \circ h \Rightarrow \mathrm{id}$$

(iii) There exists a map $g\colon B \to A$ and witnesses in the following types:

$$\eta : g \circ f \Rightarrow \mathrm{id},\ \epsilon : f \circ g \Rightarrow \mathrm{id} \text{ and } \tau : \prod_{x : A} f(\eta x) = \epsilon(f(x))$$

In [Uni13, Chapter 4] one can find an explanation, why the inverses in (ii) are not the same map and why there is „half of an adjointness“ condition in (iii): Otherwise, the corresponding type – here given explicitly for (ii) –

$$(f \text{ is an equivalence}) :\equiv \sum_{g : B \to A} \sum_{h : B \to A} (g \circ f \Rightarrow \mathrm{id}) \times (f \circ h \Rightarrow \mathrm{id})$$

would not be a proposition in the sense of 3.1.3. Most of the time, we will use the invertibility conditions (ii) and (iii) without always which one we use. For any two types A and B, we will write

$$A \simeq B$$

for the corresponding type. Here is a list of some facts about equivalences:

Lemma 3.1.6

Let A, B and C be types.

(a) We have maps for composition and inversion of equivalences:

$$\circ\colon (B \simeq C) \to (A \simeq B) \to (A \simeq C) \text{ and } \_^{-1}\colon (A \simeq B) \to (B \simeq A).$$

(b) Any map homotopic to an equivalence is an equivalence.

(c) If $f\colon A \to B$, $g\colon B \to C$ and $h\colon A \to C$ are maps such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

commutes, and if two of the maps are equivalences, the remaining one is also an equivalence.

(d) If $f\colon A \to B$ and $x, y : A$, there is an induced equivalence

$$f\colon (x =_A y) \to (f(x) =_B f(y)).$$

(e) If $P\colon A \to \mathcal{U}$ is a dependent type such that $P(a)$ is contractible for all $a : A$, then

$$\sum_{a : A} P(a) \simeq A.$$

With the axiom following, homotopies are equalities up to equivalence:

Axiom 3.1.7

The map

$$(f = g) \to (f \Rightarrow g)$$

given by taking the image equality under all evaluation maps $(f \mapsto f(x))$ is an equivalence. This is called *function extensionality*.

This will be used a lot for the lemmas about pullback squares, but we will not explicitly mention it there. And we will not care about applying it, since it holds for all models of interest.

What we do care about, is when we need to use the *univalence axiom*:

Axiom 3.1.8

For all types A and B, the map

$$(A =_{\mathcal{U}} B) \to (A \simeq B)$$

given by transporting is an equivalence.

We freely use functional extensionality everywhere, but we only use univalence for a few theorems which cannot be proved without it.

3.2 Pullbacks

Homotopy limits in Homotopy Type Theory are very pleasant to use[28], since this can be done with the basic type constructors. Homotopy limits in general and also especially pullbacks are treated in [AKL15].
What is presented in the following, could be described as the basic definitions and propositions of a „calculus of pullback squares“. To formulate all the propositions about pullbacks as propositions about pullback squares, turned out to be a great advantage for the usability of pullbacks in the formalization. [29] All the statements used in the formalization admitted a proof by „$(2,1)$-categorical“ reasoning. We will omit most proofs.

Definition 3.2.1

(a) For a cospan $f\colon A \to C \leftarrow B\colon g$, the *canonical pullback* is given as

$$P(f,g) :\equiv \sum_{a:A} \sum_{b:B} f(a) = g(b).$$

We also write $A \times_C B$ for $P(f,g)$, if the maps seem obvious enough.

(b) For any pair of maps $z_1\colon Z \to A$, $z_2\colon Z \to B$ with a homotopy $\gamma\colon f \circ z_1 \Rightarrow g \circ z_2$, we call the map

$$(z{:}Z) \mapsto (z_1(z), z_2(z), \gamma(z))\colon Z \to P(f,g)$$

the *induced map to the pullback* and write $\mathrm{ind}(\gamma)$ for the map.

(c) We define the type of *pullback squares* for a fixed boundary

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "z_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

by

$$\sum_{\gamma : f\circ z_1 \Rightarrow g \circ z_2} \mathrm{ind}(\gamma) \text{ is an equivalence}$$

Note that the important datum of a pullback square is the homotopy or 2-cell. For a given boundary, there might be 2-cells such that the square is a pullback square and 2-cells for which it is no pullback square. This means that 1-categorical reasoning about pullback squares is not always valid with our pullback squares.

Remark 3.2.2
We write

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

[28] Especially in proof assistants.

[29] The Coq-library for Homotopy Type Theory, contains at least some of the statements we need about pullbacks and also makes use of the notion of pullback-squares.

to indicate that we have a term of the pullback type with the given boundary.

Note that we would get a different boundary and therefore a different – yet canonically equivalent – type of pullback squares with this boundary, if we would exchange the upper right maps f, z_1 with the lower left maps g, z_2. This is not just a burdon of being completely formal. The information which map in the input cospan is the „right“ map of the square and which is the „bottom“ map, is important for operations like pasting and could be considered as analogous to the kind of information domain and codomain provide about arrows in a category.

Lemma 3.2.3
Let $S \equiv (\gamma, p)$ be a pullback square on

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}.$$

(a) (✓) The square may be flipped over its diagonal to get another pullback square:

$$\begin{tikzcd} Z \arrow[r, "z_2"] \arrow[d, "z_1"'] \arrow[dr, phantom, "\text{(pb)}"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}.$$

(b) (✓) If W is some type and $e \colon W \to Z$ an equivalence with maps $w_1 \colon W \to A$, $w_2 \colon W \to B$ and homotopies $H \colon z_1 \circ e \Rightarrow w_1$, $K \colon z_2 \circ e \Rightarrow w_2$, then we get a new pullback square

$$\begin{tikzcd} W \arrow[r, "w_1"] \arrow[d, "w_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}.$$

with homotopy $f(H)^{-1} \bullet \gamma_e \bullet g(K)$, where $\gamma_e(w) :\equiv \gamma(e(w))$ and $g(K)$, $f(H)$ are given by pointwise application of f and g to H and K.

(c) (✓) If there are maps f', g' and homotopies $H \colon f \Rightarrow f'$ and $K \colon g \Rightarrow g'$, then we have a pullback square:

$$\begin{tikzcd}
Z \arrow[r, "z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f'"] \\
B \arrow[r, "g'"'] & C
\end{tikzcd}.$$

(d) ($\checkmark$) An equivalent vertex of the boundary may be substituted.

We will at some point deduce the existence of equivalences we are interested in from pullback squares with the following lemma:

Lemma 3.2.4 ($\checkmark$)
If we have two pullback squares for the same cospan

$$\begin{tikzcd}
Z \arrow[r, "z_2"] \arrow[d, "z_1"'] \arrow[dr, phantom, "\text{(pb)}"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "w_2"] \arrow[d, "w_1"'] \arrow[dr, phantom, "\text{(pb)}"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

we get an equivalence of cones:

$$\begin{tikzcd}
& B & \\
Z \arrow[ur, "z_2"] \arrow[dr, "z_1"'] \arrow[rr, dashed, "\simeq"] & & W \arrow[ul, "w_2"'] \arrow[dl, "w_1"] \\
& B &
\end{tikzcd}.$$

This was one of the few lemmas with a short proof: Composing the induced equivalences to the canonical pullbacks yields the equivalence.

Lemma 3.2.5

(a) Two pullback squares of the form

$$\begin{tikzcd}
Z \arrow[r, "z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & W \arrow[d, "w_2"] \\
D \arrow[r, "h"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, "w_1"] \arrow[d, "w_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

with homotopies γ and η may be composed to get a new square

$$\begin{tikzcd}
Z \arrow[r, "w_1 \circ z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\
D \arrow[r, "g \circ h"'] & C
\end{tikzcd}$$

with homotopy $\eta_{z_1} \bullet g_\gamma$. ($\checkmark$)

(b) In the situation

$$\begin{tikzcd} Z \arrow[rr, bend left, "z_1"] \arrow[d, "z_2"'] & W \arrow[r, "w_1"'] \arrow[d, "w_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\ D \arrow[r, "h"'] & B \arrow[r, "g"'] & C \end{tikzcd}$$

where the inner square and the outer rectangle are pullbacks, we get a new pullback square

$$\begin{tikzcd} Z \arrow[r, dashed, "\psi"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & W \arrow[d, "w_2"] \\ D \arrow[r, "h"'] & B \end{tikzcd}$$

where ψ is the map induced by the pullback property of the inner square and the cone of the outer rectangle. By the definition of induced maps, we automatically have a trivial homotopy $w_1 \circ \psi \Rightarrow z_1$. ($\checkmark$)

The following lemma allows us to view all pullbacks as pullbacks of the diagonal

$$\Delta :\equiv x \mapsto (x, x)$$

of the base.

Lemma 3.2.6
For all pullback squares we have the following equivalence:

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} Z \arrow[r, "f \circ z_1"] \arrow[d, "(z_1, z_2)"'] \arrow[dr, phantom, "\text{(pb)}"] & C \arrow[d, "\Delta"] \\ A \times B \arrow[r, "f \times g"'] & C \times C \end{tikzcd}$$

We will now turn to a different view of pullback squares. A pullback square is equivalently given by an equivalence over the bottom map.

Definition 3.2.7
Let $\varphi \colon B' \to B$ be a map of types and $P \colon B \to \mathcal{U}$ and $P' \colon B' \to \mathcal{U}$ be dependent types. An *equivalence over* φ is a section in

$$\prod_{x : B'} P'(x) \simeq P(\varphi(x)).$$

Lemma 3.2.8 ($\checkmark$)
In the situation of the definition, there is a construction yielding a pullback square

$$\begin{tikzcd}
\sum P' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \sum P \arrow[d] \\
B' \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

and all pullback squares

$$\begin{tikzcd}
Z \arrow[r, "z_1"] \arrow[d, "z_2"'] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}$$

yield an equivalence over g between the dependent types.

Before we prove this, we need the following intermediate result about commutative squares.

Lemma 3.2.9
If we have a square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] \\
B \arrow[r, "g"] & Y
\end{tikzcd}$$

commutative by a homotopy $H\colon \psi \circ f \Rightarrow g \circ \varphi$, then for any $b : B$, there is an induced map

$$f_b\colon \underbrace{\left(\sum_{a:A} \varphi(a) = b\right)}_{\text{Fiber of } \varphi \text{ at } b} \to \underbrace{\left(\sum_{x:X} \psi(x) = g(b)\right)}_{\text{Fiber of } \psi \text{ at } g(b)}.$$

Proof In the situation of the lemma, let (a, γ) be a point in the fiber at b. So we have $\gamma : \varphi(a) = b$. We apply the bottom map, g, to get $f(\gamma) : g(\varphi(a)) = g(b)$. And concatenation with $H(a) : \psi(f(a)) = g(\varphi(a))$ yields a point $(f(a), H(a) \bullet f(\gamma)) \equiv: f_b((a, \gamma))$.
The map f_b defined like this, is the same as the map induced by the pullback property of the left fiber:

$$\begin{tikzcd}
\sum_{a:A} \varphi(a) = b \arrow[r] \arrow[d] \arrow[rrr, dashed, bend left, "f_b"] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\psi"] & \sum_{x:X} \psi(x) = g(b) \arrow[l] \arrow[d] \arrow[dl, phantom, "\text{(pb)}"] \\
1 \arrow[r, "b"'] \arrow[rrr, bend right, "\equiv"'] & B \arrow[r, "g"'] & Y & 1 \arrow[l, "g(b)"]
\end{tikzcd}$$

□

Proof (of the second statement in 3.2.8) This follows from the diagram in the last proof: If we paste the left and middle pullback square, we know that f_b is the induced map between two pullbacks of the same cospan and therefore an equivalence by 3.2.4.□

The following will have an important appearance in the first proof of the triviality theorem for formal disk bundles on left invertible H-spaces.

Remark 3.2.10 (✓)
If $f\colon A \to B$ is any map, $P\colon A \to \mathcal{U}$, $Q\colon B \to \mathcal{U}$ and $s\colon \prod_{a:A} P(a) \to Q(f(a))$ an equivalence over f, the induced map $\sum P \to \sum Q$ is also an equivalence.

3.3 Left invertible H-spaces

In this section we define a structure similar to the H-spaces defined in [Uni13, Definition 8.5.4], with the additional requirement that multiplication from the right with any element of the H-space is an equivalence.
The objects of interest in higher geometry we are trying to treat type-theoretically are grouplike A_∞ spaces, or ∞-groups for short. However, since the axioms for A_∞ spaces are infinitary, there is little hope for an implementation in Homotopy Type Theory. But it turned out, that for our purpose, there is no need to impose any associativity constraint at all. So we will use a larger class of structured spaces admitting a surprisingly easy proof of the one result about these structures we are interested in. This result is the triviality of the formal disk bundle which we will prove in 4.2.9 and again in 4.2.10. The first proof does not need much more from our H-spaces than the existence of a family of equivalences mapping one fixed point to any other point of the space. The second depends on a result at the end of this section.
The following axioms, lacking any associativity, turn out to be enough to prove the triviality of the formal disk bundle over a type carrying such a structure.

Definition 3.3.1
Let X be a type. A *left invertible H-space structure* on X consists of the following data:

(i) A unit $e : X$.

(ii) A multiplication map $\mu\colon X \times X \to X$.

(iii) Proof that the unit is a left and right unit, i.e. a term in each of

$$\prod_{x:X} \mu(e, x) = x \text{ and } \prod_{x:X} \mu(x, e) = x.$$

(iv) Proof that for any $a : X$ the right-translation $x \mapsto \mu(x, a)$ is an equivalence, i.e. there is a term of type

$$\prod_{a : X} (x \mapsto \mu(x, a)) \text{ is an equivalence.}$$

Let us look at some examples, where these structures may arise.

Example 3.3.2

(a) All 0-groups as defined in [Uni13, Definition 6.11.1] are examples. So especially for any type A we have a left invertible H-space $\pi_1(A)$.

(b) For any type A with a point a, the loop space $\Omega(A, a) \equiv a =_A a$ is a left invertible H-space. ✓

(c) Any connected H-space is an example by [Uni13, Lemma 8.5.5]. Two particularly interesting H-space structures are known to exist on S^1 and S^3. Both are used to construct Hopf-fibrations, the first in [Uni13, Section 8.5.3], the second in [BR16].

So (a) allows us to plug in group schemes if we interpret in the Zariski-site or Lie-Groups if we interpret in smooth sets.
Of course, we will just say X is a left invertible H-space for a type X and leave the structure implicit. Since the right-translations are invertible, we have a left inverse element for each $x : X$, by taking a preimage of the neutral element e under right-translation with x. But it turns out, that for our purposes and probably also in general, in a non-associative context, these inverses are not really useful but the actual inverses to the translations are. Let us write the operation of our H-spaces as an infix operator $\bullet$ from now on:

Definition 3.3.3
Let X be a left invertible H-space and $a : X$.

(i) We introduce the following shorthand for the right-translation:

$$\_ \bullet a :\equiv x \mapsto \mu(x, a).$$

(ii) And if $f_a : X \to X$ is an inverse to $\_ \bullet a$, we define:

$$\_ \bullet a^{-1} :\equiv f_a$$

For fixed $a, b : X$, we can use 3.1.5 to rephrase the invertibility of $\_ \bullet a$ and $\_ \bullet a^{-1}$ as follows:
The type of solutions $x : X$ of the equations $x \bullet a = b$ and $x \bullet a^{-1} = b$, namely

$$\sum_{x : X} x \bullet a = b \text{ and } \sum_{x : X} x \bullet a^{-1} = b$$

are both contractible. The following remark shows, how the notation „$\_ \bullet a^{-1}$“ may be justified and what its limitations are one has to keep in mind.

Remark 3.3.4

(a) If X is a left invertible H-space and $x : X$, the following holds for all $a : X$:

$$(x \bullet a^{-1}) \bullet a = x \text{ and } (x \bullet a) \bullet a^{-1} = x$$

(b) [30] It is important to distinguish $x \bullet a^{-1}$ from the term $\mu(x, c)$ with c an actual inverse element to a, i.e. a preimage of e under right-translation with a. In general those terms are different.

Proof (a) By definition of $(\_ \bullet a^{-1})$.

