The action of the mapping class group on the pants complex

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Abstract: For any compact surface $S = \Sigma_{g,b}$ of genus g with b boundary components, there is a cell complex attached to S, namely the *pants complex* $\mathscr{P}(S)$. The mapping class group $\mathrm{Mod}(S)$ of S acts cocompactly on this complex. In this article we calculate the orbits and stabilizers of this action.

1 Introduction

Let $S = \Sigma_{g,b}$ be a compact orientable surface of genus g with b boundary components. The *mapping class group* $\mathrm{Mod}(\Sigma_{g,b})$ is the group of all isotopy classes of orientation-preserving homeomorphisms $\Sigma_{g,b} \to \Sigma_{g,b}$. In the literature, other versions/definitions of the mapping class group can be found. For example, the *extended mapping class group* $\mathrm{Mod}^*(\Sigma_{g,b})$ is the group of isotopy classes of all (orientation-preserving and orientation-reversing) homeomorphisms $\Sigma_{g,b} \to \Sigma_{g,b}$. On the other hand, Putman [P] defines the mapping class group as the group of all isotopy classes of orientation-preserving homeomorphisms $\Sigma_{g,b} \to \Sigma_{g,b}$ which fix the boundary $\partial\Sigma_{g,b}$ up to isotopy fixing the boundary.

A *pants decomposition* of $\Sigma_{g,b}$ is a collection of isotopy classes of $3g-3+b$ pairwise disjoint simple closed curves. Any such collection is maximal and cuts the surface into *pairs of pants*, i.e. subsurfaces homeomorphic to $\Sigma_{0,3}$.

We assume that S is a surface with *negative Euler characteristic* $\chi(S) = 2-2g-b$, i.e. S is none of the surfaces $\Sigma_{0,0}$, $\Sigma_{0,1}$, $\Sigma_{0,2}$, $\Sigma_{1,0}$. In this case pants decompositions of S do exist, and each pants decomposition consists of $3g-3+b$ curves and divides S into $2g-2+b$ pairs of pants.

Two pants decompositions $\mathcal{P}, \mathcal{Q}$ differ by a *move* if there are curves $\alpha \in \mathcal{P}$, $\beta \in \mathcal{Q}$, $\alpha \neq \beta$, such that $\mathcal{P} \setminus \{\alpha\} = \mathcal{Q} \setminus \{\beta\}$ and α, β intersect minimally. If $i(\alpha,\beta) = 1$ then the move $\mathcal{P} — \mathcal{Q}$ is called a *simple move* or *S-move*. This is the case when both sides of α (resp. β) belong to the same pair of pants. If the sides of α (resp. β) belong to different pairs of pants, then $i(\alpha,\beta) = 2$ and $\mathcal{P} — \mathcal{Q}$ is called an *associativity move* or *A-move*. Compare Figure 1.

The *pants graph* $\mathscr{P}^1(S)$ is by definition a graph having all possible pants decompositions as vertices and all possible moves as edges. It was first defined in 1980 by Hatcher and Thurston [HT]. In 2000 Hatcher, Lochak and Schneps defined the pants complex $\mathscr{P}(S)$ by inserting certain 2-cells into $\mathscr{P}^1(S)$ and showed that the resulting 2-complex is connected and simply connected [HLS].

There are 2-cells of type (3S), (3A), (4C), (5A), (6AS). The definition of these cells can be found in [HLS] or in the thesis [W]. For further use, we recall the definition of a 2-cell of type (6AS). Namely, a 2-cell is inserted in any hexagon of the form

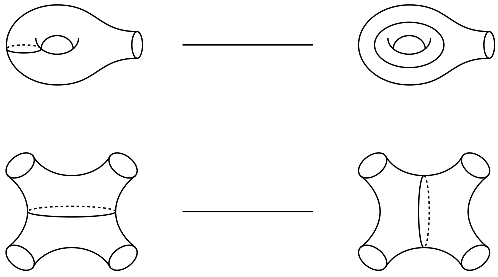


Figure 1: S-move and A-move

$$\{\gamma_1, \gamma_2\} — \{\gamma_1, \gamma_3\} — \{\gamma_3, \gamma_4\} — \{\gamma_4, \gamma_5\} — \{\gamma_4, \gamma_6\} — \{\gamma_2, \gamma_6\}$$

where all curves $\gamma_1, \ldots, \gamma_6$ lie in a common subsurface of type $\Sigma_{1,2}$, $\gamma_1 — \gamma_4$ and $\gamma_2 — \gamma_4$ being S-moves and all other moves being A-moves, compare Figure 2.

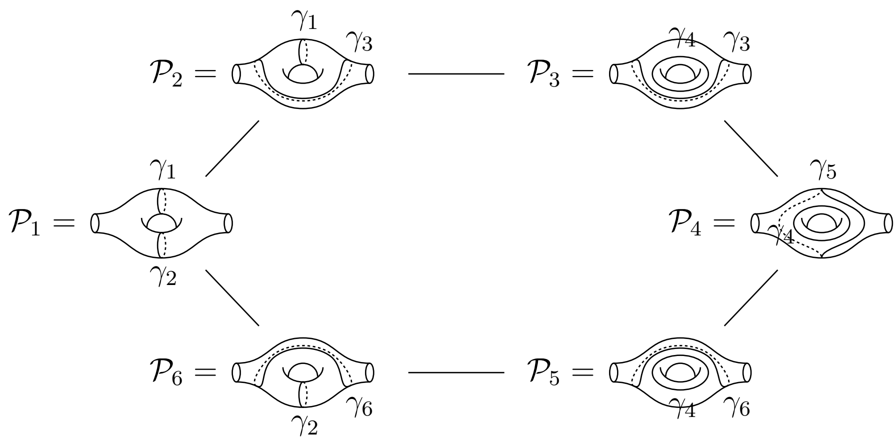


Figure 2: A hexagonal 2-cell

The mapping class group $\mathrm{Mod}(S)$ acts on the set of all isotopy classes of simple closed curves via $[f] \cdot [\alpha]_{\simeq} := [f(\alpha)]_{\simeq}$. This defines an action of $\mathrm{Mod}(S)$ on $\mathscr{P}(S)$ because any homeomorphism preserves the intersection number of two curves. In this paper we will investigate this action by calculating its orbits and stabilizers.

This paper is an excerpt of the autor's thesis "Die Aktion der Abbildungsklassengruppe auf dem Hosenkomplex" [W]. In that thesis, the stabilizers and orbits of the action are used to give a new presentation for $\mathrm{Mod}(S)$. The details of this presentation will be given in a further paper.

The author would like to thank Frank Herrlich and Gabriela Schmithüsen for their advice for both the thesis and this paper.

2 Orbits

An important tool for the calculation of the orbits under the action of $\mathrm{Mod}(S)$ on $\mathscr{P}(S)$ is the graph $\Gamma(\mathcal{P})$ which is associated to a pants decomposition $\mathcal{P}$.

Definition 2.1:
Let $\mathcal{P}$ be a pants decomposition of S. The *vertices* of the graph $\Gamma(\mathcal{P})$ are by definition all pairs of pants of $\mathcal{P}$ together with all boundary components of S. Two pairs of pants are connected by an edge if they are bounded by the same curve. A boundary component of S is connected by an edge to that pair of pants which it bounds. The resulting graph $\Gamma(\mathcal{P})$ is connected with trivalent vertices corresponding to the pairs of pants and univalent vertices corresponding to the boundary components of S. See Figure 3 for an example.

