

A Course  
on  
Convex Geometry

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# Preface

The following notes were written before and during the course on *Convex Geometry* which was held at the University of Karlsruhe in the winter term 2002/2003. Although this was the first course on this topic which was given in English, the material presented was based on previous courses in German which have been given several times, mostly in summer terms. In comparison with these previous courses, the standard program was complemented by sections on surface area measures and projection functions as well as by a short chapter on integral geometric formulas. The idea here was to lay the basis for the course on *Stochastic Geometry* which will follow in the summer term.

The exercises at the end of each section contain all the weekly problems which were handed out during the course and discussed in the weekly exercise session. Moreover, I have included a few additional exercises (some of which are more difficult) and even some severe or even unsolved problems. The list of exercises and problems is far from being complete, in fact the number decreases in the later sections due to the lack of time while preparing these notes.

I thank Matthias Heveling and Markus Kiderlen for reading the manuscript and giving hints for corrections and improvements.

Karlsruhe, February 2003

Wolfgang Weil

During repetitions of the course in 2003/2004 and 2005/2006 a number of misprints and small errors have been detected. They are corrected in the current version. Also, additional material and further exercises have been added.

Karlsruhe, August 2005

Wolfgang Weil



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SECTION HEADINGS AND PAGE NUMBERS ARE PRELIMINARY AND WILL BE UP-  
DATED AND COMPLETED DURING THE COURSE

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# Introduction

Convexity is an elementary property of sets  $A$  in a real (or complex) vector space  $V$ .  $A$  is convex if it contains all the segments joining any two points of  $A$ , i.e. if  $x, y \in A$  and  $\alpha \in [0, 1]$  implies that  $\alpha x + (1 - \alpha)y \in A$ . This simple algebraic property has surprisingly many and far-reaching consequences of geometric nature, but it also has topological consequences (if  $V$  carries a compatible topology) as well as analytical ones (if the convexity is extended to real functions  $f$  via the graph of  $f$ ). Therefore, results on convex sets and functions play a central role in many mathematical fields, in particular in functional analysis, in optimization theory and in stochastic geometry.

During this course, we shall concentrate on convex sets in  $\mathbb{R}^n$  as the prototype of a finite dimensional real vector space. In infinite dimensions totally different methods have to be used and different types of problems occur. Here, we concentrate on the classical part of convexity. Starting with convex sets and their basic properties (in Chapter 1), we discuss shortly convex functions (in Chapter 2), and then come (in Chapter 3) to the theory of convex bodies (compact convex sets). Our goal here is to present the essential parts of the Brunn-Minkowski theory (mixed volumes, quermassintegrals, Minkowski inequalities, in particular the isoperimetric inequality) as well as some more special topics (surface area measures, projection functions). The last chapter will shortly discuss some basic formulas from integral geometry.

The course starts rather elementary. Apart from a good knowledge of linear algebra (and, in Chapter 2, analysis) no deeper knowledge of other fields is required. Later we will occasionally use results from functional analysis and, in some parts, we require some familiarity with topological notions and, more important, we use some concepts and results from measure theory.



# Preliminaries and notations

Throughout the course we work in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Elements of  $\mathbb{R}^n$  are denoted by lower case letters like  $x, y, \dots, a, b, \dots$ , scalars by greek letters  $\alpha, \beta, \dots$  and (real) functions by  $f, g, \dots$ . We identify the vector space structure and the affine structure of  $\mathbb{R}^n$ , i.e. we do not distinguish between vectors and points. The coordinates of a point  $x \in \mathbb{R}^n$  are used only occasionally, therefore we indicate them as  $x = (x^{(1)}, \dots, x^{(n)})$ . We equip  $\mathbb{R}^n$  with its usual topology generated by the standard scalar product

$$\langle x, y \rangle := x^{(1)}y^{(1)} + \dots + x^{(n)}y^{(n)}, \quad x, y \in \mathbb{R}^n,$$

respectively the corresponding Euclidean norm

$$\|x\| := ((x^{(1)})^2 + \dots + (x^{(n)})^2)^{1/2}, \quad x \in \mathbb{R}^n.$$

By  $B^n$ , we denote the unit ball,

$$B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

and by

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$$

the unit sphere. Sometimes, we also make use of the Euclidean metric  $d(x, y) := \|x - y\|, x, y \in \mathbb{R}^n$ . In combined expressions, it is sometimes convenient to write  $\frac{x}{\alpha}$ , instead of  $\frac{1}{\alpha}x$ , for  $x \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$ .

Convex sets in  $\mathbb{R}^1$  are not very exciting (they are open, closed or half-open, bounded or unbounded intervalls), the results mostly are only interesting for  $n \geq 2$ . In some situations, results only make sense, if  $n \geq 2$ , although we shall not emphasize this in all cases. As a rule,  $A, B, \dots$  denote general (convex or nonconvex) sets,  $K, L, \dots$  will be used for compact convex sets (convex bodies) and  $P, Q, \dots$  for (convex) polytopes.

A number of notations will be used frequently, without further explanations:

$\text{lin } A$	linear hull of $A$
$\text{aff } A$	affine hull of $A$
$\text{dim } A$	dimension of $A$ (= dimension of $\text{aff } A$ )

$\text{int } A$	interior of $A$
$\text{rel int } A$	relative interior of $A$ (interior w.r.t. $\text{aff } A$ )
$\text{cl } A$	closure of $A$
$\text{bd } A$	boundary of $A$

If  $f$  is a function on  $\mathbb{R}^n$  with values in  $\mathbb{R}$  or in the extended real line  $[-\infty, \infty]$  and if  $A$  is a subset of the latter, we frequently abbreviate the set  $\{x \in \mathbb{R}^n : f(x) \in A\}$  by  $\{f \in A\}$ . Hyperplanes  $E \subset \mathbb{R}^n$  are therefore shortly written as  $E = \{f = \alpha\}$ , where  $f$  is a linear form and  $\alpha \in \mathbb{R}$  (note that this representation is not unique). The corresponding closed half-spaces generated by  $E$  are then  $\{f \geq \alpha\}$  and  $\{f \leq \alpha\}$ , and the open half-spaces are  $\{f > \alpha\}$  and  $\{f < \alpha\}$ .