(b) In order for us to see, that in the situation of the remark, $x \bullet a^{-1}$ and $x \bullet c$ really might be different in general, we can provide an example of some operation „$\bullet$" on the four element type $\{e, a, b, c\}$ where right-translation with c and $\_ \bullet a^{-1}$ are different maps. With „$\bullet$" given by the following table:

$\bullet$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	a	b
c	c	e	e	a

we can read of, that we have an left invertible H-space and c is an inverse element of a but we have:

$$b \bullet a^{-1} = a \neq b = b \bullet c.$$

□

Since we will, at some point, look at images of left invertible H-spaces under a modality, expressing as much as possible of this structure in diagrams will come in handy. Especially, the following characterization of the difference map ∂ defined below.

Definition+Remark 3.3.5
Let X be a left invertible H-space.

(a) The map $\partial \colon X \times X \to X$ is defined by:

$$\partial(a, b) :\equiv b \bullet a^{-1}$$

(b) The map

$$(\pi_2, \mu) \colon X \times X \to X \times X$$

is an equivalence.

(c) If $\varphi \colon X \times X \to X$ admits a homotopy commutative triangle

$$\begin{tikzcd} X \times X \arrow[rr, "(\pi_2, \mu)"] \arrow[dr, "\pi_1"'] & & X \times X \arrow[dl, "\varphi"] \\ & X & \end{tikzcd},$$

then φ is homotopic to ∂.

Proof (b) By left invertibility.

(c) Let us first state the following general fact: If $f \colon A \to B$ is an equivalence $g, h \colon B \to C$ some maps and $K \colon g \circ f \Rightarrow h \circ f$ a homotopy, we also have a homotopy between g and h by pasting with invertibility witnesses of f:

[30] Not formally verified for this thesis, but commonly known for sets.

$$\begin{tikzcd}
B \arrow[rr, "\mathrm{id}"] \arrow[dd, "\mathrm{id}"'] \arrow[dr, "f^{-1}"] & & B \arrow[dr, "h"] & \\
& A \arrow[ur, "f"] \arrow[dr, "f"] & & C \\
B \arrow[rr, "\mathrm{id}"'] \arrow[ur, "f^{-1}"] & & B \arrow[ur, "g"'] &
\end{tikzcd},$$

Now let $\varphi\colon X \times X \to X$ and $H\colon \varphi \circ (\pi_2, \mu) \Rightarrow \pi_1$ be the triangle. By the definition of ∂, we also have a triangle $\partial \circ (\pi_2, \mu) \Rightarrow \pi_1$, composing to a homotopy $\partial \circ (\pi_2, \mu) \Rightarrow \varphi \circ (\pi_2, \mu)$ with the first triangle. Now, we get the homotopy $\partial \Rightarrow \varphi$ by applying the initially stated general fact. □

We will now work towards a result needed to show the triviality of the formal disk bundle over a left invertible H-space. The result will manifest itself as a pullback square and also appears in the study of the Mayer Vietoris homotopy fiber sequence over a group. Let G be a left invertible H-space, D be an arbitrary type and $\varphi\colon D \to G$ a map. The result we are aiming at is just the statement, that the pullback of φ along $\partial\colon G \times G \to G$ is the product $G \times D$.
The following lemma and its proof originate from a comment by Mike Shulman on the nlab [Shu].

Lemma 3.3.6 (✓)
Let G be a left invertible H-space, D a type and $\varphi\colon D \to G$ a map. The following square is a pullback:

$$\begin{tikzcd}
D \times G \arrow[r, "\pi_1"] \arrow[d] & D \arrow[d, "\varphi"] \\
G \times G \arrow[r, "\partial"'] & G
\end{tikzcd}$$

where the left map is $(d, g) \mapsto (g, \varphi(d) \bullet g)$.

Proof In the situation of the lemma, we start with the canonical pullback of the cospan we are aiming at:

$$\begin{tikzcd}
P(\varphi, \partial) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & D \arrow[d, "\varphi"] \\
G \times G \arrow[r, "\partial"'] & G
\end{tikzcd}.$$

By definition, we have

$$P(\varphi, \partial) \equiv \sum_{d:D} \sum_{(g,h):G\times G} \varphi(d) = h \bullet g^{-1}.$$

Now, we know there is only a contractible space of solutions x to the equation $\varphi(d) = x \bullet g^{-1}$, or, the type

$$\sum_{h:G} \varphi(d) = h \bullet g^{-1}$$

is contractible for any given g and d. This means our pullback is a sum over contractible types:

$$\sum_{d:D} \sum_{(g,h):G\times G} \varphi(d) = h \bullet g^{-1} \simeq \sum_{d:D,\,g:G} \sum_{h:G} \varphi(d) = h \bullet g^{-1}.$$

And the latter is equivalent to its base by 3.1.6:

$$\sum_{(d,g):D\times G} \sum_{h:G} \varphi(d) = h \bullet g^{-1} \simeq D \times G.$$

All our manipulations of the pullback type except the last one were invisible to projections, so we may safely replace $P(\varphi, \partial)$ by the last sum. So we have to investigate how the last equivalence might yield a morphism of cones:

$$\begin{tikzcd}
 & D & \\
\sum_{(d,g):D\times G} \sum_{h:G} \varphi(d) = h \bullet g^{-1} \arrow[ur, bend left, "(d,g,h,\gamma) \mapsto d"] \arrow[rr, "(d,g,h,\gamma) \mapsto (d,g)"] \arrow[dr, bend right, "(d,g,h,\gamma) \mapsto (g,h)"'] & & D \times G \arrow[ul, bend right, "\pi_1"'] \arrow[dl, bend left, "?"] \\
 & G \times G &
\end{tikzcd}.$$

So the lower triangle is the only problem left. Of course, we want to use the map from the statement of the lemma: $(d, g) \mapsto (g, \varphi(d) \bullet g)$. To get the desired homotopy in the lower triangle, we need to show that if we have g, h, d and an equality $\gamma : \varphi(d) = h \bullet g^{-1}$, we also have an equality of type $(g, h) = (g, \varphi(d) \bullet g)$. The latter may be constructed from an equality between h and $\varphi(d) \bullet g$. But we get one of this kind by applying $\_ \bullet g$ to γ and concatenating the result with a witness that $\_ \bullet g$ and $\_ \bullet g^{-1}$ are inverse to each other.
So our last equivalence really was a morphism of cones in the right way and we can replace the sum with product to obtain the pullback square stated in the lemma. □

3.4 Modalities

We define modalities exactly like in [Uni13, Chapter 7] with two exceptions: The name of our modality is „$\Im$“ not „$\bigcirc$“ and for convenience we include some known consequences of the usual definition. The name is different because, apart from a few exceptions, we will use just one modality with one intended meaning and call it the *coreduction* or *infinitesimal shape modality*.
Some properties of modalities are analogous to those of the functors in ordinary category theory called *reflections*. See 5.2 for details and references. We will continue type theoretically and only occasionally hint at similarities to reflections.

Axiom 3.4.1
From this point on, we assume existence of a map $\Im\colon \mathcal{U} \to \mathcal{U}$ and maps $\iota_A\colon A \to \Im A$ for all types A, subject to the conditions (i)-(iii) below. We call a type A *coreduced*, if ι_A is an equivalence. The following conditions state, that $\Im$ is a *modality*:

(i) For all A, the map $\iota_{\Im A}$ is an equivalence.

(ii) If $B\colon \Im A \to \mathcal{U}$ is a dependent type such that for all $a : \Im A$ the type $B(a)$ is coreduced, then we can define a section of B by induction, i.e. we have a term:

$$\Im\text{-induction}\colon \left(\prod_{a:A} B(\iota_A(a))\right) \to \left(\prod_{a:\Im A} B(a)\right).$$

And we require this section to be equal to the given one in the following way:

$$\Im\text{-compute}\colon \left(s\colon \prod_{a:A} B(\iota_A(a))\right) \to \left(\prod_{a:A} s(a) = \Im\text{-induction}(s)(\iota_A(a))\right)$$

(iii) Identity types of coreduced types are coreduced, i.e. for all points x, y in a coreduced type B the map $\iota_{x=_B y}$ is an equivalence.

There are lots of easy consequences from these basic properties of $\Im$. We have a recursor and we can extend $\Im$ to maps and get a naturality square for ι:

Definition 3.4.2

(i) Let A be any type and B be coreduced. Then we have a map:

$$\Im\text{-recursion}\colon (A \to B) \to (\Im A \to B)$$

given by applying $\Im$-induction to the constant dependent type

$$(a : \Im A) \mapsto B : \Im A \to \mathcal{U}.$$

(ii) For any function $f\colon A \to B$ between arbitrary types A and B, we have a function:

$$\Im f\colon \Im A \to \Im B$$

given by $\Im$-recursion$(\iota_B \circ f)$.

(iii) For any function $f\colon A \to B$ between arbitrary types A and B, we have a homotopy called $\Im$-naturality(f) in the square:

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[d, "f"'] & \Im A \arrow[d, "\Im f"] \\
B \arrow[r, "\iota_B"] & \Im B
\end{tikzcd}$$

given by $\Im$-compute.

So $\Im$ can be applied to maps like a functor and it is also easy to prove that this application commutes with composition of maps up to homotopy and in all known cases these homotopies can be shown to be compatible in natural ways, again up to homotopy. And $\iota_{\_}$ is a natural transformation up to homotopy from the identity to $\Im$.

Remark 3.4.3
For any homotopy $H\colon f \Rightarrow g$, we have a homotopy between the images:

$$\Im H\colon \Im f \Rightarrow \Im g.$$

Proof Let H be the given homotopy. For any $a{:}A$, we get an equality

$$\iota_B(H(a))\colon \iota_B(f(a)) = \iota_B(g(a)),$$

which we turn into an equality of type $\Im f(a) = \Im g(a)$ by using $\Im$-naturality twice. Now, we have exactly what we need to get $\Im H$ of the appropriate type by applying $\Im$-induction. $\square$

Since a modality is analogous to the reflector of a reflective subcategory, some analog of the universal property of a reflector should be expected to hold:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "f"'] & \Im A \arrow[d, dashed, "\psi"] \\ & B \end{tikzcd}$$

i.e. for all $f\colon A \to B$ into a coreduced type, we get a unique ψ, where unique means here, there is a contractible choice. We already know the factorization exists – it is given by the recursor. What remains to be shown is (b) of the following lemma.

Lemma 3.4.4
Let A be a type, B coreduced and $f\colon A \to B$ a map. Then the following holds:

(a) Let $g\colon \Im A \to B$. Applying $\Im$-recursion to $\iota_A \circ g$ yields a map homotopic to g.

(b) All maps $\varphi\colon \Im A \to B$ with $\varphi \circ \iota_A \Rightarrow f$ are homotopic to the map induced by recursion.

(c) The map

$$\_ \circ \iota_A\colon\ (\Im A \to B) \to (A \to B)$$

is an equivalence with inverse given by recursion.

Proof (a) By applying $\Im$-induction to get the homotopy using that B has coreduced identity type.

(b) Let $\varphi\colon \Im A \to B$ and $H\colon \varphi \circ \iota_A \Rightarrow f$. Now, if $\tilde{f}$ is the map induced by recursion, we have $\tilde{H}\colon \tilde{f} \circ \iota_A \Rightarrow f$. Applying the factorization map to the homotopy

$$H \bullet \tilde{H}^{-1}\colon \varphi \circ \iota_A \Rightarrow \tilde{f} \circ \iota_A$$

yields a homotopy $K\colon \Im\text{-recursion}(\varphi \circ \iota_A) \Rightarrow \Im\text{-recursion}(\tilde{f} \circ \iota_A)$ between the factorizations. But by (a) the factorizations are homotopic to $\tilde{f}$ and φ.

(c) Being a right inverse is a direct application of $\Im$-compute and what remains do be shown is precisely (b). $\square$

We are now in a position to easily conclude that being coreduced is invariant under equivalence. Of course, this would follow directly with univalence, but for reasons explained in 3.1.4, we want to avoid this whenever possible.

Lemma 3.4.5

(a) If f is an equivalence, $\Im f$ is.

(b) If A is equivalent to a coreduced type, then ι_A is an equivalence.

Proof (a) By applying $\Im$ to the invertibility witnesses.

(b) Let $f\colon A \to \Im A$ be an equivalence. We just have to use that a composition of equivalences is an equivalence:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"] & \Im A \arrow[d, "\Im f"] \\ \Im A \arrow[r, "\iota_{\Im A}"'] & \Im\Im A \end{tikzcd}$$

to see that $(\Im f)^{-1} \circ \iota_{\Im A} \circ f$ is an equivalence, and therefore, ι_A is an equivalence as well. $\square$

Coreduced types have various closedness properties, which we review in the following lemma.

Lemma 3.4.6

Let A be any type and $B\colon A \to \mathcal{U}$ such that for all $a : A$ the type $B(a)$ is coreduced.

(a) Retracts of coreduced types are coreduced.

(b) The dependent product

$$\prod_{a:A} B(a)$$

is coreduced. Note that A is not required to be coreduced here and this implies all function spaces with coreduced codomain are coreduced.

(c) If A is coreduced, the sum

$$\sum_{a:A} B(a)$$

is coreduced.

Proof (a) A type R is a retract of B, if there are maps $r : B \to R$ and $\iota : R \to B$, such that $r \circ \iota$ is homotopic to the identity. For all coreduced B and retracts R of B we have the following diagram:

$$\begin{tikzcd} R \arrow[r, "\iota"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "\iota_R"] & B \arrow[r, "r"] \arrow[d, "\iota_B"] & R \arrow[d, "\iota_R"] \\ \Im R \arrow[r, "\Im\iota"'] \arrow[rr, bend right, "\mathrm{id}"'] & \Im B \arrow[r, "\Im r"'] & \Im R \end{tikzcd}$$

Since ι_B is an equivalence, it has an inverse and by the diagram, $r \circ \iota_B^{-1} \circ \Im\iota$ is a biinverse to ι_R.

(b) This is proved, up to equivalence, in [Uni13, Theorem 7.7.7].

(c) This is [Uni13, Theorem 7.7.4]. □

One immediate consequence is $\Im 1 \simeq 1$ – this is the only provably coreduced type, since the operation mapping every type to 1 is a modality. Furthermore, 3.4.6 entails the possibility to prove for some propositions about coreduced types, that the proposition is coreduced and may hence be proved by using $\Im$-induction. One important example is the proposition that a map between coreduced types is an equivalence. But first, let us call dependent types $B\colon A \to \mathcal{U}$ depending on an arbitrary type A, *coreduced dependent types*, if, for all $a{:}A$, the type $B(a)$ is coreduced.

Lemma 3.4.7 (✓)
Let A and B be coreduced types and $f\colon A \to B$ a map. The proposition that f is an equivalence is a coreduced dependent type with coreduced base.

Proof The base, $A \to B$ is coreduced since B is coreduced. Since we assume always function extensionality and coreducedness is invariant under equivalence, the result does not depend on the particular definition of equivalence we chose. Say being an equivalence means we have a witness in

$$\sum_{l,r:B\to A}\left(\prod_{x:A}(l\circ f)(x)=x\right)\times\left(\prod_{x:B}x=(f\circ r)(x)\right)$$

then the coreducedness of this type follows from applying what we know starting with the inner identity types which are coreduced by the axioms for $\Im$. Hence the dependent products are over coreduced types and thus, themselves coreduced. The same works for the inner binary product. The sum has a coreduced base and is taken over a coreduced dependent type, so it is coreduced too and therefore, the whole proposition is coreduced. □

After these general remarks, we will now aim at proving that $\Im$ preserves left invertible H-spaces and their difference maps. The first thing we need to know is that $\Im$ preserves binary products.

Lemma 3.4.8 (✓)
[31] Let X be any type.

(a) There is an equivalence $\varphi\colon \Im X \times \Im X \to \Im(X \times X)$.

(b) We have a commutative triangle:

$$\begin{tikzcd}
\Im X \times \Im X \arrow[rr, "\varphi"] & & \Im(X \times X) \\
& X \times X \arrow[ul, "\iota_X \times \iota_X"] \arrow[ur, "\iota_{X\times X}"'] &
\end{tikzcd}$$

Proof (a) This is shown in [Uni13, Corollary 7.7.4]. But we want to look into the construction of φ.
First, we curry $\iota_{X\times X}$ to get a map

$$\psi\colon X \to (X \to \Im(X \times X)).$$

[31] In agda, we do not show, that φ is an equivalence – it is just defined and used.