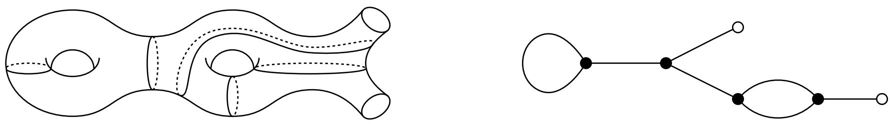


Figure 3: Example of $\mathcal{P}$ and $\Gamma(\mathcal{P})$

Theorem 2.2:
Two pants decompositions $\mathcal{P}, \mathcal{Q}$ of S are in the same orbit under the action of the mapping class group (i.e. there is an $f \in \mathrm{Mod}(S)$ such that $f(\mathcal{P}) = \mathcal{Q}$) if and only if $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q})$.

Proof:
One direction is given by the definition of the graphs $\Gamma(\mathcal{P}), \Gamma(\mathcal{Q})$: Any mapping class $f \in \mathrm{Mod}(S)$ satisfying $f(\mathcal{P}) = \mathcal{Q}$ induces an isomorphism $\varphi_f : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{Q})$ by mapping a vertex of $\mathcal{P}$ (= pair of pants or boundary component) to its image under f. On the other hand, let $\varphi : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{Q})$ be an isomorphism. φ induces a homeomorphism

$$\tilde{F} : \bigcup_{\alpha \in \mathcal{P}} \alpha \cup \partial S \to \bigcup_{\beta \in \mathcal{Q}} \beta \cup \partial S$$

in a straightforward way. We only have to be careful choosing the correct orientation of mapping a curve onto its image. $\tilde{F}$ can be extended to a homeomorphism $F : S \to S$ in the same straightforward way. The mapping class $f = [F]$ obviously fulfills $f(\mathcal{P}) = \mathcal{Q}$. □

The orbit of an edge $\mathcal{K} = (\mathcal{P} — \mathcal{Q})$ of the pants complex $\mathscr{P}(S)$ is also given by combinatorial properties of the graphs $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{Q})$. But here the situation is slightly more complicated.

Definition 2.3:

a) Let $\mathcal{K} = (\mathcal{P} — \mathcal{Q})$ be an edge in $\mathscr{P}(S)$. As in Definition 2.1, the system of curves $\mathcal{P} \cap \mathcal{Q}$ defines a graph $\Gamma(\mathcal{P} \cap \mathcal{Q})$ whose vertices are the components of the complement of the curves and the components of the boundary of S.

b) Now let $\mathcal{K} = (\mathcal{P} — \mathcal{Q})$ and $\mathcal{K}' = (\mathcal{P}' — \mathcal{Q}')$ be two edges in $\mathscr{P}(S)$ where in $\mathcal{K}$ (resp. $\mathcal{K}'$) the curve $\alpha \in \mathcal{P}$ (resp. $\alpha' \in \mathcal{P}'$) is moved to the curve $\beta \in \mathcal{Q}$ (resp. $\beta' \in \mathcal{Q}'$). If $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{P}')$ are isomorphic then any isomorphism $\varphi : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{P}')$ with $\varphi(\alpha) = \alpha'$ induces an isomorphism $\hat{\varphi} : \Gamma(\mathcal{P} \cap \mathcal{Q}) \to \Gamma(\mathcal{P}' \cap \mathcal{Q}')$. The same is valid for $\mathcal{Q}$ and $\mathcal{Q}'$.

Theorem 2.4:
Two oriented edges $\mathcal{K} = (\mathcal{P} — \mathcal{Q})$, $\mathcal{K}' = (\mathcal{P}' — \mathcal{Q}')$ as in Definition 2.3b) are in the same orbit under the action of the mapping class group (i.e. there is an $f \in \mathrm{Mod}(S)$ such that $f(\mathcal{P}) = \mathcal{P}'$ and $f(\mathcal{Q}) = \mathcal{Q}'$) if and only if there are isomorphisms $\varphi : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{P}')$ and $\psi : \Gamma(\mathcal{Q}) \to \Gamma(\mathcal{Q}')$ satisfying $\varphi(\alpha) = \alpha'$, $\psi(\beta) = \beta'$ and $\hat{\varphi} = \hat{\psi}$.

Proof:
If $f \in \mathrm{Mod}(S)$ maps $\mathcal{P}$ and $\mathcal{Q}$ to $\mathcal{P}'$ and $\mathcal{Q}'$ respectively then we have $f(\alpha) = \alpha'$, $f(\beta) = \beta'$ and $f(\mathcal{P} \setminus \{\alpha\}) = \mathcal{P}' \setminus \{\alpha'\}$, so f induces an isomorphism $\varphi_f : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{P}')$ such that $\varphi_f(\alpha) = \alpha'$ and an isomorphism $\psi_f : \Gamma(\mathcal{Q}) \to \Gamma(\mathcal{Q}')$ such that $\psi_f(\beta) = \beta'$. The induced isomorphisms $\hat{\varphi}_f, \hat{\psi}_f$ fulfill $\hat{\varphi}_f = \hat{\psi}_f$.

For the other direction, let $\varphi : \Gamma(\mathcal{P}) \to \Gamma(\mathcal{P}')$ and $\psi : \Gamma(\mathcal{Q}) \to \Gamma(\mathcal{Q}')$ be as required. Of course $\mathcal{K}$ and $\mathcal{K}'$ are moves of the same type because otherwise the graphs $\Gamma(\mathcal{P} \cap \mathcal{Q})$ and $\Gamma(\mathcal{P}' \cap \mathcal{Q}')$ would not be isomorphic.

Case 1: $\mathcal{K}$ and $\mathcal{K}'$ are S-moves.
There is a realization $g \in \mathrm{Mod}(S)$ of φ such that $g(\mathcal{P}) = \mathcal{P}'$ and $g(\alpha) = \alpha'$. The curve $g(\beta)$ is supported in the same subsurface $\Sigma'_{1,1}$ in which α' is supported. Furthermore $\alpha' — g(\beta)$ is also an S-move. In the surface $\Sigma'_{1,1}$ all edges lie in the same orbit. This is because $\mathrm{Mod}(\Sigma'_{1,1}) \cong \mathrm{SL}_2(\mathbb{Z})$ and edges are of the form $[p, q] — [r, s]$ with $ps - qr = \pm 1$ and the action is given by matrix multiplication, see [W, Chapter 2] for details. We find a mapping class $h \in \mathrm{Mod}(S)$ such that $h|_{S \setminus \Sigma'_{1,1}} = \mathrm{id}$ and $h(\alpha') = \alpha'$ and $h(g(\beta)) = \beta'$. Now we set $f := h \circ g$ and derive $f(\alpha) = \alpha'$, $f(\beta) = \beta'$, $f(\mathcal{P} \setminus \{\alpha\}) = \mathcal{P}' \setminus \{\alpha'\}$, i.e. $f(\mathcal{P}) = \mathcal{P}'$ and $f(\mathcal{Q}) = \mathcal{Q}'$.

Case 2: $\mathcal{K}$ and $\mathcal{K}'$ are A-moves.
This case is similar to the previous one; we have to use the properties of the mapping class group and the pants complex of a 4-punctured sphere. The details of this case can be found in [W, proof of 4.9]. □

Corollary 2.5:
The quotient of the pants graph $\mathscr{P}^1(S)$ modulo $\mathrm{Mod}(S)$ is finite.

Proof:
The orbits of all pants decompositions of S under the action of $\mathrm{Mod}(S)$ correspond bijectively to the isomorphism classes of all finite graphs having $2g-2+b$ trivalent vertices, b univalent vertices and $3g-3+2b$ edges, and this set is finite. (Unfortunately the number of elements of this set as a function of g and b is not known.)