The symbol  $\subset$  always includes the case of equality. The abbreviation w.l.o.g. means ‘without loss of generality’ and is used sometimes to reduce the argument to a special case. The logical symbols  $\forall$  (for all) and  $\exists$  (exists) are occasionally used in formulas.  $\square$  denotes the end of a proof.

Each section is complemented by a number of exercises. Some are very easy, but most require a bit of work. Those which are more difficult than it appears from the first look are marked by  $*$ . Occasionally, problems have been included which are either very difficult to solve or even unsolved up to now. They are indicated by P.

# Chapter 1

## Convex sets

### 1.1 Algebraic properties

**Definition.** A set  $A \subset \mathbb{R}^n$  is *convex*, if we have  $\alpha x + (1 - \alpha)y \in A$ , for all  $x, y \in A$  and  $\alpha \in [0, 1]$ .

**Examples. (1)** The simplest convex sets (apart from the points) are the segments. We denote by

$$[x, y] := \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$$

the *closed segment* between  $x$  and  $y$ ,  $x, y \in \mathbb{R}^n$ . Similarly,

$$(x, y) := \{\alpha x + (1 - \alpha)y : \alpha \in (0, 1)\}$$

is the *open segment* and we define half-open segments  $(x, y]$  and  $[x, y)$  in an analogous way.

**(2)** Other trivial examples are the affine flats in  $\mathbb{R}^n$ .

**(3)** If  $\{f = \alpha\}$  ( $f$  a linear form,  $\alpha \in \mathbb{R}$ ) is the representation of a hyperplane, the *open half-spaces*  $\{f < \alpha\}$ ,  $\{f > \alpha\}$  and the *closed half-spaces*  $\{f \leq \alpha\}$ ,  $\{f \geq \alpha\}$  are convex.

**(4)** Further convex sets are the *balls*

$$B(r) := \{x : \|x\| \leq r\}, \quad r \geq 0,$$

and their translates.

**Theorem 1.1.1.** A set  $A \subset \mathbb{R}^n$  is convex, if and only if

$$(*) \quad \alpha_1 x_1 + \cdots + \alpha_k x_k \in A$$

holds for all  $k \in \mathbb{N}$ , all  $x_1, \dots, x_k \in A$  and all  $\alpha_1, \dots, \alpha_k \in [0, 1]$  with  $\sum_{i=1}^k \alpha_i = 1$ .

**Remark.** We call  $(*)$  a *convex combination* (of the points  $x_1, \dots, x_k$ ).

*Proof.* Taking  $k = 2$ , we see that the condition on the convex combinations implies convexity.

For the other direction, assume  $A$  is convex and  $k \in \mathbb{N}$ . We use induction on  $k$ :

For  $k = 1$ , the assertion is trivially fulfilled.

For the step from  $k - 1$  to  $k$ ,  $k \geq 2$ , assume  $x_1, \dots, x_k \in A$  and  $\alpha_1, \dots, \alpha_k \in [0, 1]$  with  $\sum_{i=1}^k \alpha_i = 1$  are given. We may assume  $\alpha_i \neq 0$ ,  $i = 1, \dots, k$ , and define

$$\beta_1 := \frac{\alpha_1}{\alpha_1 + \dots + \alpha_{k-1}}, \dots, \beta_{k-1} := \frac{\alpha_{k-1}}{\alpha_1 + \dots + \alpha_{k-1}},$$

hence  $\beta_i \in [0, 1]$  and  $\sum_{i=1}^{k-1} \beta_i = 1$ . By the induction hypothesis,  $\sum_{i=1}^{k-1} \beta_i x_i \in A$ , and by the convexity

$$\left( \sum_{i=1}^{k-1} \alpha_i \right) \left( \sum_{i=1}^{k-1} \beta_i x_i \right) + \left( 1 - \sum_{i=1}^{k-1} \alpha_i \right) x_k = \sum_{i=1}^k \alpha_i x_i \in A.$$

□

**Remark and Definition.** If  $\{A_i : i \in I\}$  is an arbitrary family of convex sets (in  $\mathbb{R}^n$ ), the intersection  $\bigcap_{i \in I} A_i$  is convex. In particular, for a given set  $A \subset \mathbb{R}^n$ , the intersection of all convex sets containing  $A$  is convex, it is called the *convex hull*  $\text{conv } A$  of  $A$ .

**Theorem 1.1.2.** For  $A \subset \mathbb{R}^n$ , we have

$$\text{conv } A = \left\{ \alpha_1 x_1 + \dots + \alpha_k x_k : k \in \mathbb{N}, x_1, \dots, x_k \in A, \alpha_1, \dots, \alpha_k \in [0, 1], \sum_{i=1}^k \alpha_i = 1 \right\}.$$

In other words,  $\text{conv } A$  is the set of all convex combinations of points in  $A$ .

*Proof.* Let  $B$  denote the set on the right hand side. If  $C$  is a convex set containing  $A$ , Theorem 1.1.1 implies  $B \subset C$ . Hence, we get  $B \subset \text{conv } A$ .

On the other hand, the set  $B$  is convex since

$$\begin{aligned} & \beta(\alpha_1 x_1 + \dots + \alpha_k x_k) + (1 - \beta)(\gamma_1 y_1 + \dots + \gamma_m y_m) \\ &= \beta \alpha_1 x_1 + \dots + \beta \alpha_k x_k + (1 - \beta) \gamma_1 y_1 + \dots + (1 - \beta) \gamma_m y_m \end{aligned}$$

with coefficients in  $[0, 1]$  and

$$\begin{aligned} & \beta \alpha_1 + \dots + \beta \alpha_k + (1 - \beta) \gamma_1 + \dots + (1 - \beta) \gamma_m \\ &= \beta + (1 - \beta) = 1. \end{aligned}$$

Since  $B$  contains  $A$ , we get  $B \supset \text{conv } A$ . □

**Remarks.** (1) Trivially,  $A$  is convex, if and only if  $A = \text{conv } A$ .

(2) Later, in Section 1.2, we will give an improved version of Theorem 1.1.2 (CARATHEODORY's theorem), where the number  $k$  of points used in the representation of  $\text{conv } A$  is bounded by  $n + 1$ .