Now, all maps $\psi(x)$ are of type $X \to \Im(X \times X)$, hence factor over $\Im X$. Let $\psi'(x) \colon \Im X \to \Im(X \times X)$ be the factored map. Then ψ' is of type $X \to (\Im X \to \Im(X \times X))$ and $\Im X \to \Im(X \times X)$ is coreduced, so we get a map of type $\Im X \to (\Im X \to \Im(X \times X))$ by factoring again and finally we get the desired φ by uncurrying.

(b) For any $x_1, x_2 : X$, we want to have an equality

$$\varphi(\iota_X(x_1), \iota_X(x_2)) = \iota_{X \times X}(x_1, x_2).$$

This may be achieved by essentially using $\Im$-compute on the things we constructed in (a) with $\Im$-recursion in reversed order.
For the first $\Im$-compute step, we apply the evaluation at $\iota_X(x_2)$ to the equality of type

$$\varphi'(\iota_X(x_1)) = \psi'(x_1)$$

which we get from using $\Im$-compute for the last $\Im$-recursion in (a). This gives us – up to currying – something of type

$$\varphi'(\iota_X(x_1), \iota_X(x_2)) = \psi(x_1)(\iota_X(x_2)).$$

But the right hand side is equal to $\psi(x_1)(x_2)$ by direct application of $\Im$-compute. And the latter may be rewritten to our goal by $\psi(x_1)(x_2) \equiv \iota_{X \times X}(x_1, x_2)$. $\square$

Now, let X be a fixed left invertible H-space. We are going to construct the data turning the type $\Im X$ into a left invertible H-space. With φ, we can define, what the structural maps of the left invertible H-space structure on $\Im X$ shall be:

Definition 3.4.9
If X is a left invertible H-space with neutral element $e : X$ and multiplication $\mu \colon X \times X \to X$, we call $\Im e :\equiv \iota_X(e) : \Im X$ the neutral element of $\Im X$ and $\Im\mu \circ \varphi \colon \Im X \times \Im X \to \Im X$ the multiplication on $\Im X$. We will write $\bullet$ for the multiplication in $\Im X$.

There are two obvious ways to proceed: characterizing the properties remaining to show in diagrams or proving them directly by using $\Im-$induction. We will use the latter alternative for now and use the first later for a different problem.

Lemma 3.4.10 (✓)
(a) The neutral element defined in 3.4.9, really is left and right neutral:

$$\prod_{x : \Im X} \Im e \bullet x = x \text{ and } \prod_{x : \Im X} x \bullet \Im e = x.$$

(b) The multiplication is left invertible:

$$\prod_{a : \Im X} (\_ \bullet a) \text{ is an equivalence.}$$

Proof (a) The proofs of the two statements are entirely symmetric, so let us show just one, the left neutrality. Since the identity type is coreduced, we can reduce the problem by $\Im$-induction to

$$\prod_{x : X} \Im e \bullet \iota_X(x) = \iota_X(x).$$

If $\mu\colon X \times X \to X$ is the multiplication on X, the left hand side of the equation expands to

$$(\Im(\mu) \circ \varphi)(\iota_X(e), \iota_X(x))$$

and we can apply 3.4.8 (b), the triangle for φ, to get

$$(\Im(\mu) \circ \varphi)(\iota_X(e), \iota_X(x)) = \Im(\mu)(\iota_{X \times X}(e, x)).$$

And by the naturality we know from 3.4.2, the latter is equal to $\iota_X(\mu(e, x))$, which is just $\iota_X(x)$ by the left neutrality of the multiplication on X.

(b) In 3.4.7 we proved, that being an equivalence between coreduced types, is a coreduced dependent type. So we can use $\Im$-induction to reduce the problem to:

$$\prod_{a : X} (\_ \bullet \iota_X(a)) \text{ is an equivalence.}$$

Of course, we will use the fact, that $\_ \bullet a$ and therefore $\Im(\_ \bullet a)$ are equivalences, to solve our problem. In order to use this, we need to compare $\Im(\_ \bullet a)$ and $\_ \bullet \iota_X(a)$, since a homotopy between the two is all we need to conclude that $\_ \bullet \iota_X(a)$ is an equivalence if $\_ \bullet \iota_X(a)$ is.
To construct such a homotopy, it is enough to provide a term of type

$$\prod_{x : X} \Im(\_ \bullet a)(\iota_X(x)) = \iota_X(x) \bullet \iota_X(a).$$

We can apply the φ-triangle on the right hand side and use naturality to get an equality to $\iota_X(x \bullet a)$. Applying naturality on the left hand side yields the same term. □

Note that through the $\iota_\_$, the left invertible H-spaces on X and $\Im X$ are related in more ways than one being the $\Im$-image of the other. There are some subtleties when working with both structures at the same time. One such subtlety appearing in the second proof of the triviality of the formal disk bundle is a compatibility issue with the two difference maps ∂_X and $\partial_{\Im X}$. What we will need, is the following statement:

Lemma 3.4.11 (✓)
We have a commutative square:

$$\begin{tikzcd}
X \times X \arrow[r, "\iota_X \times \iota_X"] \arrow[d, "\partial_X"'] & \Im X \times \Im X \arrow[d, "\partial_{\Im X}"] \\
X \arrow[r, "\iota_X"'] & \Im X
\end{tikzcd}$$

Proof Let us first check, that the following variation of the square we are interested in, commutes:

$$\begin{tikzcd}
X \times X \arrow[r, "\iota_X \times \iota_X"] \arrow[dd, "\partial_X"'] & \Im X \times \Im X \arrow[d, "\varphi"] \\
& \Im(X \times X) \arrow[d, "\Im(\partial_X)"] \\
X \arrow[r, "\iota_X"'] & \Im X
\end{tikzcd}$$

But by pasting the triangle from 3.4.8 (b) to the naturality square for ∂_X, we have the square above. So we are left with showing that the right map above, $\psi :\equiv \Im(\partial_X) \circ \varphi$ is homotopic to $\partial_{\Im(X)}$. We will use the diagrammatic characterization of ∂ from 3.3.5 to prove this. This means, we have to establish the commutativity of the following triangle:

$$\begin{tikzcd}
\Im X \times \Im X \arrow[rr, "(\pi_2, \Im\mu)"] \arrow[dr, "\pi_1"'] & & \Im X \times \Im X \arrow[dl, "\Im(\partial_X) \circ \varphi"] \\
& \Im X &
\end{tikzcd}$$

We know a similar triangle holds for ∂ and apply $\Im$ to this triangle to get:

$$\begin{tikzcd}
\Im(X \times X) \arrow[rr, "\Im(\pi_2, \mu)"] \arrow[dr, "\Im\pi_1"'] & & \Im(X \times X) \arrow[dl, "\Im(\partial_X)"] \\
& \Im X &
\end{tikzcd}$$

We will paste the square:

$$\begin{tikzcd}
\Im X \times \Im X \arrow[r, "\varphi"] \arrow[d, "(\pi_2, \Im\mu)"'] & \Im(X \times X) \arrow[d, "\Im(\pi_2, \mu)"] \\
\Im X \times \Im X \arrow[r, "\varphi"'] & \Im(X \times X)
\end{tikzcd}$$

whose commutativity can be checked on $(\iota(x), \iota(x'))$ by $\Im$- and product-induction. So we have to construct an equality of type

$$\varphi(\iota(x'), \Im\mu(\varphi(\iota(x), \iota(x')))) = \Im(\pi_2, \mu)(\varphi(\iota(x), \iota(x')))$$

– we can substitute $\iota(x, x')$ for $\varphi(\iota(x), \iota(x'))$ and use naturality on the left to reduce to

$$\varphi(\iota(x'), \iota(\mu(x, x'))) = \Im(\pi_2, \mu)(\iota(x, x')).$$

And substituting on the left again solves the problem, since we are left with a naturality equality.
After pasting, the last triangle looks like this:

$$\begin{tikzcd}
\Im X \times \Im X \arrow[r, "(\pi_2, \Im\mu)"] \arrow[d, "\varphi"] & \Im X \times \Im X \arrow[d, "\varphi"] \\
\Im(X \times X) \arrow[r, "\Im(\pi_2, \mu)"] \arrow[dr, "\Im\pi_1"'] & \Im(X \times X) \arrow[d, "\Im(\partial_X)"] \\
& \Im X
\end{tikzcd}$$

And this is the triangle we wanted, up to the left map, which is again by the property of φ homotopic to π_1. $\square$

4 Cartan Geometry

A topos-theoretic analog of some of the contents of this chapter may be found in [KS17]. We assume existence of a fixed modality $\Im$ throughout this chapter. Like we already mentioned in the previous chapter, one should always think of a very special kind of modality.

4.1 Formal disks in Differential and Algebraic Geometry

The operations $\iota_A \colon A \to \Im A$ of interest could roughly be described as identifying infinitesimally close points in the space A. Of course, this hardly makes any sense in a classical setting were such infinitesimally close points do not exist as actual points. Yet if we look at the smooth model with first-order infinitesimal directions from 2.3.1, there is a nice way to think about the infinitesimals: The space of all points first-order infinitesimally close to a fixed $x_0 : X$, is nothing else than the tangent space at x_0. Hence it is possible to recover the tangent space at x_0 from the unit $\iota_A \colon A \to \Im A$ as the fiber over $\iota_A(x_0)$.
But there are also more involved concepts than tangent spaces covered in this way. Let us take a look at the algebro-geometric case of Zariski-sheaves over some field k together with the reflection $\Im$ given by

$$(\Im\mathcal{F})(\mathrm{Spec}(A)) :\equiv \mathcal{F}(\mathrm{Spec}(A_{\mathrm{red}})).$$

The unit $\iota_{\mathcal{F}} \colon \mathcal{F} \to \Im\mathcal{F}$ is given by composition with the inclusion of the reduced subscheme. Let us compute a fiber of $\iota_{\mathbb{A}^1_k}$, to get „the tangent space“ with respect to this $\Im$. The fiber $\mathcal{F}_{\tilde{0}}$ over the image $\tilde{0}$ of $0 \in \mathbb{A}^1_k$ is given as a pullback of Zariski-sheaves and computed pointwise at affine schemes as a pullback of sets. This means, the $\mathrm{Spec}(A)$-points of the fiber $\mathcal{F}_{\tilde{0}}$ are the set-theoretic pre-image of $\tilde{0}$ under $\iota_{\mathbb{A}^1_k, \mathrm{Spec}(A)}$:

$$\begin{tikzcd}
\mathcal{F}_{\tilde{0}}(\mathrm{Spec}(A)) \arrow[r] \arrow[d, hook, "\subseteq"'] & 1 \arrow[d, "\tilde{0}"] \\
\mathbb{A}^1_k(\mathrm{Spec}(A)) \arrow[r, "\iota_{\mathbb{A}^1_k}"'] & \mathfrak{I}\mathbb{A}^1_k(\mathrm{Spec}(A))
\end{tikzcd}$$

If we pretend the situation is affine and switch to rings, we get:

$$\begin{tikzcd}
\mathrm{Hom}(R, A) \arrow[r] \arrow[d, hook, "\subseteq"'] & \mathrm{Hom}(k, A) \arrow[d] \\
A \simeq \mathrm{Hom}(k[X], A) \arrow[r, "\pi \circ _"'] & \mathrm{Hom}(k[X], A_{\mathrm{red}}) \simeq A/\sqrt{0}
\end{tikzcd}$$

– where we used that $k[X]$ represents the forgetful functor from rings to sets to get natural isomorphisms. Somewhat surprisingly, we can actually solve for R in the diagram in a nice way. We need an object R, such that morphisms from R to A correspond to the elements of the nilradical $\sqrt{0} \subseteq A$. If we fix $n \in \mathbb{N}$ and ask for a ring, such that morphisms from it correspond to elements a with $a^n = 0$, we can just take $k[X]/(X^n)$. But if we want all nilpotent elements at once, there is no solution in the category of rings, but we may take the limit of all $k[X]/(X^n)$ and their projections in the category $\mathfrak{TRing}$ of topological rings to get one.[32]
We denote the limit of the $k[X]/(X^n)$ by $k[[X]]$ as its underlying ring is just the usual ring of formal power series given as sequences in k with pointwise addition and Cauchy product as multiplication. We assume the category of rings to be embedded into $\mathfrak{TRing}$ by endowing each ring with the discrete topology. Then $\mathrm{Hom}(k[[X]], A)$ is the set of continuous ring homomorphisms, which will turn out to be exactly those, mapping X to a nilpotent element.
The topology on $k[[X]]$ is generated by the neighbourhoods $a + (X^n)$ for $a \in k[[X]]$ and $n \in \mathbb{N}$. So for a continuous homomorphism φ to a discrete ring and any element $a \in k[[X]]$, there has to be some $n \in \mathbb{N}$ such that $a + (X^n)$ is mapped to a point, since $\{\varphi(a)\}$ is a neighbourhood of the image. And that just means the kernel contains (X^n) and X is mapped to a nilpotent element.
So we get the following description of the fiber $\mathcal{F}_{\tilde{0}}$:

$$\mathcal{F}_{\tilde{0}}(\mathrm{Spec}(A)) \simeq \mathrm{Hom}_{\mathfrak{TRing}}(k[[X]], A) \simeq \sqrt{0} \subseteq A.$$

The sheaf $\mathcal{F}_{\tilde{0}}$ is an example of what is called a *formal scheme* in Algebraic Geometry. And our construction is not hard to generalize to other noetherian rings.
This is an example, where $\mathfrak{I}$ does not only encode first-order differential geometric information but also what is sometimes called higher-order infinitesimal directions. Of course, this is also possible in the smooth model if we take all nilpotent algebras and not just the square-zero ones.

[32] We could also use the limit in pro-rings.

4.2 The formal disk bundle

We will now turn to working internally in the type theory with a fixed modality $\Im$ with unit ι. This modality provides us with the notion of infinitesimal proximity. To see if two points x, y in some type A are infinitesimally close to each other, we map them to $\Im A$ and ask if the images are equal.

Definition 4.2.1
Let $x, y\colon A$. Then we say x is *infinitesimally close* to y, if there is a witness in $\iota_A(x) = \iota_A(y)$.

It turns out, all morphisms of types already respect that notion of closedness, i.e. if two points are infinitesimally close to each other, their images are close as well.

Remark 4.2.2 (✓)
If $x, y\colon A$ are infinitesimally close, then for any map $f\colon A \to B$, the images $f(x)$ and $f(y)$ are infinitesimally close.

Proof We construct a map between the two types $\iota_A(x) = \iota_A(y)$ and $\iota_B(f(x)) = \iota_B(f(y))$. By 3.4.2 we can apply $\Im$ to maps and get a map $\Im f\colon \Im A \to \Im B$. So we can apply $\Im f$ to a path $\gamma\colon \iota_A(x) = \iota_A(y)$ to get a path $\Im f(\gamma)\colon \Im f(\iota_A(x)) = \Im f(\iota_A(y))$ By 3.4.2 again, we know that we have naturality square:

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[d, "f"'] & \Im A \arrow[d, "\Im f"] \\
B \arrow[r, "\iota_B"] & \Im B
\end{tikzcd}$$

and hence paths $\eta\colon \Im f(\iota_A(x)) = \iota_B(f(x))$ and $\zeta\colon \Im f(\iota_A(y)) = \iota_B(f(y))$. This yields a path of the desired type:

$$\eta^{-1} \bullet \Im f(\gamma) \bullet \zeta$$

□

We will now define the internal concept generalizing the tangent spaces and formal neighbourhoods from the introduction above.

Definition 4.2.3
Let A be a type and $a{:}A$. The type $\mathbb{D}_a$ defined below in three equivalent ways is called the *formal disk at a*.

(i) $\mathbb{D}_a$ is the sum of all points infinitesimally close to a, i.e.:

$$\mathbb{D}_a :\equiv \sum_{x:A} \iota_A(x) = \iota_A(a)$$

(ii) $\mathbb{D}_a$ is the fiber of ι_A at $\iota_A a$.