Now we suppose that there are infinitely many edges in $\mathscr{P}^1(S)/\,\mathrm{Mod}(S)$. Because there are only finitely many vertices, there is a vertex with infinitely many incident edges. So in $\mathscr{P}^1(S)$ there are infinitely many edges $\mathcal{K}_i = (\mathcal{P} - \mathcal{Q}_i)$ $(i \in \mathbb{N})$ which are pairwise not in the same orbit and which are all incident to some pants decomposition $\mathcal{P}$. Any edge is given by a move along one of the curves of $\mathcal{P}$. Since $\mathcal{P}$ has only finitely many curves, infinitely many of these edges are given by a move along the same curve $\alpha \in \mathcal{P}$. But the moves along a given curve lie in at most two orbits which gives a contradiction.
□

It is also possible to calculate the orbits of the 2-cells of $\mathscr{P}(S)$ and to show that there are only finitely many of these. This has been done in detail in [W] and will be largely omitted here. To illustrate the methods we will calculate the orbit of a 2-cell of type (6AS). For such a 2-cell $\mathcal{F} = (\mathcal{P}_1 - \ldots - \mathcal{P}_6)$, the graph $\Gamma(\mathcal{F})$ is defined similarly to Definitions 2.1 and 2.3a), with $\mathcal{P}$ in 2.1 replaced by the set $\mathcal{P}_1 \cap \ldots \cap \mathcal{P}_6$.

Lemma 2.6:
Two unoriented 2-cells $\mathcal{F}, \mathcal{F}'$ of type (6AS) are in the same orbit under the action of the mapping class group (meaning here the existence of an $f \in \mathrm{Mod}(S)$ with $f(\{\mathcal{P}_1, \ldots, \mathcal{P}_6\}) = \{\mathcal{P}'_1, \ldots, \mathcal{P}'_6\}$) if and only if $\Gamma(\mathcal{F}) \cong \Gamma(\mathcal{F}')$.

Proof:
The "only if" direction is obvious since every $f \in \mathrm{Mod}(S)$ with $f(\mathcal{F}) = \mathcal{F}'$ induces an isomorphism $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}')$. For the "if" direction let

$$\begin{aligned} \mathcal{F} &= \{\gamma_1,\gamma_2\} - \{\gamma_1,\gamma_3\} - \{\gamma_3,\gamma_4\} - \{\gamma_4,\gamma_5\} - \{\gamma_4,\gamma_6\} - \{\gamma_2,\gamma_6\} \\ &= \mathcal{P}_1 - \ldots - \mathcal{P}_6, \\ \mathcal{F}' &= \{\gamma'_1,\gamma'_2\} - \{\gamma'_1,\gamma'_3\} - \{\gamma'_3,\gamma'_4\} - \{\gamma'_4,\gamma'_5\} - \{\gamma'_4,\gamma'_6\} - \{\gamma'_2,\gamma'_6\} \\ &= \mathcal{P}'_1 - \ldots - \mathcal{P}'_6 \end{aligned}$$

and $\varphi : \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}')$ be an isomorphism. There is a realization $g \in \mathrm{Mod}(S)$ of φ with $g(\mathcal{P}_1) = \mathcal{P}'_1$, $g(\gamma_1) = \gamma'_1$ and $g(\gamma_2) = \gamma'_2$. Let $\Sigma'_{1,2}$ be the subsurface of S in which γ'_1 and γ'_2 are supported and $\Sigma'_{0,4} \subseteq \Sigma'_{1,2}$ be the subsurface of S in which γ'_2 is supported. The curves γ'_3 and $g(\gamma_3)$ both lie in $\Sigma'_{0,4}$ and are separating. Analyzing the surface $\Sigma'_{0,4}$ we find a mapping class $h_2 \in \mathrm{Mod}(S)$ which is a power of a Dehn twist about γ'_2 mapping $g(\gamma_3)$ to γ'_3. (For the definition of a Dehn twist, compare Section 3.) As such a power of a Dehn twist, h_2 fulfills $h_2(\gamma'_1) = \gamma'_1$, $h_2(\gamma'_2) = \gamma'_2$, $h_2(\gamma'_6) = \gamma'_6$, $h_2(g(\gamma_6)) = g(\gamma_6)$ and $h_2|_{S\setminus\Sigma'_{1,2}} = \mathrm{id}$. With the same arguments we get a mapping class $h_1 \in \mathrm{Mod}(S)$ with $h_1(g(\gamma_6)) = \gamma'_6$ and $h_1(\gamma'_1) = \gamma'_1$, $h_1(\gamma'_2) = \gamma'_2$, $h_1(\gamma'_3) = \gamma'_3$, $h_1|_{S\setminus\Sigma'_{1,2}} = \mathrm{id}$. Therefore the composition $h_1 \circ h_2 \circ g$ fulfills $h_1 \circ h_2 \circ g(\gamma'_i) = \gamma'_i$ for $i = 1, 2, 3, 6$.

There is exactly one curve in $\Sigma'_{1,2}$ which is nontrivial and not isotopic to a boundary curve and which is disjoint from γ'_3 and γ'_6. Since γ'_4 and $h_1 \circ h_2 \circ g(\gamma_4)$ both have these properties, we get $h_1 \circ h_2 \circ g(\gamma_4) = \gamma'_4$.

There are exactly two curves in $\Sigma'_{1,2}$ which are nontrivial and not isotopic to a boundary curve and which are disjoint from γ'_4 and have minimal intersection number with both γ'_3 and γ'_6. One of these curves is γ'_5, call the other one γ''_5. Since $h_1 \circ h_2 \circ g(\gamma_5)$ has the mentioned properties it coincides with γ'_5 or γ''_5.

Case 1: $h_1 \circ h_2 \circ g(\gamma_5) = \gamma'_5$
Then $h_1 \circ h_2 \circ g(\gamma_i) = \gamma'_i$ for all $i = 1, \dots, 6$ and therefore $h_1 \circ h_2 \circ g(\mathcal{F}) = \mathcal{F}'$.

Case 2: $h_1 \circ h_2 \circ g(\gamma_5) = \gamma''_5$
There is a mapping class $\jmath \in \operatorname{Mod}(S)$ satisfying $\jmath|_{S \setminus \Sigma'_{1,2}} = \mathrm{id}$, interchanging γ'_1 and γ'_2, γ'_3 and γ'_6, γ'_5 and γ''_5, and with $\jmath(\gamma'_4) = \gamma'_4$, compare Figure 4. Then the mapping class $f := \jmath \circ h_1 \circ h_2 \circ g$ fulfills $f(\gamma_1) = \gamma'_2$, $f(\gamma_2) = \gamma'_1$, $f(\gamma_3) = \gamma'_6$, $f(\gamma_4) = \gamma'_4$, $f(\gamma_5) = \jmath(\gamma''_5) = \gamma'_5$, $f(\gamma_6) = \gamma'_3$ and hence $f(\mathcal{F}) = \mathcal{F}'$. □

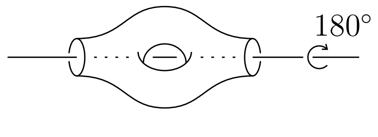


Figure 4: The mapping class $\jmath$

Example 2.7:
We give a detailed description of the quotient $\mathscr{P}(S)/\operatorname{Mod}(S)$ for the surface $S = \Sigma_{1,3}$. There are three orbits of pants decompositions corresponding to the three graphs depicted in Figure 5.