**Definition.** For sets  $A, B \subset \mathbb{R}^n$  and  $\alpha, \beta \in \mathbb{R}$ , we put

$$\alpha A + \beta B := \{\alpha x + \beta y : x \in A, y \in B\}.$$

The set  $\alpha A + \beta B$  is called a *linear combination* of the sets  $A, B$ , the operation  $+$  is called *vector addition*. Special cases get special names:

$A + B$	the <i>sum set</i>
$A + x$ (the case $B = \{x\}$ )	a <i>translate</i> of $A$
$\alpha A$	the <i>multiple</i> of $A$
$\alpha A + x$ (for $\alpha \geq 0$ )	a <i>homothetic image</i> of $A$
$-A := (-1)A$	the <i>reflection</i> of $A$ (in the origin)
$A - B := A + (-B)$	the <i>difference</i> of $A$ and $B$

**Remarks.** (1) If  $A, B$  are convex and  $\alpha, \beta \in \mathbb{R}$ , then  $\alpha A + \beta B$  is convex.

(2) In general, the relations  $A + A = 2A$  and  $A - A = \{0\}$  are **wrong**. For convex  $A$  and  $\alpha, \beta \geq 0$ , we have  $\alpha A + \beta A = (\alpha + \beta)A$ . The latter property characterizes convexity of a set  $A$ .

**Theorem 1.1.3.** Let  $A \subset \mathbb{R}^n$ ,  $B \subset \mathbb{R}^m$  be convex and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  affine. Then

$$f(A) := \{f(x) : x \in A\}$$

and

$$f^{-1}(B) := \{x : f(x) \in B\}$$

are convex.

*Proof.* Both assertions follow from

$$\alpha f(x) + (1 - \alpha)f(y) = f(\alpha x + (1 - \alpha)y).$$

□

**Corollary 1.1.4.** The projection of a convex set onto an affine subspace is convex.

The converse is obviously false, a shell bounded by two concentric balls is not convex but has convex projections.

**Definition.** (a) The intersection of finitely many closed half-spaces is called a *polyhedral set*.

(b) The convex hull of finitely many points  $x_1, \dots, x_k \in \mathbb{R}^n$  is called a (convex) *polytope*  $P$ . Each  $x_i$  which does not belong to the convex hull of the remaining  $x_j$  is called a *vertex* of  $P$ . We denote the set of all vertices of  $P$  by  $\text{vert } P$ . If the vertices are affinely independent, the polytope  $P$  is called a *simplex*. More precisely,  $P$  is called an  *$r$ -simplex*, if it has precisely  $r + 1$  affinely independent vertices.

**Remarks.** (1) A polyhedral set is closed and convex. Polytopes, as convex hulls of finite sets, are closed and bounded, hence compact. We discuss these topological questions in more generality in 1.3.

(2) For a polytope  $P$ , we have  $P = \text{conv vert } P$ . This can be seen directly from an inductive argument (see Exercise 10), but is also a special case of MINKOWSKI's theorem, which is proved in 1.5.

(3) Polyhedral sets and polytopes are somehow dual notions. We shall see later in 1.4 that the polytopes coincide with the bounded polyhedral sets.

(4) The polytope property is conserved by the usual operations. Namely, if  $P, Q$  are polytopes, then the following sets are polytopes:

- $\text{conv}(P \cup Q)$ ,
- $P \cap Q$ ,
- $\alpha P + \beta Q$ , for  $\alpha, \beta \in \mathbb{R}$ ,
- $f(P)$ , for affine  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

Here, only the second assertion is not straight forward. The proof that  $P \cap Q$  is a polytope will follow later from the mentioned connection between polytopes and bounded polyhedral sets.

(5) If  $P$  is the convex hull of affinely independent points  $x_0, \dots, x_r$ , then each  $x_i$  is a vertex of  $P$ , i.e.  $P$  is an  $r$ -simplex. An  $r$ -simplex  $P$  has dimension  $\dim P = r$ .

Simplices are characterized by the property that their points are unique convex combinations of the vertices.

**Theorem 1.1.5.** *A convex set  $A \subset \mathbb{R}^n$  is a simplex, if and only if there exist  $x_0, \dots, x_k \in A$  such that each  $x \in A$  has a **unique** representation as a convex combination of  $x_0, \dots, x_k$ .*

*Proof.* By definition,  $A$  is a simplex, if  $A = \text{conv} \{x_0, \dots, x_k\}$  with affinely independent  $x_0, \dots, x_k \in \mathbb{R}^n$ . The assertion therefore follows from Theorem 1.1.2 together with the uniqueness property of affine combinations (with respect to affinely independent points) and the well-known characterizations of affine independence (see also Exercise 11).  $\square$



**Exercises and problems**

1. (a) Show that  $A \subset \mathbb{R}^n$  is convex, if and only if  $\alpha A + \beta A = (\alpha + \beta)A$  holds, for all  $\alpha, \beta \geq 0$ .  
 (b) Which non-empty sets  $A \subset \mathbb{R}^n$  are characterized by  $\alpha A + \beta A = (\alpha + \beta)A$ , for all  $\alpha, \beta \in \mathbb{R}$ ?
- \* 2. Let  $A \subset \mathbb{R}^n$  be closed. Show that  $A$  is convex, if and only if  $A + A = 2A$  holds.
3. A set

$$R := \{x + \alpha y : \alpha \geq 0\}, \quad x \in \mathbb{R}^n, y \in S^{n-1},$$

is called a *ray* (starting in  $x$  with direction  $y$ ).

Let  $A \subset \mathbb{R}^n$  be convex and unbounded. Show that  $A$  contains a ray.

*Hint:* Show first that it is sufficient to consider the case  $A \subset \mathbb{R}^2$ .