(iii) $\mathbb{D}_a$ is defined by the following pullback square:

$$\begin{tikzcd}
\mathbb{D}_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & 1 \arrow[d, "* \mapsto \tilde{a}"] \\
A \arrow[r, "\iota_A"'] & \Im A
\end{tikzcd}$$

The characterization (c) is a verbatim translation of its topos theoretic analog [Sch13a][Definition 5.3.50] to Homotopy Type Theory.

Let us look at this construction in a simple situation in Algebraic Geometry. Let X be a Zariski-sheaf representing a scheme over a field k. A morphisms t from $\mathbb{D}_1 := \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ to X corresponds to a tangent vector at the point

$$x := t \circ \iota_{\mathrm{red}} \colon \underbrace{\mathrm{Spec}(k[\epsilon]/\epsilon^2)_{\mathrm{red}}}_{=\mathrm{Spec}(k)} \to X.$$

The element of the Zariski-tangent space at x is given as the function of type $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ mapping a linear approximation $\tilde{f} \in \mathfrak{m}_x/\mathfrak{m}_x^2$ of a germ $f \in \mathcal{O}_{X,x}$ to the value of the second projection from $k[\epsilon]/\epsilon^2 \cong k \oplus \epsilon k$. So $X(k[\epsilon]/\epsilon^2)$ may be understood as the set of tangent vectors. Now the fiber of

$$X(k[\epsilon]/\epsilon^2) \to \underbrace{\Im X(k[\epsilon]/\epsilon^2)}_{=X(k)}$$

at some point $x \in X(k)$ is precisely the set of tangent vectors at this point. But this is just the fiber at the $k[\epsilon]/\epsilon^2$-valued points. We already saw in the introduction to this chapter, what a whole fiber looks like for $X = \mathbb{A}^1_k$.

As morphisms of manifolds induce maps on tangent spaces, maps of types induce morphisms on formal disks:

Remark 4.2.4 (✓)
If $f \colon A \to B$ is a type and $a : A$, we get a map:

$$d(f)(a) \colon \mathbb{D}_a \to \mathbb{D}_{f(a)}$$

Or, dependently:

$$d(f) \colon \prod_{a : A} \mathbb{D}_a \to \mathbb{D}_{f(a)}$$

Proof To define $d(f)$ we take the sum over the map from 4.2.2:

$$d(f)(a) :\equiv (x, \gamma) \mapsto (f(x), \eta^{-1} \bullet \Im f(\gamma) \bullet \zeta)$$

– with η and ζ the paths from the naturality of ι. □

In Differential Geometry, the tangent bundle is an important basic construction consisting of all the tangent spaces in a manifold. We can mimic the construction in this abstract setting, by combining all the formal disks of a space to a bundle.

Definition 4.2.5 (✓)
Let A be a type. The type $T_\infty A$ defined in one of the equivalent ways below is called the *formal disk bundle* of A.

(i) $T_\infty A$ is the sum over all the formal disks in A:

$$T_\infty A :\equiv \sum_{a : A} \mathbb{D}_a$$

(ii) $T_\infty A$ is defined by the following pullback square:

$$\begin{tikzcd}
T_\infty A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "\iota_A"] \\
A \arrow[r, "\iota_A"'] & \Im A
\end{tikzcd}$$

Note that despite the seemingly symmetric definition, we want $T_\infty A$ to be a bundle having formal disks as its fibers, so it is important to distinguish between the two projections and their meaning. If we look at $T_\infty A$ as a bundle, meaning a morphism $p\colon T_\infty A \to A$, we always take p to be the first projection in both cases. This convention agrees with the first definition – taking the sum yields a bundle with fibers of the first projection equivalent to the $\mathbb{D}_a$ we put in.

Before we go any further, let us look at some examples in the intended models.

Remark 4.2.6

(a) In the smooth topos with first order infinitesimals, the formal disk bundle on a manifold is the tangent bundle of the manifold.

(b) As we already saw in the example at the beginning of this chapter, some formal disks correspond to formal schemes in Algebraic Geometry. This is also the case for formal disk bundles.
Let $X = \mathrm{Spec}(A)$ be an affine scheme, then we may look at its diagonal $\Delta_X\colon X \to X \times X$. This inclusion of a closed subscheme is given by the ideal

$$I = \left\{ \sum_{i=0}^{n} a_i \otimes {a_i}' \in A \otimes A \,\middle|\, \sum_{i=0}^{n} a_i \cdot {a_i}' = 0 \right\}.$$

We may look at the sequence of infinitesimal thickenings $\mathrm{Spec}(A \otimes A / I^n)$ of $\Delta_X(X)$ inside of $X \times X$ and take the colimit in the category of Zariski-sheaves. The result is usually called the formal completion of X along Δ_X in Algebraic Geometry.

For any $f\colon A \to B$ we defined the induced map $d(f)$ on formal disks. This extends to formal disk bundles.

Definition 4.2.7

For a map $f\colon A \to B$ there is an induced map on the formal disk bundles, given as

$$T_\infty f :\equiv (a, d_a) \mapsto (f(a), d(f)(d_a))$$

In Differential Geometry, the tangent bundle may or may not be trivial. This is some interesting information about a space. If we have a smooth group structure on a manifold G, i.e. a Lie-group, we may consistently translate the tangent space at the unit to any other point. This may be used to construct an isomorphism of the tangent bundle with the projection from the product of G with the tangent space at the unit.

It turns out, that this generalizes to formal disk bundles and the group structure may be replaced by the weaker notion of a left invertible H-space. We will provide two versions of a theorem stating the triviality of the formal disk bundle of a left invertible H-space in slightly differing ways. The main difference will be in the proofs. The first proof is a lot more intuitive and simpler and makes more use of features of the type theory. The second proof, found long before the first, is along the lines of [KS17, Proposition

3.18] and uses a lot of statements about pullback squares. Both proofs do not use the univalence axiom and therefore also provide a theorem for ∞-stacks with the currently known interpretation possibilities.

We will now start to work towards the first proof.

In the following we will build a family of equivalences from one formal disk of an invertible H-space to any other formal disk of the space. We start by observing how equivalences and paths act on formal disks.

Lemma 4.2.8

(a) (✓) If $f\colon A \to B$ is an equivalence, so is

$$d(f)(a)\colon \mathbb{D}_a \to \mathbb{D}_{f(a)}$$

for all $a\colon A$.

(b) (✓) Let A be a type and $x, y\colon A$ two points. For any path $\gamma\colon x = y$, we get an equivalence $\mathbb{D}_x \simeq \mathbb{D}_y$.

Proof (a) Let us first observe, that for any $x, y\colon A$ the map $\iota_A(x) = \iota_A(y) \to \iota_B(f(x)) = \iota_B(f(y))$ is an equivalence. This follows since it is homotopic to the composition of two equivalences. One is the conjugation with the paths from naturality of ι, the other is the equivalence of path spaces according to 3.1.6 induced by the equivalence $\Im f$.
Now, for a fixed $a\colon A$ we have two dependent types, $\iota_A(a) = \iota_A(x)$ and $\iota_B(f(a)) = \iota_B(f(x))$ and an equivalence over f between them. The sum of this equivalence over f is, by definition, $d(f)$ and by 3.2.10 this is an equivalence.

(b) The equivalence is just the transport of the dependent type $x \mapsto \mathbb{D}_x$. □

We are now ready to state and prove the triviality theorem.

Theorem 4.2.9 (✓)

Let V be a left invertible H-space and $\mathbb{D}_e$ the formal disk at its unit. Then the following is true:

(a) For all $x \in V$, there is an equivalence

$$\psi_x\colon \mathbb{D}_x \to \mathbb{D}_e$$

(b) $T_\infty V$ is a trivial bundle with fiber $\mathbb{D}_e$, i.e. we have an equivalence $T_\infty V \to V \times \mathbb{D}_e$ and a homotopy commutative triangle

$$\begin{tikzcd} T_\infty V \arrow[rr, "\simeq"] \arrow[dr, "\pi_1"'] & & V \times \mathbb{D}_e \arrow[dl, "\pi_1"] \\ & V & \end{tikzcd}$$

Proof (a) Let $x \in V$ be any point in V and let us denote the multiplication in V by „$\bullet$“. The right translation $t_x :\equiv y \mapsto y \bullet x$ with x given by the left invertible H-space structure on V is an equivalence. We therefore get an equivalence $\psi_x{}'\colon \mathbb{D}_e \to \mathbb{D}_{e\bullet x}$ by 4.2.8. Since e is a neutral element for $\bullet$, we get a path $x = e \bullet x$ and again with 4.2.8 an equivalence $\mathbb{D}_{e\bullet x} \to \mathbb{D}_x$. So we can compose to get the desired ψ_x.

(b) By the first definition 4.2.5 of the formal disk bundle, we have

$$T_\infty V :\equiv \sum_{x:V} \mathbb{D}_x$$

We define a morphism $\varphi\colon T_\infty V \to V \times \mathbb{D}_e$ by

$$\varphi((x, d_x)) :\equiv (x, \psi_x(d_x))$$

and its inverse by

$$\varphi^{-1}((x, d_e)) :\equiv (x, \psi_x^{-1}(d_e)).$$

Now, to see φ is an equivalence with inverse φ^{-1}, one has to provide paths of types

$$\begin{aligned} (x, d_x) &= \varphi^{-1}(\varphi(x, d_x)) = (x, \psi^{-1}(\psi(d_x))) \\ \text{and } (x, d_e) &= \varphi(\varphi^{-1}(x, d_e)) = (x, \psi(\psi^{-1}(d_e))) \end{aligned}$$

– which exist since the ψ_x are equivalences by (a). □

We turn now to the second proof. A topos-theoretic version of the following is [KS17, Proposition 3.18].

Theorem 4.2.10 (✓)
Let V be a left invertible H-space, then $T_\infty V$ is a product. More precisely, if $\mathbb{D}$ is the formal disk at the unit in V, we get a triangle:

$$\begin{tikzcd} T_\infty V \arrow[rr, "\simeq"] \arrow[dr, "{(\pi_1, \pi_2)}"'] & & \mathbb{D} \times V \arrow[dl, "{(d, g) \mapsto (\pi_1(d) \cdot g, g)}"] \\ & V \times V & \end{tikzcd}$$

Proof Let us start with the defining pullback square of the formal disk bundle $T_\infty V$ of the left invertible H-space V.

$$\begin{tikzcd} T_\infty V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & V \arrow[d, "\iota_V"] \\ V \arrow[r, "\iota_V"'] & \Im V \end{tikzcd}$$

By 3.2.6 we get a new pullback square:

$$\begin{tikzcd} T_\infty V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \Im V \arrow[d, "\Delta_{\Im V}"] \\ V \times V \arrow[r, "\iota_V \times \iota_V"'] & \Im V \times \Im V \end{tikzcd}$$

We will now start to build a different square which eventually will be pasted to what we constructed so far.
By 3.4.10 the image $\mathfrak{J}V$ of V under $\mathfrak{J}$ also carries the structure of a left invertible H-space. Hence we can apply 3.3.6 to $\mathfrak{J}V$ and the inclusion of its unit $\mathfrak{J}e\colon 1 \to \mathfrak{J}V$, to get the pullback square:

$$\begin{tikzcd}
1 \times \mathfrak{J}V \arrow[r] \arrow[d, "(_, g) \mapsto (\mathfrak{J}e \bullet g, g)"'] \arrow[dr, phantom, "\text{(pb)}"] & 1 \arrow[d, "\mathfrak{J}e"] \\
\mathfrak{J}V \times \mathfrak{J}V \arrow[r, "\partial_{\mathfrak{J}V}"'] & \mathfrak{J}V
\end{tikzcd}.$$

Since $1 \times \mathfrak{J}V \simeq \mathfrak{J}V$ and the left map is homotopic to the diagonal $\Delta_{\mathfrak{J}V}$, we get a square we can paste to what we constructed in the beginning:

$$\begin{tikzcd}
T_\infty V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathfrak{J}V \arrow[d, "\Delta_{\mathfrak{J}V}"] \\
V \times V \arrow[r, "\iota_V \times \iota_V"'] & \mathfrak{J}V \times \mathfrak{J}V
\end{tikzcd}
\quad
\begin{tikzcd}
\mathfrak{J}V \arrow[r] \arrow[d, "\Delta_{\mathfrak{J}V}"'] \arrow[dr, phantom, "\text{(pb)}"] & 1 \arrow[d, "\mathfrak{J}e"] \\
\mathfrak{J}V \times \mathfrak{J}V \arrow[r, "\partial_{\mathfrak{J}V}"'] & \mathfrak{J}V
\end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd}
T_\infty V \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & 1 \arrow[d] \\
V \times V \arrow[r, "\iota_V \times \iota_V"'] & \mathfrak{J}V \times \mathfrak{J}V \arrow[r, "\partial_{\mathfrak{J}V}"'] & \mathfrak{J}V
\end{tikzcd}.$$

The bottom map $\partial_{\mathfrak{J}V} \circ \iota_V \times \iota_V$ is precisely what was proved in 3.4.11 to be homotopic to $\iota_V \circ \partial$. If we replace the homotopic map and put in the defining square for $\mathbb{D}$, we get a new square on the left by pasting:

$$\begin{tikzcd}
T_\infty V \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbb{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & 1 \arrow[d] \\
V \times V \arrow[r, "\partial_V"'] & V \arrow[r, "\iota_V"'] & \mathfrak{J}V
\end{tikzcd}.$$

But we can also apply 3.3.6 to get a pullback of the left cospan:

$$\begin{tikzcd}
\mathbb{D} \times V \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathbb{D} \arrow[d] \\
V \times V \arrow[r, "\partial_V"'] & V
\end{tikzcd}$$

Now, by 3.2.4 we get the desired equivalence over $V \times V$ of $T_\infty V$ and $\mathbb{D} \times V$, since they are both pullbacks of the same cospan. $\square$

4.3 Fiber bundles

The role of fiber bundles for the goals of this thesis is explained at the beginning of 4.6. There, we will also give more intuition for the meaning of the classifying morphism of the formal disk bundle whose construction is the most relevant part of what follows bellow.

For the following statements about fiber bundles, we will make a lot of unavoidable use of the univalence axiom and propositional truncation[33]. A classical ∞-topos-theoretic account of the following results may be found in [NSS15].

We will frequently use that all maps of types $p\colon E \to B$ appear in a pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{(pb)}"] & \tilde{\mathcal{U}} \arrow[d] \\ B \arrow[r, "\tilde{p}"'] & \mathcal{U} \end{tikzcd},$$

where $\tilde{\mathcal{U}}$ is called the *universal family* and obtained by summing over the dependent type $(A{:}\mathcal{U}) \mapsto A$. The bottom map $\tilde{p}$ determines p up to canonical equivalence over B and is called the *classifying map* of p.

This way of using a univalent universe corresponds to looking at it as a moduli space of types, like we mentioned in the introduction. We could replace the $\mathcal{U}$ with some other moduli space to get specialized notions of fiber-bundles with additional structure on the fibers.

Before we start with the geometric content, we need some preliminaries about 1-epi- and 1-monomorphism. The notions of 1-epimorphism and 1-monomorphism we introduce in the following correspond to the $\|\_\|_{-1}$-connected and $\|\_\|_{-1}$-modal maps. [34]

Definition 4.3.1

Let $f\colon A \to B$ be a map of types.

(a) The map f is a *1-epimorphism* if[35]

$$\prod_{b:B} (\|A(b)\|_{-1} \simeq 1)\,.$$

We write $f\colon A \twoheadrightarrow B$ in this case.

(b) The map f is a *1-monomorphism* if

$$\prod_{b:B} (A(b) \text{ is -1-truncated})\,.$$

We write $f\colon A \hookrightarrow B$ in this case.

[33] Instead of the latter, equivalently, we could postulate existence of 1-images.

[34] In [Uni13, chapter 7], these maps are called (-1)-connected and (-1)-truncated. Topos theoretic analogs are defined in [Lur09b, pp. 6.5.1.10, 5.5.6.8] and are called 0-connective and (-1)-truncated. The terminology used here coincides with the terminology of Urs Schreiber used for example in [Sch13b] and [nLab].

[35] We write $A(b)$ for the fiber of f at $b{:}B$.

Example 4.3.2
Let $f\colon A \to B$ be an equivalence of types. Then f is a 1-epimorphism and a 1-monomorphism since all fibers of f are contractible by 3.1.5.