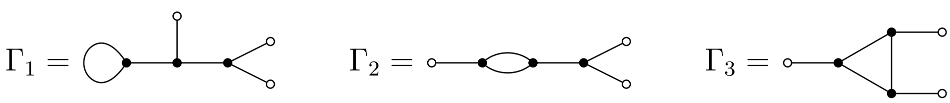


Figure 5: Graphs of signature (1,3)

An easy calculation shows that two oriented edges $\mathcal{K}_1 = (\mathcal{P}_1 — \mathcal{Q}_1)$, $\mathcal{K}_2 = (\mathcal{P}_2 — \mathcal{Q}_2)$ lie in the same orbit if and only if $\Gamma(\mathcal{P}_1) \cong \Gamma(\mathcal{P}_2)$ and $\Gamma(\mathcal{Q}_1) \cong \Gamma(\mathcal{Q}_2)$. In particular, any edge $\mathcal{K} = (\mathcal{P} — \mathcal{Q})$ corresponding to an S-move is inverted because $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q}) \cong \Gamma_1$, and any edge $\mathcal{K} = (\mathcal{P} — \mathcal{Q})$ corresponding to an A-move with $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{Q})$ is inverted. These edges are therefore barycentrically subdivided so that the quotient can be defined. All other edges connect pants decompositions of different types and are therefore not inverted by the action. Altogether the quotient graph $\mathscr{P}^1(\Sigma_{1,3})/\operatorname{Mod}(\Sigma_{1,3})$ has the form shown in Figure 6.

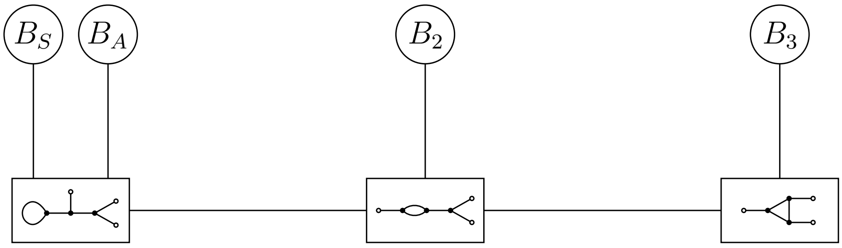


Figure 6: The quotient graph $\mathscr{P}^1(\Sigma_{1,3})/\operatorname{Mod}(\Sigma_{1,3})$

The calculation of the orbits of all 2-cells is done in detail in the thesis [W]. We will skip the development here and restrict ourselves to the enumeration of these cells. Most of the 2-cells have to be barycentrically subdivided leading to 16 orbits of 2-cells. It is virtually impossible to put all of them into a single picture, therefore we split them in two complexes. The result can be found in Figure 7. The new vertices labelled $3S$, $4C$, $3A$, $5A_1$ and $5A_2$ arise from the barycentric subdivision of some of the 2-cells.

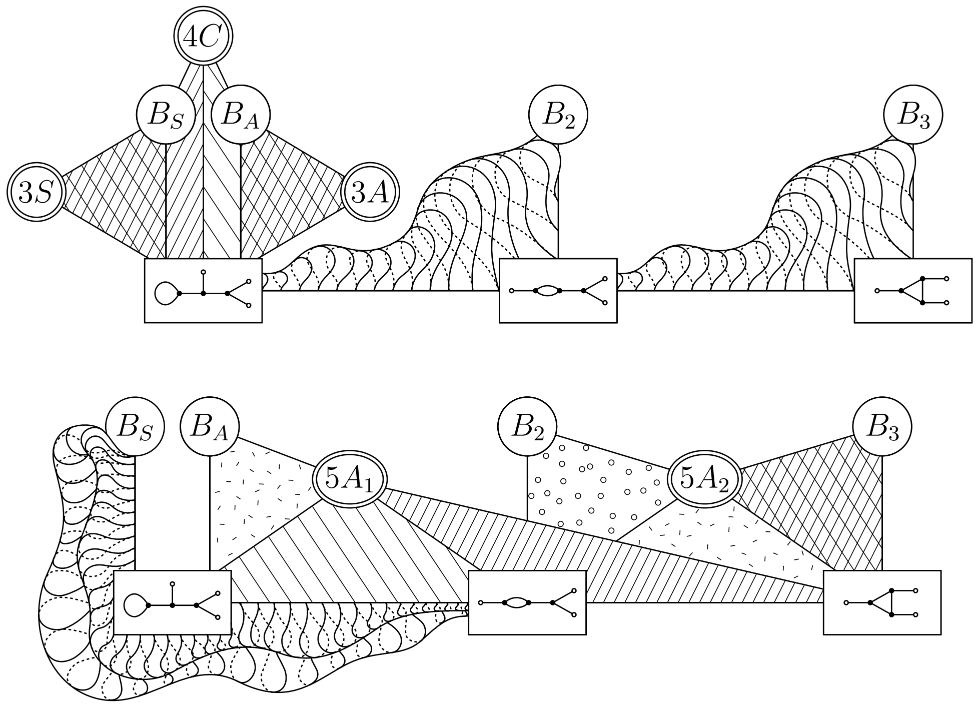


Figure 7: The quotient complex $\mathscr{P}(\Sigma_{1,3})/\operatorname{Mod}(\Sigma_{1,3})$

3 Stabilizers

In this section we discuss the stabilizers of the action of $\mathrm{Mod}(S)$ on $\mathscr{P}(S)$. That means, we calculate the stabilizers of all vertices, edges and 2-cells of the pants complex. For this purpose let S be a surface which (has negative Euler characteristic and) is none of the surfaces $\Sigma_{0,3}$, $\Sigma_{0,4}$, $\Sigma_{1,2}$, $\Sigma_{2,0}$.

Let $\mathcal{P} = \{\alpha_1, \ldots, \alpha_n\}$ be a pants decomposition of S. By choosing a suitable numeration we can assume that $\alpha_1, \ldots, \alpha_r$ are all genus-1-separating curves and $\alpha_{r+1}, \ldots, \alpha_s$ are all 2-separating curves. Here *genus-k-separating* resp. *k-separating* means that the curve is separating and that one component of its complement is a $\Sigma_{k,1}$ resp. a $\Sigma_{0,k+1}$. For any genus-1-separating curve and any 2-separating curve α we have an associated *half twist* σ_α, that is a mapping class twisting the $\Sigma_{1,1}$ resp. $\Sigma_{0,3}$ by 180 degrees and fixing the rest of the surface. The square of any such half twist is a *Dehn twist.* A Dehn twist t_α about a simple closed curve α is defined by cutting S open along α, twisting one side of an appropriate neighbourhood of α by 360 degrees, and reglueing the surface.

Definition 3.1:
$\mathrm{Stab}(\mathcal{P}) := \{f \in \mathrm{Mod}(S) \,:\, f(\mathcal{P}) = \mathcal{P}\}$
$\mathrm{Stab}_{pw}(\mathcal{P}) := \{f \in \mathrm{Mod}(S) \,:\, f(\alpha_i) = \alpha_i \text{ for all } \alpha_i \in \mathcal{P}\}$
$\mathrm{Stab}^*_{pw}(\mathcal{P}) := \{f \in \mathrm{Mod}(S) \,:\, f(\alpha_i) = \alpha_i \text{ for all } \alpha_i \in \mathcal{P} \text{ and } f(\partial) = \partial \text{ for all components } \partial \text{ of } \partial S\}$

Lemma 3.2:
$\mathrm{Stab}_{pw}(\mathcal{P}) = \langle \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_r}, \sigma_{\alpha_{r+1}}, \ldots, \sigma_{\alpha_s}, t_{\alpha_{s+1}}, \ldots, t_{\alpha_n} \rangle$,
$\mathrm{Stab}^*_{pw}(\mathcal{P}) = \langle \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_r}, t_{\alpha_{r+1}}, \ldots, t_{\alpha_s}, t_{\alpha_{s+1}}, \ldots, t_{\alpha_n} \rangle$.