4. For a set  $A \subset \mathbb{R}^n$ , the *polar*  $A^\circ$  is defined as

$$A^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall y \in A\}.$$

Show that:

- (a)  $A^\circ$  is closed, convex and contains 0.
  - (b) If  $A \subset B$ , then  $A^\circ \supset B^\circ$ .
  - (c)  $(A \cup B)^\circ = A^\circ \cap B^\circ$ .
  - (d) If  $P$  is a polytope,  $P^\circ$  is polyhedral.
5. (a) If  $\|\cdot\|' : \mathbb{R}^n \rightarrow [0, \infty)$  is a norm, show that the corresponding unit ball  $B' := \{x \in \mathbb{R}^n : \|x\|' \leq 1\}$  is convex and symmetric (i.e.  $B' = -B'$ ).
  - (b) Show that

$$\|\cdot\|_1 : \mathbb{R}^n \rightarrow [0, \infty), \quad x = (x^{(1)}, \dots, x^{(n)}) \mapsto \sum_{i=1}^n |x^{(i)}|,$$

and

$$\|\cdot\|_\infty : \mathbb{R}^n \rightarrow [0, \infty), \quad x = (x^{(1)}, \dots, x^{(n)}) \mapsto \max_{i=1, \dots, n} |x^{(i)}|,$$

are norms. Describe the corresponding unit balls  $B_1$  and  $B_\infty$ .

- (c) Show that for an arbitrary norm  $\|\cdot\|' : \mathbb{R}^n \rightarrow [0, \infty)$  there are constants  $\alpha, \beta, \gamma > 0$  such that

$$\alpha \|\cdot\|_1 \leq \beta \|\cdot\|_\infty \leq \|\cdot\|' \leq \gamma \|\cdot\|_1.$$

Describe these inequalities in terms of the corresponding unit balls  $B_1, B_\infty, B'$ .

*Hint:* Show first the last inequality. Then prove that

$$\inf\{\|x\|_\infty : x \in \mathbb{R}^n, \|x\|' = 1\} > 0,$$

and deduce the second inequality from that.

(d) Use (c) to show that all norms on  $\mathbb{R}^n$  are equivalent.

6. For a set  $A \subset \mathbb{R}^n$  let

$$\ker A := \{x \in A : [x, y] \subset A \text{ for all } y \in A\}$$

be the *kernel* of  $A$ . Show that  $\ker A$  is convex. Show by an example that  $A \subset B$  does not imply  $\ker A \subset \ker B$ .

7. Let  $A \subset \mathbb{R}^n$  be a *locally finite* set (this means that  $|A \cap B(r)| < \infty$ , for all  $r \geq 0$ ). For each  $x \in A$ , we define the *Voronoi cell*

$$C(x, A) := \{z \in \mathbb{R}^n : \|z - x\| \leq \|z - y\| \forall y \in A\},$$

consisting of all points  $z \in \mathbb{R}^n$  which have  $x$  as their nearest point (or one of their nearest points) in  $A$ .

(a) Show that the Voronoi cells  $C(x, A), x \in A$ , are closed and convex.

(b) If  $\text{conv } A = \mathbb{R}^n$ , show that the Voronoi cells  $C(x, A), x \in A$ , are bounded and polyhedral, hence they are convex polytopes.

*Hint:* Use Exercise 3.

(c) Show by an example that the condition  $\text{conv } A = \mathbb{R}^n$  is not necessary for the boundedness of the Voronoi cells  $C(x, A), x \in A$ .

8. Show that the set  $\mathcal{A}$  of all convex subsets of  $\mathbb{R}^n$  is a complete lattice with respect to the inclusion order.

*Hint:* Define

$$A \wedge B := A \cap B,$$

$$A \vee B := \text{conv}(A \cup B),$$

$$\inf \mathcal{M} := \bigcap_{A \in \mathcal{M}} A, \quad \mathcal{M} \subset \mathcal{A},$$

$$\sup \mathcal{M} := \text{conv} \left( \bigcup_{A \in \mathcal{M}} A \right), \quad \mathcal{M} \subset \mathcal{A}.$$

9. Show that, for  $A, B \subset \mathbb{R}^n$ , we have  $\text{conv}(A + B) = \text{conv } A + \text{conv } B$ .

10. Let  $P = \text{conv} \{x_1, \dots, x_k\}$  and w.l.o.g. assume  $x_1, \dots, x_r$  are the vertices of  $P$ . Show by induction on  $k \geq r$  that  $P = \text{conv} \{x_1, \dots, x_r\}$ .

11. Assume that  $x_1, \dots, x_k \in \mathbb{R}^n$  are such that each  $x \in \text{conv} \{x_1, \dots, x_k\}$  is a unique convex combination of  $x_1, \dots, x_k$ . Show that  $x_1, \dots, x_k$  are affinely independent.

12. Let  $P = \text{conv} \{x_0, \dots, x_n\}$  be an  $n$ -simplex in  $\mathbb{R}^n$ . Denote by  $E_i$  the affine hull of  $\{x_0, \dots, x_n\} \setminus \{x_i\}$  and by  $H_i$  the closed half-space bounded by  $E_i$  and with  $x_i \in H_i$ ,  $i = 0, \dots, n$ .

(a) Show that  $x_i \in \text{int } H_i$ ,  $i = 0, \dots, n$ .

- (b) Show that  $P = \bigcap_{i=0}^n H_i$ .
- (c) Show that  $P \cap E_i$  is an  $(n - 1)$ -simplex.

## 1.2 Combinatorial properties

Naturally, combinatorial problems arise in connection with polytopes. In the following, however, we discuss problems of general convex sets which are called combinatorial, since they involve the cardinality of points or sets. The most important results in this part of convex geometry (which is called *Combinatorial Geometry*) are the theorems of CARATHÉODORY, HELLY and RADON.