The following two lemmata are also true if $\|\_\|_{-1}$ is replaced by any other modality.

Lemma 4.3.3
In a homotopy-commutative square like

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, two heads, "e"'] & X \arrow[d, hook, "m"] \\ B \arrow[r, "\psi"] & Y \end{tikzcd}$$

where e is a 1-epimorphism and m a 1-monomorphism, there is a unique diagonal $d\colon B \to X$ making the upper and the lower triangle commute up to some homotopy.

Proof Let there be a square like in the statement of the lemma. Simply put, the diagonal map is constructed by applying φ to the fiber over some point in B.
We know from 3.2.9 that for any $b\!:\!B$, we have an induced map $\varphi_b\colon A(b) \to X(\psi(b))$ between the fibers. This provides us with a map

$$\|\varphi_b\|_{-1}\colon \|A(b)\|_{-1} \to X(\psi(b))$$

by applying $\|\_\|_{-1}$-recursion and using that m is 1-monomorphic.
Now we use that e is 1-epimorphic: $\|A(b)\|_{-1}$ is contractible and we can map $b\!:\!B$ to the center of this contraction. If we put this together with the map above and the inclusions of the fibers $X(\psi(b))$ into X, we get the diagonal $d\colon B \to X$.
The lower triangle commutes since the composition of the morphism m with one of its fiber-inclusions factors through a point meaning it does not matter which point in the fiber we are mapping to.
The upper triangle commutes, since, if we map some $a\!:\!A$ down to B, it is certainly in the fiber over its image, and therefore equal to the contraction center of the fiber.
Now, suppose we have some diagonal $d'\colon B \to X$ making the triangles commute. Then, for any $b\!:\!B$, the point $d'(b)$ is in the fiber over $\psi(b)$. This means the fiber is contractible and therefore, $d'(b) = d(b)$. With function extensionality, we get an equality $d = d'$, which proves the desired uniqueness. □

Lemma 4.3.4
For any map $f\colon A \to B$ there is a unique triangle:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \\ & \text{1-image}(f) \arrow[ur, hook, "m"'] & \end{tikzcd}$$

where e is a 1-epimorphism, m a 1-monomorphism and 1-image(f) is given by

$$\text{1-image}(f) :\equiv \sum_{b:B} \left\| \sum_{a:A} f(a) = b \right\|_{-1} .$$

Proof The uniqueness is a direct application of 4.3.3. And a proof of the general case of $\|\_\|_n$ may be found in [Uni13, chapter 7.6]. $\square$

In Topology, an F-fiber bundle is a map $p\colon E \to B$, that is locally trivial and all its fibers are isomorphic to F. Local triviality means, that B may be covered by open sets U_i, such that on each U_i the restricted map $p_{|p^{-1}(U_i)}$ is isomorphic to the projection $F \times U_i \to U_i$.
We may rephrase this in a more diagrammatic way: A cover of B may equivalently be given as the surjective map $w\colon \coprod_{i\in I} U_i \to B$. Then p is locally trivial, if and only if, the pullback of p along w is one projection of a product with second factor F. In a topos of set-valued sheaves, we may replace the surjective map resembling a cover by an epimorphism. In 5.1 we explain in more detail why 1-epimorphisms are a good notion of cover for smooth manifolds.

Definition 4.3.5
Let $p\colon E \to B$ be a map of types. For another map $w\colon W \to B$ we say w trivializes p, if w is a 1-epimorphism and there are pullback squares:

$$\begin{tikzcd} F \arrow[d] & W \times F \arrow[l] \arrow[r] \arrow[d, "\pi_1"] \arrow[dl, phantom, "\text{(pb)}"] \arrow[dr, phantom, "\text{(pb)}"] & E \arrow[d, "p"] \\ 1 & W \arrow[l] \arrow[r, two heads, "w"'] & B \end{tikzcd}$$

The map p is called an *F-fiber bundle* in this case.

Note that this definition does not provide us with a nice way to define the *type* of F-fiber bundles, since we ask for a non-unique datum, the trivializing map. Of course, we could truncate appropriately to remove the choice of map. But we will later see that we could have defined F-fiber bundles more easily with their classifying maps to a type called $\mathrm{BAut}(F)$, providing us directly with the type of F-fiber bundles.
It would be possible to start with the latter as the definition, but from the geometric point of view, this would not be very convincing, since the fact that we may represent and detect fiber bundles by mapping into $\mathrm{BAut}(F)$, is an important and more advanced *theorem* about fiber bundles. And since one of our goals is using Homotopy Type Theory to have easier proofs of geometric theorems, using the theorem as a definition, leaving us with no theorem to prove, would be counter productive.
So, therefore, the definition above should be read as telling us, whether one fixed map is an F-fiber bundle - but not what the type of F-fiber bundles is.
We will now work towards the theorem mentioned above and start by reviewing the definition of the type $\mathrm{BAut}(F)$.

Definition 4.3.6
Let F be a type and $t_F\colon 1 \to \mathcal{U}$ the map given by $* \mapsto F$.

(a) Let $\mathrm{BAut}(F) :\equiv 1\text{-image}(t_F)$.

(b) We also have the 1-monomorphism $\iota_{\mathbf{BAut(F)}}\colon \mathrm{BAut}(F) \to \mathcal{U}$.

(c) The map $\pi\colon F//\mathrm{Aut}(F) \to \mathrm{BAut}(F)$ is given as the first projection of the dependent sum over
$$((F', |\varphi|) : \mathrm{BAut}(F)) \mapsto F'$$

The map $\pi \colon F//\mathrm{Aut}(F) \to \mathrm{BAut}(F)$ is the universal F-fiber bundle, meaning all F-fiber bundles with any base will turn out to be pullbacks of this map. We are now ready to state the central theorem about fiber bundles we need:

Theorem 4.3.7
A map $p \colon E \to B$ is an F-fiber bundle, if and only if there is a map $\chi \colon B \to \mathrm{BAut}(F)$, such that there is a pullback square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{(pb)}"] & F//\mathrm{Aut}(F) \arrow[d, "\pi"] \\ B \arrow[r, "\chi"'] & \mathrm{BAut}(F) \end{tikzcd}.$$

In this case, χ is called the *classifying map* of p.

We will not prove this immediately. The theorem will be split up into two lemmata. In 4.3.9, a trivializing map is constructed from a classifying map. And the other implication is the statement of 4.3.10.
We start with an easy consequence of having a classifying map:

Remark 4.3.8 (✓)
If $p \colon E \to B$ has a classifying map $\chi \colon B \to \mathrm{BAut}(F)$, then all fibers are merely equivalent to F:

$$\prod_{b:B} \| E(b) \simeq F \|_{-1}.$$

Proof From the pullback square witnessing that χ classifies p, we get an equivalence over χ. So a fiber $E(b)$ of p is equivalent to $\pi(\chi(b))$. Since $\chi(b)$ is in $\mathrm{BAut}(F)$ it is equal to some (F', γ) in $\mathrm{BAut}(F)$ with $\gamma \colon \|F' = F\|_{-1}$. Truncating the universe-transport map, we get $\|F' \simeq F\|_{-1}$ and we already know $F' \simeq \pi(\chi(b))$ since F' and $\pi(\chi(b))$ are equal by applying the inclusion from $\mathrm{BAut}(F)$ into the universe to the equality between $\chi(b)$ and (F', γ). □

The next step is the construction of a trivializing map from a classifying map. This construction is very similar to the construction of the universal covering $\sum_{x:X} x = x_0$ of a pointed type (X, x_0), where the paths $x = x_0$ are replaced by equivalences „$E(b) \simeq F$“ in the following lemma.

Lemma 4.3.9
Let $p \colon E \to B$ be an F-fiber bundle, then p is locally trivial with fiber F and is trivialized by

$$\pi_1 \colon \left(\sum_{b:B} E(b) \simeq F \right) \to B.$$

Proof Let $W :\equiv \sum_{b:B} E(b) \simeq F$ and w be the projection to B. By 4.3.8 we know that w is a 1-epimorphism. It remains to show, that we have a span of pullbacks:

$$\begin{tikzcd} F \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & ? \arrow[l] \arrow[r] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\text{(pb)}"] & E \arrow[d, "p"] \\ 1 & W \arrow[l] \arrow[r, "w"'] & B \end{tikzcd}$$

We chose ? to be the canonical pullback and replace along the following equivalences:

$$\begin{aligned}
&\sum_{(b,\varphi):W}\ \sum_{e:E} b = p(e) \\
\simeq &\sum_{(b,\varphi):W}\ \sum_{(b',e):\sum_{b:B} E(b)} b = b' \\
\simeq &\sum_{(b,\varphi):W}\ \sum_{b':B}\ \sum_{e:E(b')} b = b' \\
\simeq &\sum_{(b,\varphi):W}\ \sum_{b':B} E(b') \times (b = b') \\
\simeq &\sum_{(b,\varphi):W} E(b)
\end{aligned}$$

So we need to show that

$$\begin{tikzcd}
F \arrow[d] & \sum_{(b,\varphi):W} E(b) \arrow[l] \arrow[d, "\pi_1"] \\
1 & W \arrow[l]
\end{tikzcd}$$

is a pullback square. But we have an equivalence over $W \to 1$: For any $(b, \varphi) : W$, we find an equivalence from $E(b)$ to F, so we can take just φ. □

Now let us see that, conversely, all F-fiber bundles are represented by a classifying morphism to $\mathrm{BAut}(F)$.

Lemma 4.3.10 (✓)
Let $p\colon E \to B$ be locally trivial with fiber F. Then there is a map $\chi\colon B \to \mathrm{BAut}(F)$ such that

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{(pb)}"] & F//\mathrm{Aut}(F) \arrow[d, "\pi"] \\
B \arrow[r, "\chi"'] & \mathrm{BAut}(F)
\end{tikzcd}.$$

Proof Let us start with the squares witnessing the local triviality of p:

$$\begin{tikzcd}
F \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & W \times F \arrow[l] \arrow[r] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\text{(pb)}"] & E \arrow[d, "p"] \\
1 & W \arrow[l] \arrow[r, two heads, "w"'] & B
\end{tikzcd}.$$

The map p may be written as a pullback of the universal family over the universe along some map $\tilde{p}$. We paste this square to the squares above and look at some triangles pasted to the bottom and shown in the following to commute:

$$\begin{tikzcd}
F \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & W \times F \arrow[l] \arrow[r] \arrow[d, "\pi_1"] \arrow[dr, phantom, "\text{(pb)}"] & E \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\text{(pb)}"] & \tilde{U} \arrow[d] \\
1 \arrow[rrr, bend right=50, "t_F"'] & W \arrow[l] \arrow[r, two heads, "w"] \arrow[rr, bend right, "_ \mapsto F"'] & B \arrow[r, "\tilde{p}"] & U
\end{tikzcd}$$
.

We know by pasting of the right squares, that $\tilde{p} \circ w$ classifies the projection $\pi_1 \colon W \times F \to W$. Hence $\tilde{p} \circ w$ has to be the constant map with value F. The lowest map in the diagram is the classifying map for $F \to 1$, thus again, the constant map, letting the lowest triangle commute.
The map t_F is the map we factored into an 1-epimorphism and a 1-monomorphism in the definition of $\mathrm{BAut}(F)$. So let us paste this factorization triangle to t_F and rearrange:

$$\begin{tikzcd}
W \arrow[r, two heads, "w"] \arrow[d] & B \arrow[r, "\tilde{p}"] & U \\
1 \arrow[urr, "t_F"] \arrow[rr, two heads] & & \mathrm{BAut}(F) \arrow[u, hook]
\end{tikzcd}$$

Now, 4.3.3 yields a map χ as the diagonal in

$$\begin{tikzcd}
W \arrow[r] \arrow[d, two heads, "w"] & \mathrm{BAut}(F) \arrow[d, hook, "\iota_{\mathrm{BAut}(F)}"] \\
B \arrow[r, "\tilde{p}"'] \arrow[ur, dashed, "\chi"] & U
\end{tikzcd}$$
.

Let us concentrate on the lower right triangle to see that χ is the classifying map for p. If we pullback the universal family over U along $\tilde{p}$ and $\iota_{\mathrm{BAut}(F)}$, we get the maps p and π and, by pullback pasting, a pullback square between them:

$$\begin{tikzcd}
E \arrow[r, dashed] \arrow[d, "p"] \arrow[rr, bend left] & F//\mathrm{Aut}(F) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\text{(pb)}"] & \tilde{U} \arrow[d] \\
B \arrow[r, "\chi"'] & \mathrm{BAut}(F) \arrow[r] & U
\end{tikzcd}$$
. □

This also finishes the proof of the theorem 4.3.7. From now on, we will call p an F-fiber bundle whenever we have established one of the equivalent conditions of the theorem.

We will finish our study of fiber bundles by looking at $\mathrm{Aut}(F)$-principal bundles and start with a construction generalizing the universal cover over $\mathrm{BAut}(F)$ to fibers not necessarily sets.

Definition 4.3.11
Let $P\colon \mathrm{Aut}(F)//\mathrm{Aut}(F) \to \mathrm{BAut}(F)$ be defined by summing over

$$((F', |\gamma|) \colon \mathrm{BAut}(F)) \mapsto F \simeq F'.$$

We know by univalence, that the total space of P is contractible. So we could have defined P as the inclusion of the point given by $(F, |\mathrm{refl}|)$ and we will use both versions in the following.
There are additional, easy to see similarities to the universal cover:

Lemma 4.3.12
The map $P\colon \mathrm{Aut}(F)//\mathrm{Aut}(F) \to \mathrm{BAut}(F)$

(a) is a 1-epimorphism and

(b) an $\mathrm{Aut}(F)$-fiber bundle.

Proof (a) We have to show, that the (-1)-truncated fibers of P are „not empty". Let $\gamma\colon F =_U F'$ be a path between the fiber and another type, and $x :\equiv (F', |\gamma|)$ the corresponding point in $\mathrm{BAut}(F)$. Then $(*, \gamma)$ is in

$$P(x) \equiv \left(\sum_{y:1} P(y) = x \right) \simeq \left(\sum_{y:1} F = F' \right).$$

So we can map the path $F =_U F'$ into the fiber $P((F', |\gamma|))$ and by applying $\|\_\|_{-1}$ to this map, we get one between the truncations, thus yielding for any point $(F', |\gamma|)$ in $\mathrm{BAut}(F)$ a point in the fiber over it.

(b) $\mathrm{Aut}(F)$ denotes the loop space $(F, |\mathrm{refl}|) =_{\mathrm{BAut}(F)} (F, |\mathrm{refl}|)$. We show local triviality with the loop space pullback square:

$$\begin{tikzcd}
\mathrm{Aut}(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & 1 \arrow[d, two heads, "P"] \\
1 \arrow[r, two heads, "P"'] & \mathrm{BAut}(F)
\end{tikzcd}$$

□

Definition 4.3.13
A map $p\colon X \to B$ is called *principal* $\mathrm{Aut}(F)$*-bundle*, if there is a pullback square:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Aut}(F)//\mathrm{Aut}(F) \arrow[d, two heads, "P"] \\
B \arrow[r, "\chi"'] & \mathrm{BAut}(F)
\end{tikzcd}$$

By switching to an equivalence over the χ in the pullback square above, we see that p is a 1-epimorphism, since all its fibers are equivalent to fibers of P, which have the property required of 1-epimorphisms. And, since pullbacks of fiber bundles are again fiber bundles with the same fiber type, we have the following:

Remark 4.3.14
All principal $\mathrm{Aut}(F)$-bundles are 1-epimorphic $\mathrm{Aut}(F)$-fiber bundles.

We get the following important relation between F-fiber bundles and $\mathrm{Aut}(F)$-principal bundles:

Remark 4.3.15
Any F-fiber bundle $p\colon E \to B$ is associated to an $\mathrm{Aut}(F)$-principal bundle $\hat{p}$, given as pullback of $P\colon \mathrm{Aut}(F)//\mathrm{Aut}(F) \to \mathrm{BAut}(F)$ along the classifying morphism χ of p:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\hat{p}"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Aut}(F)//\mathrm{Aut}(F) \arrow[d] \\
B \arrow[r, "\chi"] & \mathrm{BAut}(F) \\
E \arrow[r] \arrow[u, "p"] \arrow[ur, phantom, "\text{(pb)}"] & F//\mathrm{Aut}(F) \arrow[u]
\end{tikzcd}$$

One classical example of an associated principal bundle is the *frame bundle*, which is associated to the tangent bundle of a smooth n-manifold. The automorphism group in this case, is the automorphism group of a tangent space, hence $\mathrm{GL}_n(\mathbb{R})$. The frame bundle has as its fibers all possible choices of a basis of the tangent space at this point, so the fibers are isomorphic to $\mathrm{GL}_n(\mathbb{R})$.