Proof:
In both cases the inclusion "$\supseteq$" is obvious. Let $f \in \mathrm{Stab}_{pw}(\mathcal{P})$. Suppose first that there is a curve $\alpha_i \in \mathcal{P}$ which is mapped onto itself by f, but with reversed orientation. Consider the following cases:

Case 1: Both sides of α_i belong to the same pair of pants H. Then we find a curve $\beta \in \mathcal{P}$ such that $\partial H = \alpha_i \cup \beta$. The mapping class $f \circ \sigma_\beta$ preserves the orientation of α_i, and all other curves are not changed by σ_β.

Case 2: α_i bounds two different pairs of pants H_1 and H_2. H_1 and H_2 are interchanged by f because f reverses the orientation of α_i. f is an element of $\mathrm{Stab}_{pw}(\mathcal{P})$, i.e. all curves of $\mathcal{P}$ are mapped to itself. These two facts lead to the conclusion that S is one of the surfaces $\Sigma_{0,4}$, $\Sigma_{1,2}$, $\Sigma_{2,0}$ which are all excluded.

So we may assume that all curves of $\mathcal{P}$ are mapped orientation-preservingly to themselves by f.

If $f \in \mathrm{Stab}^*_{pw}(\mathcal{P})$ then every boundary component is mapped to itself by f. If $f \in \mathrm{Stab}_{pw}(\mathcal{P})$, suppose that $\partial_1, \partial_2 \subseteq \partial S$ are different boundary components of S such that $f(\partial_1) = \partial_2$. Then the pair of pants H_1 next to ∂_1 is mapped onto the pair of pants

H_2 next to ∂_2. Due to the assumptions on S there are curves $\alpha_i, \alpha_j \in \mathcal{P}$ such that α_i bounds H_1 and α_j bounds H_2 and $f(\alpha_i) = \alpha_j$, hence $\alpha_j = f(\alpha_i) = \alpha_i$ and $H_1 = H_2$. In this case the mapping class $f \circ \sigma_{\alpha_i}$ maps ∂_1 to ∂_1 and ∂_2 to ∂_2. Therefore we can assume that f maps all boundary components of S to themselves.

Altogether f is a mapping class which maps every curve of $\mathcal{P}$ and ∂S orientation-preservingly to itself. That means that f is a composition of Dehn twists about the curves of $\mathcal{P}$ and ∂S. Because Dehn twists about the curves of ∂S are trivial, f is an element of $\langle t_{\alpha_1}, \dots, t_{\alpha_n} \rangle$ as required. $\square$

Every element f of the stabilizer of a pants decomposition $\mathcal{P}$ induces an automorphism $\tilde{f}$ of the associated graph $\Gamma(\mathcal{P})$. (For the definition of $\Gamma(\mathcal{P})$, compare definition 2.1.) Furthermore any automorphism γ of $\Gamma(\mathcal{P})$ induces a permutation of the edges of $\Gamma(\mathcal{P})$, hence an element of the symmetric group $S_{3g-3+2b}$. This gives rise to a group homomorphism

$$\gamma_{\mathcal{P}} \,:\, \mathrm{Stab}(\mathcal{P}) \to \mathrm{Aut}(\Gamma(\mathcal{P})) \to S_{3g-3+2b}.$$

Let $T_{\mathcal{P}}$ be the image of $\gamma_{\mathcal{P}}$. Then we have

Proposition 3.3:
The following sequence is exact:

$$1 \longrightarrow \mathrm{Stab}^*_{pw}(\mathcal{P}) \hookrightarrow \mathrm{Stab}(\mathcal{P}) \xrightarrow{\gamma_{\mathcal{P}}} T_{\mathcal{P}} \longrightarrow 1$$

Proof:
$f \in \ker(\gamma_{\mathcal{P}}) \Leftrightarrow f$ induces the trivial permutation on the edges of $\Gamma(\mathcal{P})$
$\Leftrightarrow f(\alpha_i) = \alpha_i$ for any $\alpha_i \in \mathcal{P}$ and $f(\partial) = \partial$ for any boundary component ∂
$\Leftrightarrow f \in \mathrm{Stab}^*_{pw}(\mathcal{P})$ $\square$

By Lemma 3.2 and Proposition 3.3 the stabilizer $\mathrm{Stab}(\mathcal{P})$ is a group extension of a finite group by a free abelian group. Be careful that the exact sequence in 3.3 in general does not split (it is easy to find counterexamples).

At the beginning of the section we excluded some special surfaces. In those cases there are *more* elements in the pointwise stabilizers $\mathrm{Stab}_{pw}(\mathcal{P})$ and $\mathrm{Stab}^*_{pw}(\mathcal{P})$. For example, there are nontrivial mapping classes which map every curve on S to itself, called *hyperelliptic involutions*. It is an easy exercise to calculate the stabilizers and find the appropriate short exact sequences in these cases (see [W]).

Now we turn to the stabilizer of an edge of the pants complex. So let $\mathcal{K} = (\mathcal{P} — \mathcal{Q}) = (\alpha — \beta)$ be such an edge with $\mathcal{P} = \{\alpha, \alpha_2, \dots, \alpha_n\}$ and $\mathcal{Q} = \{\beta, \alpha_2, \dots, \alpha_n\}$. We define the following special stabilizer groups.

Definition 3.4:

a) $\mathrm{Stab}^+_{or}(\mathcal{K}) := \{f \in \mathrm{Mod}(S) \,:\, f(\mathcal{P}) = \mathcal{P} \text{ and } f(\mathcal{Q}) = \mathcal{Q}\}$
$\mathrm{Stab}^-_{or}(\mathcal{K}) := \{f \in \mathrm{Mod}(S) \,:\, f(\mathcal{P}) = \mathcal{Q} \text{ and } f(\mathcal{Q}) = \mathcal{P}\}$
$\mathrm{Stab}_{nor}(\mathcal{K}) := \{f \in \mathrm{Mod}(S) \,:\, f(\{\mathcal{P},\mathcal{Q}\}) = \{\mathcal{P},\mathcal{Q}\}\} = \mathrm{Stab}^+_{or}(\mathcal{K}) \cup \mathrm{Stab}^-_{or}(\mathcal{K})$

b) $\mathrm{Stab}(\{\alpha_2,\ldots,\alpha_n\}) := \{f \in \mathrm{Mod}(S) \,:\, f(\{\alpha_2,\ldots,\alpha_n\}) = \{\alpha_2,\ldots,\alpha_n\}\}$
$\mathrm{Stab}_{pw}(\{\alpha_2,\ldots,\alpha_n\}) := \{f \in \mathrm{Mod}(S) \,:\, f(\alpha_i) = \alpha_i \text{ for all } i = 2,\ldots,n\}$
$\mathrm{Stab}^*_{pw}(\{\alpha_2,\ldots,\alpha_n\}) := \{f \in \mathrm{Mod}(S) \,:\, f(\alpha_i) = \alpha_i \text{ for all } i = 2,\ldots,n \text{ and } f(\partial) = \partial \text{ for all components } \partial \text{ of } \partial S\}$

c) $F^+_{\alpha,\beta} := \{f \in \mathrm{Mod}(S) \,:\, f(\alpha) = \alpha \text{ and } f(\beta) = \beta\}$
$F^-_{\alpha,\beta} := \{f \in \mathrm{Mod}(S) \,:\, f(\alpha) = \beta \text{ and } f(\beta) = \alpha\}$
$\mathrm{Stab}(\{\alpha,\beta\}) := \{f \in \mathrm{Mod}(S) \,:\, f(\{\alpha,\beta\}) = \{\alpha,\beta\}\} = F^+_{\alpha,\beta} \cup F^-_{\alpha,\beta}$

The following lemma is very easy to prove, so we omit the proof.