**Theorem 1.2.1 (RADON).** *Let  $A \subset \mathbb{R}^n$  be a set of at least  $n+2$  points. Then there exists a partition  $A = B \cup C$ ,  $B \cap C = \emptyset$ , such that*

$$\text{conv } B \cap \text{conv } C \neq \emptyset.$$

*Proof.* It is sufficient to discuss the case  $A = \{x_1, \dots, x_{n+2}\}$ . For the coordinates  $x_i = (x_i^{(1)}, \dots, x_i^{(n)})$ ,  $i = 1, \dots, n+2$ , we consider the system of linear equations

$$\begin{aligned} \sum_{i=1}^{n+2} \alpha_i x_i^{(j)} &= 0, \quad j = 1, \dots, n, \\ \sum_{i=1}^{n+2} \alpha_i &= 0. \end{aligned}$$

These are  $n+1$  equations for the variables  $\alpha_1, \dots, \alpha_{n+2}$ , hence there exists a non-trivial solution. For a fixed solution  $(\alpha_1, \dots, \alpha_{n+2}) \neq (0, \dots, 0)$ , let

$$B := \{x_i : \alpha_i \geq 0\}, \quad C := \{x_i : \alpha_i < 0\}.$$

Then  $B, C$  build a partition of  $A$  and

$$y := \frac{\sum_{i:\alpha_i \geq 0} \alpha_i x_i}{\sum_{i:\alpha_i \geq 0} \alpha_i} = \frac{\sum_{i:\alpha_i < 0} (-\alpha_i) x_i}{\sum_{i:\alpha_i < 0} (-\alpha_i)} \in \text{conv } B \cap \text{conv } C.$$

□

As a consequence, we next derive HELLY's Theorem (in a particular version).

**Theorem 1.2.2 (HELLY).** *Let  $\mathcal{A}$  be a finite family of at least  $n+1$  convex sets  $A \subset \mathbb{R}^n$ . If each  $n+1$  of the sets in  $\mathcal{A}$  have a nonempty intersection, then*

$$\bigcap_{A \in \mathcal{A}} A \neq \emptyset.$$

*Proof.* We proceed by induction on  $k = |\mathcal{A}|$ ,  $k \geq n+1$ .

For  $k = n+1$ , the assumption trivially implies the assertion.

Let  $k \geq n + 2$  and assume that the assertion holds for all families of  $k - 1$  sets. Then, to each  $A \in \mathcal{A}$ , there exists a point

$$x_A \in \bigcap_{A' \in \mathcal{A} \setminus \{A\}} A'.$$

If  $x_A = x_{\tilde{A}}$ , for two (different) sets  $A, \tilde{A} \in \mathcal{A}$ , then

$$x_A \in \bigcap_{A' \in \mathcal{A}} A'$$

and we are done.

Otherwise, all points  $x_A, A \in \mathcal{A}$ , are different; therefore the set  $\bar{\mathcal{A}} := \{x_A : A \in \mathcal{A}\}$  has  $k \geq n + 2$  points. By Theorem 1.2.1, there exists a partition  $\bar{\mathcal{A}} = B \cup C$ ,  $B \cap C = \emptyset$ , and a  $z \in \text{conv } B \cap \text{conv } C$ . This gives rise to two subfamilies  $\mathcal{A}_1 := \{A \in \mathcal{A} : x_A \in B\}$  and  $\mathcal{A}_2 := \{A \in \mathcal{A} : x_A \in C\}$  with  $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$  and  $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ . Since  $\text{conv } B \subset A'$ , for all  $A' \in \mathcal{A}_2$  and  $\text{conv } C \subset A'$ , for all  $A' \in \mathcal{A}_1$ , we conclude that  $z \in A$ , for all  $A \in \mathcal{A}$ . Therefore,

$$\bigcap_{A \in \mathcal{A}} A \neq \emptyset.$$

□

HELLY's Theorem has interesting applications. For some of them, we refer to the exercises. In general, the theorem cannot be extended to infinite families of convex sets (see Exercise 1). An exception is the case of compact sets.

**Theorem 1.2.3 (HELLY).** *Let  $\mathcal{A}$  be a family of at least  $n + 1$  compact convex sets in  $\mathbb{R}^n$  ( $\mathcal{A}$  may be infinite) and assume that any  $n + 1$  sets in  $\mathcal{A}$  have a non-empty intersection. Then, there is a point  $x \in \mathbb{R}^n$  which is contained in all sets of  $\mathcal{A}$ .*

*Proof.* By Theorem 1.2.2, every finite subfamily of  $\mathcal{A}$  has a non-empty intersection. For compact sets, this implies

$$\bigcap_{A \in \mathcal{A}} A \neq \emptyset.$$

In fact, if  $\bigcap_{A \in \mathcal{A}} A = \emptyset$ , then

$$\bigcup_{A \in \mathcal{A}} (\mathbb{R}^n \setminus A) = \mathbb{R}^n.$$

By the covering property, any compact  $A_0 \in \mathcal{A}$  is covered by finitely many open sets  $\mathbb{R}^n \setminus A_1, \dots, \mathbb{R}^n \setminus A_k, A_i \in \mathcal{A}$ . This implies

$$\bigcap_{i=0}^k A_i = \emptyset,$$

a contradiction. □

The following result will be frequently used later on.

**Theorem 1.2.4 (CARATHÉODORY).** *For a set  $A \subset \mathbb{R}^n$  and  $x \in \mathbb{R}^n$  the following two assertions are equivalent:*

- (a)  $x \in \text{conv } A$ ,
- (b) *there is an  $r$ -simplex  $P$  ( $0 \leq r \leq n$ ) with vertices in  $A$  and such that  $x \in P$ .*

*Proof.* (b) $\Rightarrow$ (a): Since  $\text{vert } P \subset A$ , we have  $x \in P = \text{conv vert } P \subset \text{conv } A$ .