4.4 Formally étale maps

In the following we will introduce formally étale maps resembling some class of maps including the local diffeomorphisms in the smooth setting and étale maps in the algebro-geometric world[36]. The definition uses a modality $\Im$ like in 3.4, which we assume to be given in everything that follows. The formally étale maps are also the right maps of a factorization system existing for any modality, but we will not make use of this fact.

Definition 4.4.1
Let $f\colon A \to B$ be a map of types and $\varphi\colon A \to B \times_{\Im B} \Im A$ the morphism induced by the naturality square for $\Im$:

[36] The latter is at least true in a non relative setting.

$$\begin{tikzcd}
A \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\iota_A"] \arrow[ddr, bend right, "f"'] & & \\
& B \times_{\Im B} \Im A \arrow[r] \arrow[d] & \Im A \arrow[d, "\Im f"] \\
& B \arrow[r, "\iota_B"'] & \Im B
\end{tikzcd}$$

(a) We call f *formally unramified*, if φ is a 1-monomorphism.

(b) We call f *formally smooth*, if φ is a 1-epimorphism.

(c) We call f *formally étale*, if φ is an equivalence and write $f\colon A - \text{ét} \to B$.

Remark 4.4.2
If $\Im$ were left exact, formally étale maps could equivalently be defined as maps $f\colon A \to B$, such that $d(f)$ is an equivalence over f, i.e. all the induced maps

$$d(f)(a)\colon \mathbb{D}_a \to \mathbb{D}_{f(a)}$$

are equivalences. Yet in our more general situation, this is a weaker notion but might still work as a definiton.

Before we continue our type theoretic study of formally étale maps, let us explore the definition for sheaves on the Zariski-site. On the Zariski-site, we want $\Im$ to be the functor given pointwise by mapping the set of A-valued points to the set of A_{red}-valued points. For a morphism $f\colon X \to Y$ of sheaves, being formally étale amounts to turning all naturality squares

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{Zar}}(\mathrm{Spec}(A), X) \arrow[r, "_ \circ \iota_{\text{red}}"] \arrow[d, "f \circ _"] & \mathrm{Hom}_{\mathfrak{Zar}}(\mathrm{Spec}(A_{\text{red}}), X) \arrow[d, "f \circ _"] \\
\mathrm{Hom}_{\mathfrak{Zar}}(\mathrm{Spec}(A), Y) \arrow[r, "_ \circ \iota_{\text{red}}"'] & \mathrm{Hom}_{\mathfrak{Zar}}(\mathrm{Spec}(A_{\text{red}}), Y)
\end{tikzcd}$$

into pullbacks. This is just a pullback of sets and we can rephrase this condition to get the following lifting property: For any commutative square of the form indicated below, there is a unique diagonal lift d letting both triangles commute.

$$\begin{tikzcd}
\mathrm{Spec}(A_{\text{red}}) \arrow[r] \arrow[d, "\iota_{\text{red}}"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "d"] & Y
\end{tikzcd}$$

If we take A to be $k[X]/(X^2)$ for some field k this just means that we can uniquely lift tangent vectors at k-points for any choice of lift of the k-point.
In [EGAIV4, § 17], a formally étale map is defined as a morphism of schemes $f\colon X \to Y$, such that for any ring A and nilpotent ideal $N \subseteq A$, any square

$$\begin{tikzcd} \mathrm{Spec}(A/N) \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "d"'] & Y \end{tikzcd}$$

has exactly one morphism d making both triangles commute[37]. Note that $\sqrt{0}$ does not need to be nilpotent itself. Hence the class of maps formally étale in the sense of [EGAIV4] and the maps internally defined as formally étale above might disagree and there might be no inclusion of one into the other. But we can also choose Zariski-sheaves on noetherian affine schemes as a model. In this case, our[38] class of formally étale maps are those lifting against inclusions of reduced subschemes of affine schemes. Therefore, the formally étale maps from [EGAIV4] are included, since the nilradical of a noetherian ring is always nilpotent. So we can apply the type theoretic theorems about formally étale maps to maps between noetherian schemes, formally étale in the sense of [EGAIV4]. Note that we have to restrict all other schemes appearing in those statements to be noetherian. Of course, we can conclude analogous statements for formally smooth and formally unramified maps.
In the case of formally étale and formally unramified maps, the type theoretic and the definition of [EGAIV4] agree – in the sense that the lifting property for the nilradical implies the corresponding lifting property for any nilpotent ideal. For formally unramified maps, we just have to note, that uniqueness of lifts d in a given square

$$\begin{tikzcd} \mathrm{Spec}(A/N) \arrow[r] \arrow[d, "\iota_N"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, "d"'] & Y \end{tikzcd} \tag{1}$$

is implied by uniqueness in the square extended by composing with $\iota_{\mathrm{red}}\colon \mathrm{Spec}(A_{\mathrm{red}}) \to$

[37]In [EGAIV4], the definition is not stated as a lifting diagram, but the requirement that composition with ι is bijective.
[38]Meaning those defined type-theoretically with $\mathfrak{I}$.

Spec(A):

$$
\begin{tikzcd}
\mathrm{Spec}(A_{\mathrm{red}}) \arrow[r, "\iota_{\mathrm{red}}"] \arrow[d, "\iota_{\mathrm{red}}"'] & \mathrm{Spec}(A/N) \arrow[r] & X \arrow[dd, "f"] \\
\mathrm{Spec}(A/N) \arrow[d, "\iota_N"'] & & \\
\mathrm{Spec}(A) \arrow[rr] \arrow[uurr, "d"] & & Y
\end{tikzcd}
\tag{2}
$$

– which is the case by the type theoretic definition of f being formally unramified.
Now, let f be formally étale in the type theoretic sense. Then, for any lifting problem square like (4.1), we have a lift d like in the extended square (4.2). Composition with $\iota_N\colon \mathrm{Spec}(A) \to \mathrm{Spec}(A/N)$ yields a new map d' letting all triangles commute in the following diagram:

$$
\begin{tikzcd}
\mathrm{Spec}(A_{\mathrm{red}}) \arrow[r, "\iota_{\mathrm{red}}"] \arrow[d, "\iota_{\mathrm{red}}"'] & \mathrm{Spec}(A/N) \arrow[r] & X \arrow[dd, "f"] \\
\mathrm{Spec(A/N)} \arrow[d, "\iota_N"'] \arrow[urr, "d'"] & & \\
\mathrm{Spec}(A) \arrow[rr] \arrow[uurr, "d"] & & Y
\end{tikzcd}
$$

So d' is a lift for the square with left $\iota_{\mathrm{red}}\colon \mathrm{Spec}(A_{\mathrm{red}}) \to \mathrm{Spec}(A/N)$. But $\varphi \circ \mathrm{id}_{\mathrm{Spec}(A/N)}$ is also a lift for this square and since such a lift is unique, we have

$$d \circ \iota_N = d' = \varphi \circ \mathrm{id}_{\mathrm{Spec}(A/N)} = \varphi.$$

Hence d is a solution to the original lifting problem and f is étale.
In the case of formal smooth sets, a morphism between two smooth manifolds is formally étale, if and only if, it is a local diffeomorphism in the traditional sense. This is [KS17, Proposition 3.2].

Now, let us return to the type theory. There are some immediate consequences of the definition and facts about pullback squares.

Lemma 4.4.3

(a) If $f\colon A \to B$ and $g\colon B \to C$ are formally étale, their composition $g \circ f$ is formally étale. If the composition $g \circ f$ and g are formally étale, then f is formally étale.

(b) Equivalences are formally étale. (✓)

(c) Maps between coreduced types are formally étale.

(d) All fibers of a formally étale map are coreduced.

Proof (a) By pullback pasting.

(b) The naturality square for an equivalence is a commutative square with equivalences on opposite sides. Those squares are always pullback squares.

(c) This is, again, a square with equivalences on opposite sides.

(d) The pullback square witnessing $f\colon A \to B$ being formally étale yields an equivalence over ι_B. So, each fiber of f is equivalent to some fiber of $\Im f$. But fibers of maps between coreduced types are always coreduced by 3.4.6 (c), hence each fiber of f is equivalent to a coreduced type, thus itself coreduced. □

Remark 4.4.4
If $\Im$ is a left exact modality, as it is the case for the intended applications, we also get the converse of 4.4.3 (d): If all fibers of a map are coreduced, it is formally étale.

If $\Im$ is left exact[39], we also have the following example:

Example 4.4.5
Let A be a type and $a{:}A$. In 4.2.3 we defined the formal disk at a and may now look at its inclusion into A:

$$\varphi_a\colon \mathbb{D}_a \to A$$

given as the projection from the defining pullback square, or, more explicitly as

$$\varphi_a\left((x, \gamma\colon \iota_A(x) = \iota_A(a))\right) :\equiv x.$$

So the fiber of φ_a at $x{:}A$ is equivalent to $\iota_A(x) = \iota_A(a)$, therefore coreduced and φ_a is formally étale by 4.4.4.

In Differential Geometry, a local diffeomorphism induces isomorphisms on tangent spaces. So by analogy, formally étale maps should induce equivalences on formal disks. We will prove a bundle-version of this statement.

Lemma 4.4.6 (✓)
Let $\varphi\colon A \to B$ be formally étale. Pulling back the formal disk bundle on B along φ yields a pullback square:

$$\begin{tikzcd}
T_\infty A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty B \arrow[d] \\
A \arrow[r, "\varphi"'] & B
\end{tikzcd}$$

Proof We start with pasting the pullback square witnessing φ being formally étale to the defining square of $T_\infty A$:

$$\begin{tikzcd}
T_\infty A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & A \arrow[d, "\iota_A"] \\
A \arrow[r, "\iota_A"'] & \Im A
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\iota_A"'] \arrow[dr, phantom, "\text{(pb)}"] & B \arrow[d, "\iota_B"] \\
\Im A \arrow[r, "\Im\varphi"'] & \Im B
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
T_\infty A \arrow[r] \arrow[d] & A \arrow[r, "\varphi"] \arrow[d, phantom, "\text{(pb)}"] & B \arrow[d, "\iota_B"] \\
A \arrow[r, "\iota_A"'] & \Im A \arrow[r, "\Im\varphi"'] & \Im B
\end{tikzcd}$$

[39] Note that internal and external left exactness are not the same.

By pasting a naturality square to the bottom map, we get:

$$\begin{tikzcd}
T_\infty A \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & B \arrow[d, "\iota_B"] \\
A \arrow[r, "\varphi"'] & B \arrow[r, "\iota_B"'] & \mathfrak{I}B
\end{tikzcd}.$$

Now, we plug in the defining square of $T_\infty B$ on the right and get the desired result by pasting:

$$\begin{tikzcd}
T_\infty A \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & B \arrow[d, "\iota_B"] \\
A \arrow[r, "\varphi"'] & B \arrow[r, "\iota_B"'] & \mathfrak{I}B
\end{tikzcd}.$$

$\square$

Remark 4.4.7
This immediately gives us a dependent version. If $f\colon A \to B$ is formally étale, $d(f)$ is an equivalence over f:

$$d(f)\colon \prod_{a:A} \mathbb{D}_a \simeq \mathbb{D}_{f(a)}.$$

Like customary for a lot of properties in topology and geometry, we can carry our definitions from morphisms to objects:

Definition 4.4.8
Let A be a type, then A is called

(a) formally étale

(b) formally smooth

(c) formally unramified

if the map $A \to 1$ has this property.

Unwinding the definition, we get a specialized pullback square:

$$\begin{tikzcd}
A \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\iota_A"] \arrow[ddr, bend right] & & \\
& 1 \times_{\mathfrak{I}1} \mathfrak{I}A \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathfrak{I}A \arrow[d] \\
& 1 \arrow[r, "\simeq"] & \mathfrak{I}1
\end{tikzcd}$$

And since $\mathfrak{I}1 \simeq 1$ we have an equivalence at the bottom and therefore at the top as well, like indicated in the diagram. So φ is 1-epimorphic, 1-monomorphic or an equivalence if and only if ι_A is. This proves the following remark:

Remark 4.4.9
Let A be a type.

(i) A is formally étale, if and only if, it is coreduced.

(ii) A is formally smooth, if and only if, ι_A is a 1-epimorphism.

In some models, particularly those arising from Differential Geometry, ι_A is always a 1-epimorphism. But in general, there are counterexamples. The last statement is supported by the following example from Algebraic Geometry.

Example 4.4.10
If we work with Zariski-sheaves and unwind the definition of our formally smooth maps, we end up with the following lifting property:
A map $f\colon X \to Y$ is formally smooth, if and only if, for any ring A there is a diagonal lift in all commutative squares like the following:

$$\begin{tikzcd}
\mathrm{Spec}(A_{\mathrm{red}}) \arrow[r] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "\exists"'] & Y
\end{tikzcd}$$

We will now turn to an easy example of a non smooth variety over some field k. Let $R := k[X,Y]/(XY)$ and $V := \mathrm{Spec}(R)$, then V may be visualized as the union of the coordinate axis in the affine plane over k. The variety V is not smooth at the intersection of the two axis.
Now, the only tricky thing is to chose the right ring for A, since lifts do always exist for canonical candidates like $k[\epsilon]/(\epsilon^2)$. The solution[40] is to forget about the single tangent vector and use an infinitesimal thickening of V as a subscheme of $\mathrm{Spec}(k[X,Y])$. A first-order infinitesimal thickening is given as $\mathrm{Spec}(k[X,Y]/(XY)^2)$ and on the side of rings, it is easy to calculate, that the identity on R cannot be lifted along the reduced subscheme inclusion $\iota_{\mathrm{red}}\colon \mathrm{Spec}(R) \to \mathrm{Spec}(k[X,Y]/(XY)^2)$: Suppose we had a lift:

$$\begin{tikzcd}
k[X,Y]/(XY) & k[X,Y]/(XY) \arrow[l] \arrow[dl, "f"] \\
k[X,Y]/(XY)^2 \arrow[u, "\pi"] & k \arrow[l] \arrow[u]
\end{tikzcd}$$

[40]The author has to thank Tobias Columbus for suggesting this to him.

then we get

$$\begin{aligned}
0 =& f(XY + (XY)) \\
=& f(X + (XY)) \cdot f(Y + (XY)) \\
=& (X + P \cdot XY + (XY)^2) \cdot (Y + Q \cdot XY + (XY)^2) \\
=& XY(1 + P \cdot Y + Q \cdot X) + (XY)^2 \\
\neq& 0.
\end{aligned}$$

Hence, such an f cannot exist and we know there are non formally smooth Zariski-sheaves.

4.5 V-manifolds

A topological n-manifold is a topological space X admitting a cover with open sets U_i such that each U_i is homeomorphic to $\mathbb{R}^n$. The definition of a V-manifold below is somewhat analogous, where $\mathbb{R}^n$ is replaced by some left invertible H-space V.
But the comparison has some problem – we want to think of our V-manifolds as carrying *smooth* structure, not just topological structure. So smooth manifolds might seem the better analogy, but the usual way of defining those is a bit misleading in this context. Usually, the structure of a smooth manifold is given by the specific cover U_i and the smooth structure on the U_i. And in general there are different ways for a fixed topological space to be a smooth manifold, but being a topological manifold is just a property of the underlying topological space. While we want to define the first concept, the way of defining it is more similar to the definition of the latter, since our spaces already carry smooth structure – we accepted in 4.2.5, that we can define the formal disk bundle for any space and that it is supposed to generalize the tangent bundle.