Lemma 3.5:
$\mathrm{Stab}^+_{or}(\mathcal{K}) = F^+_{\alpha,\beta} \cap \mathrm{Stab}(\{\alpha_2,\ldots,\alpha_n\})$
$\mathrm{Stab}^-_{or}(\mathcal{K}) = F^-_{\alpha,\beta} \cap \mathrm{Stab}(\{\alpha_2,\ldots,\alpha_n\})$
$\mathrm{Stab}_{nor}(\mathcal{K}) = \mathrm{Stab}(\{\alpha,\beta\}) \cap \mathrm{Stab}(\{\alpha_2,\ldots,\alpha_n\})$

Proposition 3.6:
As above, let $\alpha_2,\ldots,\alpha_r$ be all genus-1-separating curves and $\alpha_{r+1},\ldots,\alpha_s$ be all 2-separating curves. Then

$$F^+_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\ldots,\alpha_n\}) = \langle \sigma_{\alpha_2},\ldots,\sigma_{\alpha_r},\sigma_{\alpha_{r+1}},\ldots,\sigma_{\alpha_s},t_{\alpha_{s+1}},\ldots,t_{\alpha_n}\rangle$$
$$F^+_{\alpha,\beta} \cap \mathrm{Stab}^*_{pw}(\{\alpha_2,\ldots,\alpha_n\}) = \langle \sigma_{\alpha_2},\ldots,\sigma_{\alpha_r},t_{\alpha_{r+1}},\ldots,t_{\alpha_s},t_{\alpha_{s+1}},\ldots,t_{\alpha_n}\rangle$$

Proof:
Proposition 3.6 is proved similarly to Lemma 3.2. For a given mapping class $f \in F^+_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\ldots,\alpha_n\})$ we first show that without loss of generality we may assume that every curve α_i $(i = 2,\ldots,n)$ and every boundary curve $\partial \subseteq \partial S$ is mapped orientation-preservingly to itself. (To achieve this assumption, we possibly have to modify f by half twists about some appropriate curves.) As in 3.2, f is a composition of Dehn twists about the curves $\alpha_2,\ldots,\alpha_n$ and of a special mapping class supported in a regular neighbourhood Σ of $\alpha \cup \beta$. Σ is one of the surfaces $\Sigma_{1,1}$ or $\Sigma_{0,4}$. In both cases, the identity is the only mapping class preserving the boundary of Σ and the curves α,β (compare [W, section 2]). This means that f is a composition of Dehn twists about the curves $\alpha_2,\ldots,\alpha_n$. □

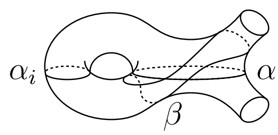
Proposition 3.7:
Again, let $\alpha_2,\ldots,\alpha_r$ be all genus-1-separating curves and $\alpha_{r+1},\ldots,\alpha_s$ be all 2-separating curves.

a) Let $\mathcal{K}$ be an S-move. Then $F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\}) \neq \emptyset$ and for any $f_{\alpha,\beta} \in F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\})$ we have

$$F^{-}_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = f_{\alpha,\beta} \circ \langle \sigma_{\alpha_2},\dots,\sigma_{\alpha_r},\sigma_{\alpha_{r+1}},\dots,\sigma_{\alpha_s},t_{\alpha_{s+1}},\dots,t_{\alpha_n}\rangle,$$
$$F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = f_{\alpha,\beta} \circ \langle \sigma_{\alpha_2},\dots,\sigma_{\alpha_r},t_{\alpha_{r+1}},\dots,t_{\alpha_s},t_{\alpha_{s+1}},\dots,t_{\alpha_n}\rangle.$$

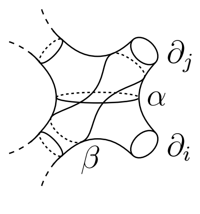
b) Let $\mathcal{K}$ be an A-move and let α,β be nonseparating. If there is a subsurface $\Sigma_{1,2} \subseteq S$ and a curve $\alpha_i \in \{\alpha_2,\dots,\alpha_n\}$ such that $\alpha_i \cup \alpha \cup \beta \subseteq \Sigma_{1,2}$ then $F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\}) \neq \emptyset$ and for any $f_{\alpha,\beta} \in F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\})$ we have

$$F^{-}_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = f_{\alpha,\beta} \circ \langle \sigma_{\alpha_2},\dots,\sigma_{\alpha_r},\sigma_{\alpha_{r+1}},\dots,\sigma_{\alpha_s},t_{\alpha_{s+1}},\dots,t_{\alpha_n}\rangle,$$
$$F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = f_{\alpha,\beta} \circ \langle \sigma_{\alpha_2},\dots,\sigma_{\alpha_r},t_{\alpha_{r+1}},\dots,t_{\alpha_s},t_{\alpha_{s+1}},\dots,t_{\alpha_n}\rangle.$$

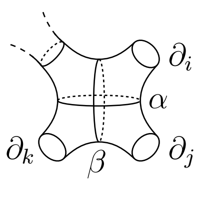


c) Let $\mathcal{K}$ be an A-move and let $\Sigma_{0,4} \subseteq S$ be the component of $S \setminus (\alpha_2 \cup \ldots \cup \alpha_n)$ containing α and β. Furthermore, assume that one of the following properties is fulfilled:

- Exactly two of the boundary components of $\Sigma_{0,4}$ are also boundary components of S, and these two components neither lie in the same component of $\Sigma_{0,4} \setminus \alpha$ nor in the same component of $\Sigma_{0,4} \setminus \beta$.



- Exactly three of the boundary components of $\Sigma_{0,4}$ are also boundary components of S.



Then $F^{-}_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\dots,\alpha_n\}) \neq \emptyset$ and for any $f_{\alpha,\beta} \in F^{-}_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\dots,\alpha_n\})$ we have

$$F^{-}_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = f_{\alpha,\beta} \circ \langle \sigma_{\alpha_2},\dots,\sigma_{\alpha_r},\sigma_{\alpha_{r+1}},\dots,\sigma_{\alpha_s},t_{\alpha_{s+1}},\dots,t_{\alpha_n}\rangle,$$
$$F^{-}_{\alpha,\beta} \cap \mathrm{Stab}^{*}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = \emptyset.$$

d) In all other cases

$$F^-_{\alpha,\beta} \cap \mathrm{Stab}_{pw}(\{\alpha_2,\dots,\alpha_n\}) = \emptyset,$$
$$F^-_{\alpha,\beta} \cap \mathrm{Stab}^*_{pw}(\{\alpha_2,\dots,\alpha_n\}) = \emptyset.$$

The proof is a little tedious and will be skipped here. It can be found in [W, Proposition 3.11].

As for vertices, it is possible to arrange the stabilizer groups of an edge $\mathcal{K}$ of $\mathscr{P}(S)$ into suitable short exact sequences. Namely any element of $\mathrm{Stab}_{nor}(\mathcal{K})$ induces an automorphism of $\Gamma(\mathcal{P}\cap\mathcal{Q})$, and any automorphism of the latter graph induces a permutation of the edges, hence an element of the symmetric group $S_{3g-4+2b}$. This gives rise to a group homomorphism

$$\gamma_{\mathcal{P},\mathcal{Q}} : \mathrm{Stab}_{nor}(\mathcal{K}) \to \mathrm{Aut}(\Gamma(\mathcal{P}\cap\mathcal{Q})) \to S_{3g-4+2b}.$$

Let $T^{nor}_{\mathcal{P},\mathcal{Q}}$ and $T^{or}_{\mathcal{P},\mathcal{Q}}$ be the images of $\mathrm{Stab}_{nor}(\mathcal{K})$ and $\mathrm{Stab}^+_{or}(\mathcal{K})$ respectively under $\gamma_{\mathcal{P},\mathcal{Q}}$. Like in Proposition 3.3, we get the following short exact sequences:

$$1 \longrightarrow \mathrm{Stab}(\{\alpha,\beta\}) \cap \mathrm{Stab}^*_{pw}(\{\alpha_2,\dots,\alpha_n\}) \hookrightarrow \mathrm{Stab}_{nor}(\mathcal{K}) \longrightarrow T^{nor}_{\mathcal{P},\mathcal{Q}} \longrightarrow 1$$
$$1 \longrightarrow \langle \sigma_{\alpha_1},\dots,\sigma_{\alpha_r}, t_{\alpha_{r+1}},\dots,t_{\alpha_s}, t_{\alpha_{s+1}},\dots,t_{\alpha_n}\rangle \hookrightarrow \mathrm{Stab}^+_{or}(\mathcal{K}) \longrightarrow T^{or}_{\mathcal{P},\mathcal{Q}} \longrightarrow 1$$

The details can be found in the thesis [W].