(a) $\Rightarrow$ (b): By Theorem 1.1.2,  $x = \alpha_1 x_1 + \cdots + \alpha_k x_k$  with  $k \in \mathbb{N}$ ,  $x_1, \dots, x_k \in A$ ,  $\alpha_1, \dots, \alpha_k \in [0, 1]$  and  $\sum_{i=1}^k \alpha_i = 1$ . We assume that  $k$  is the minimal number for which such a representation is possible, i.e.  $x$  is not in the convex hull of any  $k - 1$  points of  $A$ . We also may assume that  $\dim \text{conv } \{x_1, \dots, x_k\} = n$  (otherwise we choose aff  $\{x_1, \dots, x_k\}$  as the new space and  $A' := \{x_1, \dots, x_k\}$  as the new set). Then we have to show that  $k = n + 1$ . Assume  $k \geq n + 2$ . As in the proof of Theorem 1.2.1, we obtain numbers  $\beta_1, \dots, \beta_k \in \mathbb{R}$ , not all vanishing, such that

$$\sum_{i=1}^k \beta_i x_i = 0$$

and

$$\sum_{i=1}^k \beta_i = 0.$$

Let  $J$  be the set of indices  $i \in \{1, \dots, k\}$ , for which  $\beta_i < 0$  and choose  $i_0 \in J$  such that

$$\frac{\alpha_{i_0}}{(-\beta_{i_0})} = \min_{i \in J} \frac{\alpha_i}{(-\beta_i)}.$$

Then, we have

$$x = \sum_{i=1}^k \left( \alpha_i - \frac{\alpha_{i_0}}{\beta_{i_0}} \beta_i \right) x_i$$

with  $\alpha_i - \frac{\alpha_{i_0}}{\beta_{i_0}} \beta_i \geq 0$  and  $\sum_{i=1}^k \left( \alpha_i - \frac{\alpha_{i_0}}{\beta_{i_0}} \beta_i \right) = 1$ , but  $\alpha_{i_0} - \frac{\alpha_{i_0}}{\beta_{i_0}} \beta_{i_0} = 0$ . This is a contradiction to the minimality of  $k$ .  $\square$

### Exercises and problems

1. Show by an example that Theorem 1.2.3 is wrong if the sets in  $\mathcal{A}$  are only assumed to be closed (and not necessarily compact).

2. In an old German fairy tale, a tailor claimed the fame to have ‘killed seven with one stroke’. A closer examination showed that the victims were in fact flies which had landed on a toast covered with jam. The tailor had used a fly-catcher of convex shape for his sensational victory. As the remains of the flies on the toast showed, it was possible to kill any three of them with one stroke of the (suitably) shifted fly-catcher without even turning the direction of the handle.

Is it possible that the tailor told the truth?

3. Let  $\mathcal{F}$  be a family of finitely many parallel closed segments in  $\mathbb{R}^2$ ,  $|\mathcal{F}| \geq 3$ . Suppose that for any three segments in  $\mathcal{F}$  there is a line intersecting all three segments.

Show that there is a line in  $\mathbb{R}^2$  intersecting all the segments in  $\mathcal{F}$ .

\* Show that the above result remains true without the finiteness condition.

4. Prove the following version of CARATHÉODORY’s theorem:

Let  $A \subset \mathbb{R}^n$  and  $x_0 \in A$  be fixed. Then  $\text{conv } A$  is the union of all simplices with vertices in  $A$  and such that  $x_0$  is one of the vertices.

- \* 5. Prove the following generalization of CARATHÉODORY’s theorem (Theorem of BUNDT):

Let  $A \subset \mathbb{R}^n$  be a connected set. Then  $\text{conv } A$  is the union of all simplices with vertices in  $A$  and dimension at most  $n - 1$ .

### 1.3 Topological properties

Although convexity is a purely algebraic property, it has a variety of topological consequences. One striking property of convex sets is that they always have (relative) interior points. In order to prove that, we first need an auxiliary result.

**Proposition 1.3.1.** *If  $P = \text{conv} \{x_0, \dots, x_k\}$  is a  $k$ -simplex in  $\mathbb{R}^n$ ,  $1 \leq k \leq n$ , then*

$$\text{rel int } P = \{\alpha_0 x_0 + \dots + \alpha_k x_k : \alpha_i \in (0, 1), \sum \alpha_i = 1\}.$$

*Proof.* W.l.o.g. we may assume  $k = n$  and  $x_0 = 0$ . Then we have

$$P = \{\alpha_1 x_1 + \dots + \alpha_n x_n : \alpha_i \in [0, 1], \sum \alpha_i \leq 1\},$$

and we need to show that

$$\text{int } P = \{\alpha_1 x_1 + \dots + \alpha_n x_n : \alpha_i \in (0, 1), \sum \alpha_i < 1\}.$$

Notice that  $x_1, \dots, x_n$  is a basis of  $\mathbb{R}^n$ . Therefore, a point  $x \in P$ ,  $x = \alpha_1 x_1 + \dots + \alpha_n x_n$  with  $\alpha_i \in (0, 1)$  and such that  $\sum \alpha_i < 1$  has a neighborhood  $U$ , such that each  $y \in U$  has a representation  $y = \beta_1 x_1 + \dots + \beta_n x_n$  with  $\beta_i \in (0, 1)$  and  $\sum \beta_i < 1$ . This implies  $U \subset P$  and  $x$  is an interior point of  $P$ .

Conversely, if  $x \in P$  has a representation  $x = \alpha_1 x_1 + \dots + \alpha_n x_n$ , where one  $\alpha_i$  vanishes, say  $\alpha_1 = 0$ , then  $x$  lies in the (linear) hyperplane  $\text{lin} \{x_2, \dots, x_n\}$ . Each neighborhood of  $x$  therefore contains points  $y$  which have (at least) one negative coefficient with respect to the basis  $x_1, \dots, x_n$ . By Theorem 1.1.5,  $y \notin P$  and therefore  $x$  is a boundary point of  $P$ . Analogously, we may treat the case  $x = \alpha_1 x_1 + \dots + \alpha_n x_n$ ,  $\sum_{i=1}^n \alpha_i = 1$ . In that case,  $x$  lies in the (affine) hyperplane  $\text{aff} \{x_1, \dots, x_n\}$ , thus in each neighborhood of  $x$  there are points  $y$  which have a representation  $y = \beta_1 x_1 + \dots + \beta_n x_n$  with  $\beta_1 + \dots + \beta_n > 1$ . Again  $y \notin P$  and therefore  $x$  is a boundary point.  $\square$

**Theorem 1.3.2.** *For a convex set  $A \subset \mathbb{R}^n$ ,  $A \neq \emptyset$ , we have  $\text{rel int } A \neq \emptyset$ .*

*Proof.* If  $\dim A = k$ , then  $A$  contains  $k + 1$  affinely independent points and hence a  $k$ -simplex  $P$ . By Proposition 1.3.1,  $P$  has relative interior points  $x$ . Each such  $x$  fulfills  $x \in \text{rel int } A$ .  $\square$

Theorem 1.3.2 shows that, for the investigation of a fixed convex set  $A$ , it is useful to consider the affine hull of  $A$ , as the basic space, since then  $A$  has interior points. We will often take advantage of this fact by assuming that the affine hull of  $A$  is the whole space  $\mathbb{R}^n$ . Therefore, proofs in the following frequently start with the sentence, that we may assume (w.l.o.g.) that the convex set under consideration has dimension  $n$ .