Now let us work towards the definition of a V-manifold by transforming the definition of a smooth n-manifold, before we state the type theoretic version[41]. So let M have an atlas $(\varphi_i : \mathbb{R}^n \to M)_{i\in I}$ with smooth transition maps.
As maps of smooth manifolds, the charts φ_i are local diffeomorphisms. For $W := \coprod_{i\in I} \mathbb{R}^n$ the map $\varphi : W \to M$ given by the φ_i is also a local diffeomorphism.
In the definition below, we will require existence of an 1-epimorphism $w : W \to M$ and the discussion in 5.1 shows, that covers by charts provide an example. The condition that M locally looks like $\mathbb{R}^n$ is generalized to M admitting a formally étale span

$$\begin{tikzcd}
& W \arrow[dl, "\text{ét}"'] \arrow[dr, two heads, "\text{ét}"] & \\
V & & M
\end{tikzcd}$$

where V is a left invertible H-space taking the role of $\mathbb{R}^n$. If M is a smooth manifold, the formally étaleness condition just means M is locally diffeomorphic to $\mathbb{R}^n$. In general this condition just tells us, that M is *infinitesimally* modeled on the formal disk at the unit in V.
All propositions we will prove about V-manifolds just use the triviality of the formal disk bundle of V – we could replace left invertible H-spaces in the following definition with spaces with this property, but so far, there has been no need to do so.

Definition 4.5.1
Let V be a left invertible H-space. A type M is a *V-manifold*, if there is a type W and formally étale maps

[41] We repeat an argument in [KS17, Example 2.4].

$$\begin{tikzcd} & W \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}", two heads] & \\ V & & M \end{tikzcd}.$$

Let us look at some rather trivial yet important example:

Example 4.5.2
Let V be a left invertible H-space. Since all equivalences are formally étale by 4.4.3, the identity on V is formally étale and a 1-epimorphism by 4.3.2. So the span

$$\begin{tikzcd} & V \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}", two heads] & \\ V & & V \end{tikzcd}$$

witnesses that V is a V-manifold.

The following theorem establishes that the formal disk bundle on a V-manifold is a fiber bundle and generalizes the frame bundle construction from Differential Geometry to our abstract setting. In the smooth first order case, the frame bundle of a n-manifold is a $\mathrm{GL}(n, \mathbb{R})$-principle bundle with fibers given as the possible choices of basis of the tangent space at a point. If we allow arbitrary infinitesimals, this principle bundle is usually called the jet frame bundle.

Theorem 4.5.3
Let M be a V-manifold and $\mathbb{D}$ be the formal disk at the unit in V. Then the following holds:

(a) The formal disk bundle $T_\infty M$ is a $\mathbb{D}$-fiber bundle.

(b) The fiber bundle $T_\infty M$ is associated to an $\mathrm{Aut}(\mathbb{D})$-principal bundle.

Proof (b) is a direct consequence of (a) by 4.3.15. And (a) is a corollary of the following lemma stating precisely what we required of a fiber bundle in our first definition 4.3.5.□

Lemma 4.5.4
The cover of a V-manifold trivializes its formal disk bundle:
If M is a V-manifold witnessed by the maps

$$V \leftarrow W \twoheadrightarrow M$$

we have pullback squares:

$$\begin{tikzcd} \mathbb{D} \arrow[d] & T_\infty W \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "(\mathrm{pb})"] & T_\infty M \arrow[d] \arrow[dl, phantom, "(\mathrm{pb})"] \\ 1 & W \arrow[l] \arrow[r, two heads, "w"'] & M \end{tikzcd}.$$

Proof We saw in 4.4.6 that formally étale base-changes induce pullback squares between formal disk bundles. So we get two squares from the witnesses for M being a V-manifold:

$$\begin{tikzcd} T_\infty V \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty W \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty M \arrow[d] \\ V & W \arrow[l] \arrow[r, two heads] & M \end{tikzcd}.$$

In 4.2.10 we proved, that the formal disk bundle on V is trivial, so we can paste a pullback square witnessing this on the left:

$$\begin{tikzcd} \mathbb{D} \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty V \arrow[l] \arrow[d] \\ 1 & V \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} T_\infty V \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty W \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & T_\infty M \arrow[d] \\ V & W \arrow[l] \arrow[r, two heads] & M \end{tikzcd}$$

yielding the desired result. □

By applying our result 4.3.10 about general fiber bundles, we get a classifying map for the formal disk bundle on any V-manifold:

Remark 4.5.5 (✓)
For any V-manifold M, we have a map τ_M and a pullback square:

$$\begin{tikzcd} T_\infty M \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\text{(pb)}"] & \mathbb{D}_e//\mathrm{Aut}(\mathbb{D}_e) \arrow[d, "\pi"] \\ M \arrow[r, "\tau_M"'] & \mathrm{BAut}(\mathbb{D}_e) \end{tikzcd}.$$

Or, equivalently, we have a homotopy commutative triangle over the universe $\mathcal{U}$:

$$\begin{tikzcd} M \arrow[rr, "\tau_M"] \arrow[dr, "\tau_M{}'"'] & & \mathrm{BAut}(\mathbb{D}_e) \arrow[dl, "\iota"] \\ & \mathcal{U} & \end{tikzcd}$$

Note that both statements are independent of the equivalent definitions we gave in 4.2.3 for the formal disks and the formal disk bundle. Yet the diagrams are not strictly the same, if we replace $\mathbb{D}_e$ by another version obtained by a different definition. This is proven by straight forward calculations which we will not mention here, but are done in the agda-version.
We state one last consequence of 4.5.3, immediate by applying 4.3.8.

Remark 4.5.6
All formal disks of M are merely equivalent:

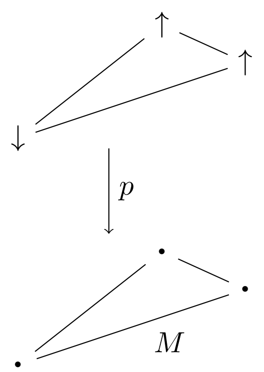
$$\prod_{x:M} \|\mathbb{D}_x \simeq \mathbb{D}_e\|_{-1}.$$

4.6 G-structures

We will now work towards the definition of what are called *torsion free G-structures* and present some concepts subsumed by this definition.
The structure group of an F-fiber bundle is just $\mathrm{Aut}(F)$, the loop space of $\mathrm{BAut}(F)$. A reduction of the structure group of a fiber bundle $p\colon E \to M$ is nothing else than a lift of its classifying morphism χ_p along the representation of the inclusion of a subgroup of the structure group, or along a more general morphism $\mathrm{B}\varphi\colon \mathrm{BG} \to \mathrm{BAut}(F)$:

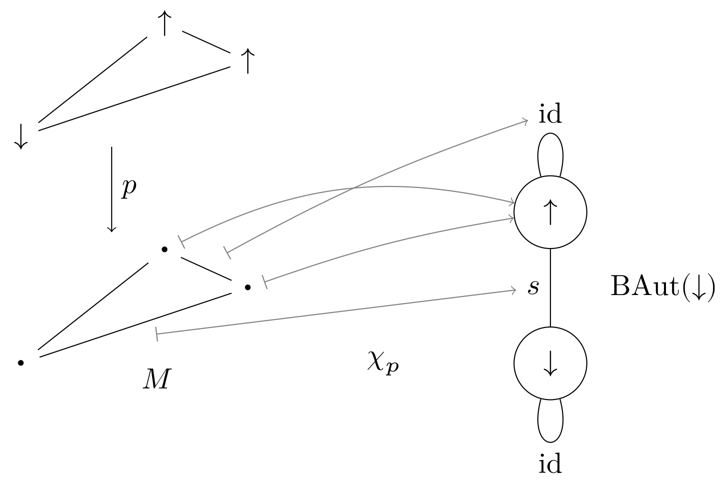
$$\begin{tikzcd} & \mathrm{BG} \arrow[d, "\mathrm{B}\varphi"] \\ M \arrow[ur, dashed, "\psi"] \arrow[r, "\chi_p"'] & \mathrm{BAut}(F) \end{tikzcd}$$

In the case of smooth manifolds and first order infinitesimals, for example, this will allow us to define pseudo-Riemannian and symplectic structures on an n-manifold, under the assumption that there are types representing the classifying spaces BO and BSp with morphisms corresponding to the subgroup inclusions into $\mathrm{BAut}(\mathbb{D}) \simeq \mathrm{BGL}(n)$.
We will look at a contrived example to give some intuition what such a lift means and why it might be interesting.
Let M be the boundary of a triangle and even if this does not quite fit the setting of smooth manifolds, let us imagine M to be given by three points and edges, or equalities, joining the points. Now, let $p{:}E \to M$ be a bundle over M with two equivalent types as fibers, which we denote by „$\downarrow$“ and „$\uparrow$“, mapped to M like indicated in the following picture:



We pretend our fibers have $\mathbb{Z}/2\mathbb{Z}$ as their automorphism group containing the identity and one automorphism s, switching the direction of the arrow. Let us look at the classifying morphism for this bundle, constructed to prove 4.3.7. To construct its codomain, $\mathrm{BAut}(\downarrow)$, we need to pick a fiber, say $\downarrow$ and construct its 1-image in the universe, $\mathrm{BAut}(\downarrow)$. $\mathrm{BAut}(\downarrow)$ may be thought of as consisting of two points, which we denote by the types $\downarrow$ and $\uparrow$ they represent. For any equivalence of the types represented in $\mathrm{BAut}(\downarrow)$ we have an equality in $\mathrm{BAut}(\downarrow)$. These equalities are indicated by arrows in the picture below.
Now, the classifying map χ_p maps each point $x \in M$ to the point in $\mathrm{BAut}(\downarrow)$ representing the type of the fiber $p^{-1}(x)$. The equalities $\gamma{:}x = y$, or the edges of M, are mapped to the equalities in $\mathrm{BAut}(\downarrow)$ representing the equivalence of the fiber we get by traveling

along γ lifted to the total space of p. Some mapping relations of χ_p are indicated in the picture below.



The picture also clarifies the result in 4.3.10, that the bundle p may be recovered from its classifying map – if we are given χ_p we may look at its images on points and put a corresponding fiber over each point. Now we can glue these fibers with the equivalences prescribed by χ_p. Note that, if M were a filled triangle, the 2-cell would be mapped to a relation between the equalities in BAut(↓), imposing the condition that the equality obtained by moving along the boundary has to be mapped to the trivial equivalence. The example above would still be possible, but bundles switching the fiber direction [42] on each egde would be impossible with a filled triangle.

Apart from providing enough data to reconstruct the bundle, the classifying map χ_p may also be used to discover properties of the bundle. For instance, we can ask if all the equivalences of fibers appearing in our bundle are contained in a subgroup of BAut(↓). In our example, this is the case up to homotopy – or in other words, χ_p is homotopic to a constant map. Yet the situation is still not entirely trivial. The homotopy from χ_p to a constant map contains the information exposing p to be a trivial fiber bundle. And for more complex bundles and spaces, this information can be interesting. Some known structures that can be expressed as G-structures on smooth manifolds are listed in the following example.

Example 4.6.1
We give a list of examples, what group morphisms – which are almost always inclusions of subgroups – encode structures on a smooth n-manifold as G-structures. Some of the examples assume $n = 2d$.

[42] Note that it is not possible to provide a picture of such a bundle in the same way a picture for p was drawn above.

$G \to GL(n)$	G-structure
$O(n) \to GL(n)$	Riemannian metric
$GL^+(n) \to GL(n)$	orientation
$O(n-1,1) \to GL(n)$	pseudo-Riemannian metric
$SO(n-1,1) \to GL(n)$	metric and orientation
$SO(n,2) \to GL(n)$	conformal structure
$GL(d,\mathbb{C}) \to GL(2d,\mathbb{R})$	almost complex structure
$U(d) \to GL(2d,\mathbb{R})$	almost Hermitian structure
$Sp(d) \to GL(2d,\mathbb{R})$	almost symplectic structure
$Spin(n) \to SO(n) \to GL(n)$	spin structure

For a classical definition of $O(n)$- and $GL(d,\mathbb{C})$-structures, see [Che66]. Note that in all of the above examples, G is a 0-group, yet our theory also supports higher groups. The string 2-group and the fivebrane 6-group are examples of higher G-structures of interest in physics. See [SSS09] for details and references.

At the end of the introduction above, we mentioned that the homotopy from $\chi_p \colon M \to \mathrm{BAut}(F)$ to a constant map proving that the bundle is trivial is itself a non trivial datum. We will construct such a trivializing homotopy for the formal disk bundle on a left invertible H-space V using the trivialization theorem 4.2.9. This will finally allow us to define what it means for G-structures on a V-manifold M to be trivial on each formal disk of M. In the smooth first order case, this means the G-structure is *torsion free*. In [Gui65] it was discovered, that for some G-structures, torsion freeness corresponds to an integrability condition. Furthermore, Guillemin calls a G-structure flat of order k, if it is trivial in the sense described above on k-th order infinitesimal disks and flat if it is trivial on open neighbourhoods.
This terminology comes from the $O(n)$-structures. A torsion free $O(n)$-structure on a smooth n-manifold admits an affine metric connection. The type theoretic notion is a natural analogue of a metric connection – it will state that the formal disk bundle looks infinitesimally like the disk bundle on V, hence like the formal disk bundle on an „affine space“.

Let V be a fixed left invertible H-space and M a V-manifold for the remainder of the chapter. We start by defining what a G-structure on M is. In the following, we will take loop spaces [43] of types typically denoted „BG“ and call them *groups*. All group objects[44] in an $(\infty,1)$-topos are equivalent to loop spaces by [Lur09a, Theorem 1.3.16], a generalization of Peter May's recognition theorem. So it is reasonable to assume, that with univalence, the situation is the same in the type theory, at least in the sense that we can assume all the groups in the model to be equivalent to loop spaces. It is not clear, if the construction can be done internally since there is no known way to define an A_∞-structure on a type. For 0-groups, deloopings were constructed internally in [LF14]. We will only talk about groups in a very special situation – if they come with a morphism to the structure group of the formal disk bundle. Such a morphism is given by a pointed type $(\mathrm{B}e, \mathrm{BG})$, a map $\mathrm{B}\varphi{:}\mathrm{BG} \to \mathrm{BAut}(\mathbb{D}_e)$ and an equality $\gamma \colon \mathrm{B}\varphi(\mathrm{Be}) = *$, where $*$ denotes the point in $\mathrm{BAut}(\mathbb{D}_e)$ given by $\mathbb{D}_e$ and refl. The equality ensures, that we have a map between the actual groups, i.e. the loop spaces of BG and $\mathrm{BAut}(\mathbb{D}_e)$ can be

[43] The loop space of a pointed type (A,a) is just the identity type $a =_A a$.
[44] More precisely, the A_∞-spaces such that π_0 gives a 0-group.

constructed from $\mathrm{B}\varphi$ but it is also necessary as a datum fixing the conjugation class of the map between the groups. So when we identify groups over $\mathrm{BAut}(\mathbb{D}_e)$ with the data described above, this should be seen as an instance of abuse of notation, since the group over $\mathrm{BAut}(\mathbb{D}_e)$ really is the data turned into a map between the loop spaces.

Definition 4.6.2 (✓)

(a) A *group over* $\mathrm{BAut}(\mathbb{D}_e)$ is given by a type BG together with a map $\mathrm{B}\varphi\colon \mathrm{BG} \to \mathrm{BAut}(\mathbb{D}_e)$ and an equality $\gamma : \mathrm{B}\varphi(\mathrm{Be}) = *$, where $*$ is the point in $\mathrm{BAut}(\mathbb{D}_e)$ given by $\mathbb{D}_e$ and refl.

(b) Let G be a group over $\mathrm{BAut}(\mathbb{D}_e)$. The type of G*-structures* on the V-manifold M is given as the sum

$$\sum_{\psi : M \to BG} \mathrm{B}\varphi \circ \psi \Rightarrow \tau_M,$$

i.e. it is the type of homotopy commutative triangles

$$\begin{tikzcd} & \mathrm{BG} \arrow[d, "\mathrm{B}\varphi"] \\ M \arrow[ur, dashed, "\psi"] \arrow[r, "\tau_M"'] & \mathrm{BAut}(\mathbb{D}_e) \end{tikzcd}.$$

We have the following two, seemingly trivial, examples requiring surprisingly non trivial constructions.

Remark 4.6.3 (✓)

(a) On the V-manifold V, we have a 1-structure for the trivial group 1 given by the delooping $\mathrm{B}1 \simeq 1$.