In the last part of this section we treat the stabilizers of 2-cells.

Definition 3.8:
For a 2-cell $\mathcal{F}$ of $\mathscr{P}(S)$ we define

$\mathrm{Stab}_{pw}(\mathcal{F}) := \{f \in \mathrm{Mod}(S) : f(\mathcal{P}) = \mathcal{P} \text{ for all } \mathcal{P} \in \mathcal{F}\}$,
$\mathrm{Stab}_{or}(\mathcal{F}) := \{f \in \mathrm{Mod}(S) : f(\mathcal{F}) = \mathcal{F} \text{ and the orientation of } \mathcal{F} \text{ is preserved}\}$,
$\mathrm{Stab}_{nor}(\mathcal{F}) := \{f \in \mathrm{Mod}(S) : f(\mathcal{F}) = \mathcal{F}\}$.

Clearly $\mathrm{Stab}_{pw}(\mathcal{F}) \subseteq \mathrm{Stab}_{or}(\mathcal{F}) \subseteq \mathrm{Stab}_{nor}(\mathcal{F})$.

We follow the policy of Section 2 and skip a detailed analysis of these stabilizer groups. Like in Section 2, we illustrate the methods by proving the following lemma.

Lemma 3.9:
Let $\mathcal{F}$ be a 2-cell of $\mathscr{P}(S)$ of type (6AS). Then

$$\mathrm{Stab}_{pw}(\mathcal{F}) = \mathrm{Stab}_{or}(\mathcal{F}) = \mathrm{Stab}_{nor}(\mathcal{F}).$$

Proof:
Let us write

$$\begin{aligned}\mathcal{F} &= \{\gamma_1,\gamma_2\} — \{\gamma_1,\gamma_3\} — \{\gamma_3,\gamma_4\} — \{\gamma_4,\gamma_5\} — \{\gamma_4,\gamma_6\} — \{\gamma_2,\gamma_6\}\\ &= \mathcal{P}_1 — \mathcal{P}_2 — \mathcal{P}_3 — \mathcal{P}_4 — \mathcal{P}_5 — \mathcal{P}_6\end{aligned}$$

as in Figure 2. Let $f \in \mathrm{Stab}_{nor}(\mathcal{F})$. Since γ_5 is the only curve which occurs in precisely one of the pants decompositions of $\mathcal{F}$, we conclude $f(\gamma_5) = \gamma_5$ and hence $f(\gamma_4) = \gamma_4$. Therefore we also have $f(\{\gamma_1,\gamma_2\}) = \{\gamma_1,\gamma_2\}$ and $f(\{\gamma_3,\gamma_6\}) = \{\gamma_3,\gamma_6\}$. If $f(\gamma_1) = \gamma_1$ then $f(\gamma_i) = \gamma_i$ for all $i = 1,\ldots,6$ and $f \in \mathrm{Stab}_{pw}(\mathcal{F})$.

Now assume that $f(\gamma_1) = \gamma_2$ and hence $f(\gamma_2) = \gamma_1$, $f(\gamma_3) = \gamma_6$, $f(\gamma_6) = \gamma_3$. As in case 2 of the proof of Lemma 2.6 there is a mapping class $\jmath \in \mathrm{Mod}(S)$ with the following properties:

- $\jmath|_{S\setminus\Sigma_{1,2}} = \mathrm{id}$ where $\Sigma_{1,2}$ is the subsurface of S in which $\mathcal{F}$ is supported,
- $\jmath(\gamma_1) = \gamma_2$, $\jmath(\gamma_2) = \gamma_1$, $\jmath(\gamma_3) = \gamma_6$, $\jmath(\gamma_4) = \gamma_4$, $\jmath(\gamma_6) = \gamma_3$.
- $\jmath^2 = t_{\partial_1} \circ t_{\partial_2}^{-1}$ where ∂_1, ∂_2 are the boundary curves of the subsurface $\Sigma_{1,2}$.

For an illustration of $\jmath$, compare again Figure 4. $\jmath$ is defined in such a way that $\jmath \circ f(\gamma_1) = \gamma_1$, $\jmath \circ f(\gamma_2) = \gamma_2$, $\jmath \circ f(\gamma_3) = \gamma_3$, $\jmath \circ f(\gamma_4) = \gamma_4$, $\jmath \circ f(\gamma_5) = \jmath(\gamma_5) \neq \gamma_5$, $\jmath \circ f(\gamma_6) = \gamma_6$.

Case 1: $f(\partial_1) = \partial_1$ and $f(\partial_2) = \partial_2$
In this case also $\jmath \circ f(\partial_1) = \partial_1$ and $\jmath \circ f(\partial_2) = \partial_2$. This means that the restriction $\jmath \circ f|_{\Sigma_{1,2}}$ of $\jmath \circ f$ to the subsurface $\Sigma_{1,2}$ is a mapping class of $\Sigma_{1,2}$ which preserves the curves γ_1, γ_2 and the boundary curves ∂_1, ∂_2 of $\Sigma_{1,2}$. As in the proof of Lemma 3.2 this means that $\jmath \circ f|_{\Sigma_{1,2}}$ is a composition of Dehn twists about the curves $\gamma_1, \gamma_2, \partial_1, \partial_2$. Because of $\jmath \circ f(\gamma_3) = \gamma_3$ and $\jmath \circ f(\gamma_6) = \gamma_6$ we conclude that $\jmath \circ f|_{\Sigma_{1,2}}$ is a composition of Dehn twists about the curves ∂_1, ∂_2, whence $\jmath \circ f(\gamma_5) = \gamma_5$, a contradiction.

Case 2: $f(\partial_1) = \partial_2$ and $f(\partial_2) = \partial_1$
In this second case there is a mapping class $\imath \in \mathrm{Mod}(S)$ satisfying the following, compare Figure 8:

- $\imath|_{S\setminus\Sigma_{1,2}} = f|_{S\setminus\Sigma_{1,2}}$,
- $\imath(\alpha) = \alpha$ for any curve α in $\Sigma_{1,2}$,
- $\imath^2|_{\Sigma_{1,2}} = \mathrm{id}$.