A further consequence of convexity is that topological notions like interior or closure of a (convex) set can be expressed in purely geometric terms.



**Theorem 1.3.3.** *If  $A \subset \mathbb{R}^n$  is convex, then*

$$\text{cl } A = \{x \in \mathbb{R}^n : \exists y \in A \text{ with } (x, y] \subset A\}$$

and

$$\text{int } A = \{x \in \mathbb{R}^n : \forall y \in \mathbb{R}^n, y \neq x, \exists z \in (x, y) \text{ with } [x, z] \subset A\}.$$

Again, we first need an auxiliary result.

**Proposition 1.3.4.** *If  $A \subset \mathbb{R}^n$  is convex,  $x \in \text{cl } A$ ,  $y \in \text{relint } A$ , then  $[y, x) \subset \text{relint } A$ .*

*Proof.* As we explained above, we may assume  $\dim A = n$ . Since  $x \in \text{cl } A$ , there exists a sequence  $x_k \rightarrow x$ ,  $x_k \in A$ ,  $k = 1, 2, \dots$ . Let  $z \in (y, x)$ , i.e.  $z = \alpha y + (1 - \alpha)x$ ,  $\alpha \in (0, 1)$ . Then  $y_k := \frac{1}{\alpha}(z - (1 - \alpha)x_k)$  converges towards  $y$ , as  $k \rightarrow \infty$ . Since  $y \in \text{int } A$ , there is an open ball  $U$  around  $y$  with  $U \subset A$ . For  $k$  large enough we have  $y_k \in U$ . Then, there exists an open ball  $V$  around  $y_k$  with  $V \subset U \subset A$ . The convexity of  $A$  implies  $z \in \alpha V + (1 - \alpha)x_k \subset A$ . Since  $\alpha V + (1 - \alpha)x_k$  is open,  $z$  is in  $\text{int } A$ .  $\square$

*Proof of Theorem 1.3.3.* The case  $A = \emptyset$  is trivial, hence we assume now that  $A \neq \emptyset$ .

Concerning the first equation, we may assume  $\dim A = n$  since the sets on both sides depend only on  $\text{aff } A$ . Let  $B$  be the set on the right hand side, then we obviously have  $B \subset \text{cl } A$ . To show the converse inclusion, let  $x \in \text{cl } A$ . By Theorem 1.3.2 there is a point  $y \in \text{int } A$ , hence by Proposition 1.3.4 we have  $[y, x) \subset \text{int } A \subset A$ . Therefore,  $x \in B$ .

The second equation is trivial for  $\dim A < n$  since then both sides are empty. Hence, let  $\dim A = n$ . We denote the set on the right hand side by  $C$ , then the inclusion  $\text{int } A \subset C$  is obvious. For the converse, let  $x \in C$ . Again, we choose  $y \in \text{int } A$  by Theorem 1.3.2,  $y \neq x$ . The definition of  $C$  implies that there exists  $z \in A$  with  $x \in (y, z)$ . Proposition 1.3.4 then shows that  $x \in \text{int } A$ .  $\square$

**Remark.** Theorem 1.3.3 shows that (and how) topological notions like the interior and the closure of a set can be defined for convex sets  $A$  on a purely algebraic basis, without that a topology has to be given in the underlying space. This can be used in arbitrary real vector spaces  $V$  (without a given topology) to introduce and study topological properties of convex sets.

**Corollary 1.3.5.** *For convex  $A \subset \mathbb{R}^n$ , the sets  $\text{relint } A$  and  $\text{cl } A$  are convex.*

*Proof.* In view of the above remark, we deduce the results from Theorem 1.3.3, instead of giving a direct proof based on the topological notions  $\text{relint}$  and  $\text{cl}$ .

The convexity of  $\text{relint } A$  follows immediately from Proposition 1.3.4.

For the convexity of  $\text{cl } A$ , let  $A \neq \emptyset$ ,  $x_1, x_2 \in \text{cl } A$ ,  $\alpha \in (0, 1)$ . From Theorem 1.3.3, we get points  $y_1, y_2 \in A$  with  $(x_1, y_1] \subset A$ ,  $(x_2, y_2] \subset A$ . Hence

$$\alpha(x_1, y_1] + (1 - \alpha)(x_2, y_2] \subset A.$$

Since

$$(\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_2] \subset \alpha(x_1, y_1] + (1 - \alpha)(x_2, y_2],$$

we obtain  $\alpha x_1 + (1 - \alpha)x_2 \in \text{cl } A$ , again from Theorem 1.3.3.  $\square$

**Corollary 1.3.6.** *For convex  $A \subset \mathbb{R}^n$ , we have*

$$\text{cl } A = \text{cl rel int } A$$

and

$$\text{rel int } A = \text{rel int cl } A.$$

*Proof.* The inclusion

$$\text{cl rel int } A \subset \text{cl } A$$

is obvious. Let  $x \in \text{cl } A$ . By Theorem 1.3.2 there is a  $y \in \text{rel int } A$  and by Proposition 1.3.4 we have  $[y, x) \subset \text{rel int } A$ . Since  $\text{rel int } A$  is convex (Corollary 1.3.5), Theorem 1.3.3 implies  $x \in \text{cl rel int } A$ .