(b) Again, on the V-manifold V, we have a G-structure for any group G canonically induced by the 1-structure above. We call this G-structure the trivial G-structure on V.

Proof (a) We chose the constant map $(x \mapsto *)$ as a candidate for the lift and are hence left to find a homotopy letting the following diagram commute:

$$\begin{tikzcd} & \mathrm{B}1 \simeq 1 \arrow[d, "(x \mapsto *)"] \\ V \arrow[ur, dashed, "(x \mapsto *)"] \arrow[r, "\tau_V"'] & \mathrm{BAut}(\mathbb{D}_e) \end{tikzcd}.$$

The proposition follows, if we have a homotopy from τ_V to the inclusion of the canonical point $* : \mathrm{BAut}(\mathbb{D}_e)$. In 4.2.9 we proved that for any $x : V$, we have a map

$$\psi_x\colon \mathbb{D}_x \to \mathbb{D}_e.$$

By applying univalence this yields essentially the homotopy we need. We can circumvent the construction of equalities in $\mathrm{BAut}(\mathbb{D}_e)$, by using that we have a homotopy between $\xi :\equiv (x \mapsto \mathbb{D}_e)$ and $\zeta :\equiv (x \mapsto \mathbb{D}_x)$ in the universe where both maps commute with $\iota_{\mathrm{BAut}(\mathbb{D}_e)} \circ (x \mapsto *)$ and $\iota_{\mathrm{BAut}(\mathbb{D}_e)} \circ \tau_V$ respectively:

$$\begin{tikzcd}
V \arrow[rr, shift left, "\tau_V"] \arrow[rr, shift right, "(x \mapsto *)"'] \arrow[dr, shift right, "\xi"'] \arrow[dr, shift left, "\zeta"] & & \mathrm{BAut}(\mathbb{D}_e) \arrow[dl, hook, "\iota_{\mathrm{BAut}(\mathbb{D}_e)}"] \\
& \mathcal{U} &
\end{tikzcd}.$$

Since $\mathrm{BAut}(\mathbb{D}_e)$ was constructed by using the 1-image factorization 4.3.4, we know its inclusion into the universe is a 1-monomorphism. And all 1-monomorphisms f have the following „monomorphic“ property:

$$f \circ x \Rightarrow f \circ y \text{ implies } x \Rightarrow y \tag{$*$}$$

So we get the homotopy $\tau_V \Rightarrow (x \mapsto *)$ by applying this to the 1-monomorphism $\iota_{\mathrm{BAut}(\mathbb{D}_e)}$ and the homotopy $\xi \Rightarrow \zeta$.
Let us finish the proof by showing the general property $(*)$ of 1-monomorphisms $f{:}A \to B$:
If $x, y{:}C \to A$ are two maps and $H{:}f \circ x \Rightarrow f \circ y$ is a homotopy, then for any $c{:}C$, $x(c)$ and $y(c)$ are, by using the homotopy, both in the (homotopy) fiber over $f(x(c))$. Since f is a 1-monomorphism, the fiber is a proposition, so $x(c) = y(c)$ as points in the fiber and therefore, by applying the inclusion of the fiber to this equality, we have $x(c) = y(c)$ in A.

(b) We need a lift ψ and a homotopy letting the following commute:

$$\begin{tikzcd}
& \mathrm{BG} \arrow[d, "\mathrm{B}\varphi"] \\
V \arrow[ur, dashed, "\psi"] \arrow[r, "\tau_V"'] & \mathrm{BAut}(\mathbb{D}_e)
\end{tikzcd}.$$

The equality $\gamma\colon \mathrm{B}\varphi(\mathrm{B}e) = *$ is implicitly present since we have a group over $\mathrm{BAut}(\mathbb{D}_e)$ lets the following triangle commute:

$$\begin{tikzcd}
\mathrm{B1} \arrow[rr, "_ \mapsto \mathrm{B}e"] \arrow[dr, "_ \mapsto *"'] & & \mathrm{BG} \arrow[dl, "\mathrm{B}\varphi"] \\
& \mathrm{BAut}(\mathbb{D}_e &
\end{tikzcd}$$

Pasting the triangle from (a) to the left yields a triangle of the desired type:

$$\begin{tikzcd}
V \arrow[r, "_ \mapsto *"] \arrow[dr, "\tau_V"'] & \mathrm{B1} \arrow[r, "_ \mapsto \mathrm{B}e"] \arrow[d] & \mathrm{BG} \arrow[dl, "\mathrm{B}\varphi"] \\
& \mathrm{BAut}(\mathbb{D}_e) &
\end{tikzcd}$$

□

We will continue with a brief discussion, why the definition of the type of G-structures as given in 4.6.2 is already a good candidate for a moduli type of G-structures. There certainly is a witness in our type

$$G_M :\equiv \sum_{\psi : M \to BG} \mathrm{B}\varphi \circ \psi \Rightarrow \tau_M$$

of G-structures on M if we have something we want to call a G-structure - what is not clear, is that we do not have multiple witnesses for the G-structures we consider to be the same.

In the sum type, an equality of two triangles corresponds to a 3-cell between the two triangles. We briefly provide another view on the situation, from the perspective of $\mathrm{BAut}(\mathbb{D}_e)$ actions, indicating that G_M is the correct moduli type of G-structures on M. We can replace the sum equivalently by a dependent product type:

Lemma 4.6.4
We have an equivalence of types:

$$\left(\sum_{\psi : M \to BG} \mathrm{B}\varphi \circ \psi \Rightarrow \tau_M \right) \simeq \left(\prod_{x : \mathbf{BAut}(\mathbb{D}_e)} \tau_M^{-1}(x) \to \mathrm{B}\varphi^{-1}(x) \right)$$

Proof This is 3.2.8 for the following square:

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\tau_M"'] & \mathrm{BG} \arrow[d, "\mathrm{B}\varphi"] \\ \mathrm{BAut} \arrow[r, "\mathrm{id}"'] & \mathrm{BAut}(\mathbb{D}_e) \end{tikzcd}$$

where we drop the pullback property on one side and the restriction to equivalences over the identity on the other. □

Hence, this type of G-structures corresponds to the invariants under the $\mathrm{BAut}(\mathbb{D}_e)$ action on the morphisms over $\mathrm{BAut}(\mathbb{D}_e)$.

Let us now turn to our final goal of defining special G-structures, called *torsion free* in the smooth, first order case and *formally flat* in a more general context. If we require the G-structure in the examples 4.6.1 to be torsion free, we get more interesting examples of structures on smooth manifolds. For example, if an almost complex structure is torsion free, it is already a complex structure. We will again give a list of examples, which concepts might be expressed as torsion free G-structures in 4.6.8.

Let G be a fixed group over $\mathrm{BAut}(\mathbb{D}_e)$ until we have finished this definition. Let ψ be a lift

$$\begin{tikzcd} & \mathrm{BG} \arrow[d, "\mathrm{B}\varphi"] \\ M \arrow[ur, dashed, "\psi"] \arrow[r, "\tau_M"'] & \mathrm{BAut}(\mathbb{D}_e) \end{tikzcd}.$$

with homotopy $H\colon \varphi\circ\psi \Rightarrow \tau_M$, i.e. a witness of a G-structure on M. Then, for any $x\in M$, we can construct a new triangle

$$\begin{tikzcd} & \mathrm{BG} \arrow[d, "\mathrm{B}\varphi"] \\ \mathbb{D}_x \arrow[ur, dashed, "\psi\circ\iota_{\mathbb{D}_x}"] \arrow[r, "\tau_M\circ\iota_{\mathbb{D}_x}"'] & \mathrm{BAut}(\mathbb{D}_e) \end{tikzcd}. \tag{3}$$

by whiskering ψ to H, which is easily constructed:

$$H' :\equiv (d:\mathbb{D}_x)\mapsto H(\iota_{\mathbb{D}_x}(d)).$$

Definition 4.6.5
The *restriction* of a G-structure to the formal disk at $x:M$, is the triangle (4.3).

Basically, a G-structure is formally flat, if all its restrictions to the formal disks of M are equivalent to the trivial G-structure – which is still to define. In 4.6.3 we defined the trivial G-structure on V and now, we call its restriction to $\mathbb{D}_e$ the trivial structure on $\mathbb{D}_e$. The homotopy in the corresponding triangle tells us that the map $\mathbb{D}_e\to\mathrm{BAut}(\mathbb{D}_e)$ is homotopic to the inclusion of a point. Yet the homotopy is in general an important datum – depending on the situation it contains some information on why the G-structure on V was trivial.
We will now transport this notion of triviality along equivalences of formal disks to the other restrictions. To be able to achieve this, we will first define a family over $\mathrm{BAut}(\mathbb{D}_e)$ of all the triangles in question and possibly containing more triangles than we need. Remember for the following definition that the points of $\mathrm{BAut}(\mathbb{D}_e)$ can be written in the form $(\mathbb{D},|\gamma|)$, where $\mathbb{D}$ is some type equivalent to $\mathbb{D}_e$ and $\gamma:\mathbb{D}=\mathbb{D}_e$ an equality in $\mathcal{U}$ witnessing the equivalence.

Definition 4.6.6 (✓)
Let $\Delta\colon \mathrm{BAut}(\mathbb{D}_e)\to\mathcal{U}$ be the dependent type given by

$$\Delta((\mathbb{D},|\gamma|)) :\equiv \sum_{\psi\,:\,\mathbb{D}\to\mathrm{BAut}(\mathbb{D}_e)}\ \sum_{\psi'\,:\,\mathbb{D}\to\mathrm{BG}} \mathrm{B}\varphi\circ\psi' \Rightarrow \psi$$

This enables us to transport triangles along equivalences of $\mathbb{D}_e$ to some type $\mathbb{D}$, since they are equalities in the universe, which we may lift along $\iota_{\mathrm{BAut}(\mathbb{D}_e)}$. This is enough to compare all restrictions to formal disks of M:

Definition 4.6.7
Let $T(x)$ denote the triangle obtained by restricting a G-structure on M to $\mathbb{D}_x$ and T_e the trivial triangle defined above. Then the G-structure is called *torsion free* or *formally flat*, if for all $x:M$ and all equalities $\gamma:(\mathbb{D}_x,|\eta|)=(\mathbb{D}_e,|\mathrm{refl}|)$ in $\mathrm{BAut}(\mathbb{D}_e)$

$$\|(\text{transport of }\Delta\text{ along }\gamma)(T(x)) = T_e\|_{-1}$$

Example 4.6.8
We give some known examples of torsion free G-structures in the smooth, first order case:

$\mathrm{G} \to \mathrm{GL}(n)$	torsion free G-structure
$\mathrm{GL}(d,\mathbb{C}) \to \mathrm{GL}(2d,\mathbb{R})$	complex structure
$\mathrm{U}(d) \to \mathrm{GL}(2d,\mathbb{R})$	Hermitian structure
$\mathrm{Sp}(d) \to \mathrm{GL}(2d,\mathbb{R})$	symplectic structure

5 Appendix

5.1 Epimorphisms as covers

We will show, that good open covers of a smooth manifold yield an epimorphism in the category of formal smooth sets. In the latter topos, epimorphisms are detectable on stalks, so we will start by describing stalks.

Let X be a formal smooth set and $n \in \mathbb{N}$ a fixed natural number. For $\delta > 0$, the inclusion $B^n_{\delta,0} \subseteq \mathbb{R}^n$ of the open n ball of radius δ at the origin induces a quotient map $X(\mathbb{R}^n) \to X(B^n_{\delta,0})$ identifying all smooth maps $\mathbb{R}^n \to X$ differing only outside the n-ball. We make all of these identifications at once by passing to the colimit

$$n^*X := \mathrm{colim}_{\delta>0} X(B^n_{\delta,0})$$

along all maps induced by the inclusions $B^n_{\delta,0} \subseteq B^n_{\delta\prime,0}$. The result is the set of germs at 0 of smooth function $\mathbb{R}^n \to X$. And mapping all formal smooth sets X to n^*X yields a geometric morphism from the topos of sets to the formal smooth sets. In [KS17, Proposition 2.4] it is shown, that with these geometric morphisms the topos of formal smooth sets has enough points.

So a map $w\colon W \to M$ is an epimorphism of formal smooth sets, if and only if, it induces[45] an epimorphism

$$w_k\colon k^*W \to k^*M$$

for all $k \in \mathbb{N}$. Now, suppose M is a smooth n-manifold and $(\varphi_i\colon \mathbb{R}^n \to M)_{i\in I}$ a smooth atlas. Let $W := \coprod_{i\in I} \mathbb{R}^n$ then the map $w\colon W \to M$ given by the φ_i turns out to be an epimorphism.

In order to see this, we need to find a preimage for any germ in any k^*M. Any $\psi\colon k^*M$ is represented by some $\tilde{\psi}\colon B^k_{\delta,0} \to M$. The point $\tilde{\psi}(0)$ has to have some neighbourhood contained in the image of one of the open maps φ_i. So we may assume, $\tilde{\psi}$ to factor through some chart φ_{i_0} and a new map ζ:

$$\begin{tikzcd} B^k_{\delta,0} \arrow[rr, "\tilde{\psi}"] \arrow[dr, "\zeta"'] & & M \\ & \mathbb{R}^n \arrow[ur, "\varphi_{i_0}"'] & \end{tikzcd}$$

The new map ζ is smooth since φ_{i_0} is a chart of a smooth manifold and therefore a local diffeomorphism. So ζ represents a germ in k^*W mapped to $\tilde{\psi}$ by w and we know w is an epimorphism.

If we are given any epimorphism $w\colon W \to X$ of general formal smooth sets W and X, we can pass to the stalks and lift germs of smooth maps $B^k_{\delta,0} \to X$ to W like above.

In toposes of ∞-stacks, an atlas of a manifold provides an example of a 1-epimorphism, since the epimorphisms of 0-truncated objects are precisely the 1-epimorphisms.

[45] By taking the colimit of the relevant components $w_{B^k_{\delta,0}}$.

5.2 Reflective subcategories and factorization systems

In this section, we merely list a few definitions and statements, which might be helpful in understanding Let C be a category with terminal object 1.

Definition 5.2.1
A subcategory $U \subseteq C$ is *reflective*, if it is full and the inclusion admits a left adjoint $R\colon C \to U$, the *reflector*.

Examples are everywhere:

Example 5.2.2

(a) The functor mapping all objects to the terminal object is a reflection.

(b) Sheafification is a reflector to the subcategory of sheaves inside the category of set-valued presheaves on a topological space or site. Moreover, this is a left exact reflection – it preserves all finite limits.

(c) Mapping a scheme or a Zariski-sheaf to the spectrum of the ring of its global sections is a reflection and the affine schemes form a reflective subcategory.

(d) The coreductions in all the models mentioned in 2.3.

Let $U \subseteq C$ be a reflective subcategory with reflector R from now on. Furthermore, let η denote the unit of the adjunction between R and the inclusion of U into C. This means η appears in the following universal property:
For any object $X \in C$ and any morphism $\varphi\colon X \to Y$ with Y in U, the dashed morphism exists uniquely:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr] & R(X) \arrow[d, dashed] \\
 & Y
\end{tikzcd}$$

Up to isomorphism, membership of U may be characterized by the unit of the adjunction:

Remark 5.2.3
Any object $X \in C$ is isomorphic to some object in U, if and only if, η_X is an isomorphism.

Let now C be a category with all pullbacks R be a left exact reflector of a reflective subcategory U of C with unit η.

Remark 5.2.4
There are induced reflective subcategories on all slices of C. The units and the reflectors are given by the induced maps from naturality squares to pullbacks: If $f\colon A \to X$ is an object in the slice C/X we have the following diagram:

$$
\begin{tikzcd}
A \arrow[dr, dashed, "\eta_f^X"] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\
& B \times_{R(B)} R(A) \arrow[r] \arrow[d] & R(A) \arrow[d, "R(f)"] \\
& B \arrow[r, "\eta_B"'] & R(B)
\end{tikzcd}
$$

This is a hint, why only left exact reflectors are allowed on our models – to have an operation inside the type theory, we have to be able to apply it in any context, that means in any slice. Furthermore, we have to be able to do it in a way consistent with substitution, that means applying the reflector in some slice should commute with pullbacks. This is the case for left exact reflectors.

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