For this mapping class we have $f \circ \imath(\partial_1) = \partial_1$ and $f \circ \imath(\partial_2) = \partial_2$ and $f \circ \imath(\alpha) = f(\alpha)$ for any curve α in $\Sigma_{1,2}$. Replacing f by $f \circ \imath$ and applying case 1 gives the contradiction. □

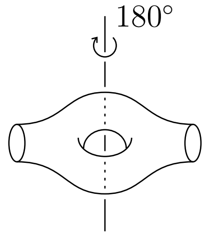


Figure 8: The mapping class $\imath|_{\Sigma_{1,2}}$

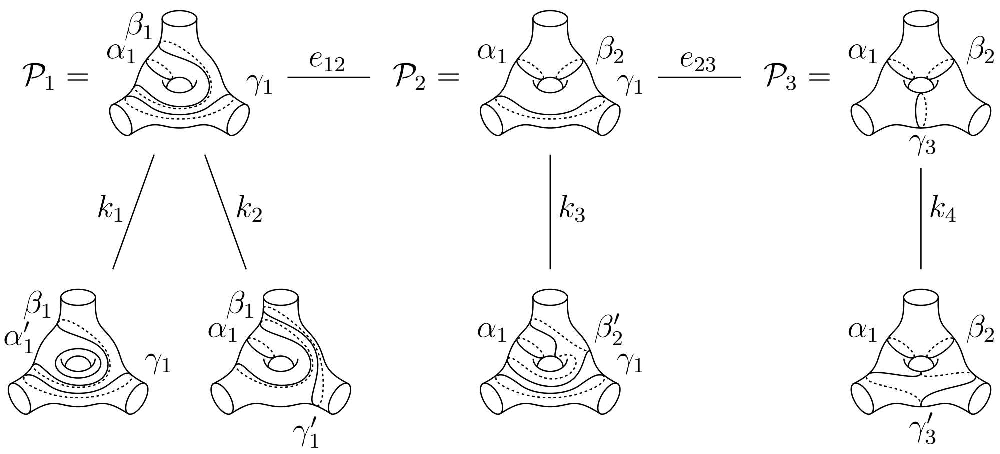


Figure 9: A subtree of $\mathscr{P}(S)$

Example 3.10:
We resume the example of 2.7 and calculate the stabilizers of representatives of all orbits. To begin with, we consider the subtree of $\mathscr{P}(S)$ depicted in Figure 9.

In the parametrization of Figure 9, we also consider two special mapping classes of finite order, compare Figure 10.

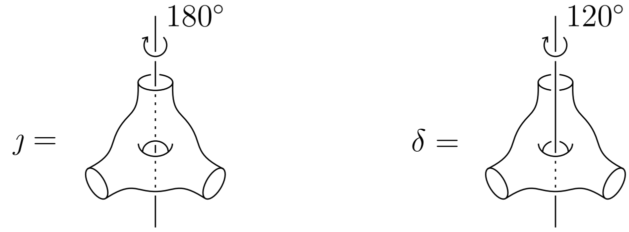


Figure 10: The mapping classes $\jmath$ and δ

Then we have:

$$\begin{aligned}\mathrm{Stab}(\mathcal{P}_1) &= \langle t_{\alpha_1}, \sigma_{\beta_1}, \sigma_{\gamma_1} \rangle \\ \mathrm{Stab}(\mathcal{P}_2) &= \langle t_{\alpha_1}, t_{\beta_2}, \sigma_{\gamma_1}, \jmath \rangle \\ \mathrm{Stab}(\mathcal{P}_3) &= \langle t_{\alpha_1}, t_{\beta_2}, t_{\gamma_3}, \jmath, \delta \rangle\end{aligned}$$

Furthermore:

$$\begin{aligned}\mathrm{Stab}^+_{or}(e_{12}) = \mathrm{Stab}_{nor}(e_{12}) = \langle t_{\alpha_1}, \sigma_{\gamma_1}\rangle \\ \mathrm{Stab}^+_{or}(e_{23}) = \mathrm{Stab}_{nor}(e_{23}) = \langle t_{\alpha_1}, t_{\beta_2}, \jmath\rangle\end{aligned}$$

There is a mapping class $a \in \mathrm{Mod}(\Sigma_{1,3})$ which is a root of σ_{β_1}, interchanging the curves α_1 and α_1', namely $a := t_{\alpha_1} \circ t_{\alpha_1'} \circ t_{\alpha_1}$. Also there is a mapping class $b \in \mathrm{Mod}(\Sigma_{1,3})$ with $b^2 = \mathrm{id}$, interchanging β_2 and β_2', namely $b := \sigma_{\gamma_1}^{-1} \circ t_{\alpha_1}^{-1} \circ \sigma_{\beta_1} \circ t_{\beta_2}$. Furthermore, there is a half twist c_1 about β_1 interchanging γ_1 and γ_1' with $c_1^2 = t_{\beta_1}$. Finally $c_3 := t_{\alpha_1} \circ t_{\beta_2} \circ \sigma_{\gamma_1}^{-1} \circ t_{\gamma_3}^{-1}$ interchanges γ_3 and γ_3' and is a square root of the mapping class $t_{\alpha_3} \circ t_{\beta_2}$. With these four mapping classes we get for the stabilizers of the inverted edges $k_1, \ldots, k_4$:

$$\begin{aligned}\mathrm{Stab}_{nor}(k_1) &= \langle a, \sigma_{\gamma_1}\rangle \\ \mathrm{Stab}^+_{or}(k_1) &= \langle \sigma_{\beta_1}, \sigma_{\gamma_1}\rangle \\ \mathrm{Stab}_{nor}(k_2) &= \langle t_{\alpha_1}, \sigma_{\beta_1}, c_1\rangle \\ \mathrm{Stab}^+_{or}(k_2) &= \langle t_{\alpha_1}, \sigma_{\beta_1}\rangle \\ \mathrm{Stab}_{nor}(k_3) &= \langle t_{\alpha_1}, \sigma_{\gamma_1}, b\rangle \\ \mathrm{Stab}^+_{or}(k_3) &= \langle t_{\alpha_1}, \sigma_{\gamma_1}\rangle \\ \mathrm{Stab}_{nor}(k_4) &= \langle t_{\alpha_1}, t_{\beta_2}, \jmath, c_3\rangle \\ \mathrm{Stab}^+_{or}(k_4) &= \langle t_{\alpha_1}, t_{\beta_2}, \jmath\rangle\end{aligned}$$

At the end of our example consider Figure 11 where a 2-cell $\mathcal{F}$ of type (6AS) is sketched.

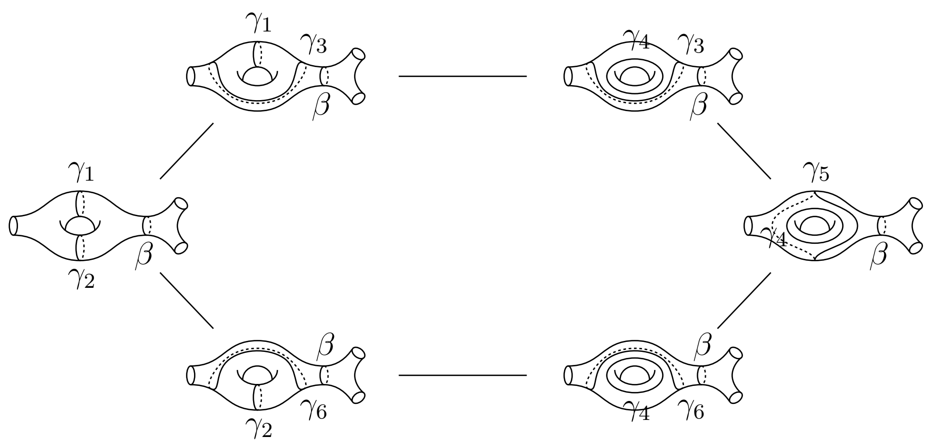


Figure 11: A 2-cell of type (6AS) in $\mathscr{P}(\Sigma_{1,3})$

An easy calculation shows

$$\mathrm{Stab}_{pw}(\mathcal{F}) = \mathrm{Stab}_{or}(\mathcal{F}) = \mathrm{Stab}_{nor}(\mathcal{F}) = \mathrm{Stab}(\{\gamma_1, \gamma_3, \beta\}) \cap \mathrm{Stab}(\{\gamma_2, \gamma_6, \beta\}) = \langle \sigma_\beta\rangle.$$

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