The inclusion

$$\text{rel int } A \subset \text{rel int cl } A$$

is again obvious. Let  $x \in \text{rel int cl } A$ . Since  $\text{cl } A$  is convex (Corollary 1.3.5), we can apply Theorem 1.3.3 to  $\text{cl } A$ . Therefore, for  $y \in \text{rel int } A$ ,  $y \neq x$ , (which exists by Theorem 1.3.2), we obtain  $z \in \text{cl } A$  such that  $x \in (y, z)$ . By Proposition 1.3.4,  $x \in \text{rel int } A$ .  $\square$

We finally study the topological properties of the convex hull operator. For a closed set  $A \subset \mathbb{R}^n$ , the convex hull  $\text{conv } A$  need not be closed. A simple example is given by the set

$$A := \{(t, \frac{1}{t}) : t > 0\} \cup \{(0, 0)\} \subset \mathbb{R}^2.$$

However, the convex hull operator behaves well with respect to open and compact sets.

**Theorem 1.3.7.** *If  $A \subset \mathbb{R}^n$  is open,  $\text{conv } A$  is open. If  $A \subset \mathbb{R}^n$  is compact,  $\text{conv } A$  is compact.*

*Proof.* Let  $A$  be open and  $x \in \text{conv } A$ ,  $x = \alpha_1 x_1 + \cdots + \alpha_k x_k$ ,  $x_i \in A$ ,  $\alpha_i \in [0, 1]$ ,  $\sum \alpha_i = 1$ . We can choose a ball  $U$  around the origin such that  $x_i + U \subset A$ ,  $i = 1, \dots, k$ . Since

$$\alpha_1(U + x_1) + \cdots + \alpha_k(U + x_k) = U + x \subset \text{conv } A,$$

we have  $x \in \text{int conv } A$ , hence  $\text{conv } A$  is open.

Now let  $A$  be compact. Since  $A$  is contained in a ball  $B(r)$ , we have  $\text{conv } A \subset B(r)$ , i.e.  $\text{conv } A$  is bounded. In order to show that  $\text{conv } A$  is closed, let  $x_k \rightarrow x$ ,  $x_k \in \text{conv } A$ ,  $k = 1, 2, \dots$ . By Theorem 1.2.4, each  $x_k$  has a representation

$$x_k = \alpha_{k0}x_{k0} + \dots + \alpha_{kn}x_{kn}$$

with

$$\alpha_{k0}, \dots, \alpha_{kn} \in [0, 1]$$

and

$$\sum_{i=0}^n \alpha_{ki} = 1.$$

Because  $A$  and  $[0, 1]$  are compact, we find a subsequence  $(k_r)_{r \in \mathbb{N}}$  in  $\mathbb{N}$  such that the  $2n + 2$  sequences  $(x_{k_r j})_{r \in \mathbb{N}}, j = 0, \dots, n$ , and  $(\alpha_{k_r j})_{r \in \mathbb{N}}, j = 0, \dots, n$ , all converge. We denote the limits by  $y_j$  and  $\beta_j$ ,  $j = 0, \dots, n$ . Then,  $y_j \in A$ ,  $\beta_j \in [0, 1]$ ,  $\sum \beta_j = 1$  and  $x = \beta_0 y_0 + \dots + \beta_n y_n$ . Hence,  $x \in \text{conv } A$ .  $\square$

**Remark.** The last theorem shows, in particular, that a convex polytope  $P$  is compact; a fact, which can of course be proved in a simpler, more direct way.

### Exercises and problems

1. Let  $P = \text{conv} \{a_0, \dots, a_n\}$  be an  $n$ -simplex in  $\mathbb{R}^n$  and  $x \in \text{int } P$ .

Show that the polytopes

$$P_i := \text{conv} \{a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n\}, \quad i = 0, \dots, n,$$

are  $n$ -simplices with pairwise disjoint interiors and that

$$P = \bigcup_{i=0}^n P_i.$$

2. Show that, for  $A \subset \mathbb{R}^n$ ,

$$\text{cl conv } A = \bigcap \{B \subset \mathbb{R}^n : B \supset A, B \text{ closed and convex}\}.$$

3. Let  $A, B \subset \mathbb{R}^n$  be convex.

- (a) Show that  $\text{rel int } (A + B) = \text{rel int } A + \text{rel int } B$ .
- (b) If  $A$  (or  $B$ ) is bounded, show that  $\text{cl } (A + B) = \text{cl } A + \text{cl } B$ .
- (c) Show by an example that (b) is wrong, if neither  $A$  nor  $B$  are assumed to be bounded.

4. Let  $A, B \subset \mathbb{R}^n$  be convex,  $A$  closed,  $B$  compact. Show that  $A + B$  is closed (and convex). Give an example, that shows that the compactness of one of the sets  $A, B$  is necessary for this statement.

## 1.4 Support and separation theorems

Convex sets are sets which contain with their elements also all convex combinations. In this section, we consider a description of convex sets which is of a dual nature, in that it describes convex sets  $A$  as intersections of half-spaces. For such a result, we have to assume that  $A$  is a closed set.

We start with results on the metric projection which are of independent interest.

**Theorem 1.4.1.** *Let  $A \subset \mathbb{R}^n$  be nonempty, convex and closed. Then for each  $x \in \mathbb{R}^n$ , there is a unique point  $p(A, x) \in A$  satisfying*

$$\|p(A, x) - x\| = \inf_{y \in A} \|y - x\|.$$

**Definition.** The mapping  $p(A, \cdot) : \mathbb{R}^n \rightarrow A$  is called the *metric projection* (onto  $A$ ).

*Proof.* For  $x \in A$ , we obviously have  $p(A, x) = x$ . For  $x \notin A$ , there is a ball  $B(r)$  such that

$$A \cap (x + B(r)) \neq \emptyset.$$

Then,

$$\inf_{y \in A} \|y - x\| = \inf_{y \in A \cap (x + B(r))} \|y - x\|.$$

Since  $A_r := A \cap (x + B(r))$  is compact and  $f : y \mapsto \|y - x\|$  continuous, there is a point  $p(A, x) := y_0 \in A$  realizing the minimum of  $f$  on  $A_r$ .

If  $y_1 \in A$  is a second point realizing this minimum, with  $y_1 \neq y_0$ , then  $y_2 := \frac{1}{2}(y_0 + y_1) \in A$  and  $\|y_2 - x\| < \|y_0 - x\|$ , by the triangle inequality. This is a contradiction and hence the metric projection  $p(A, x)$  is unique.  $\square$

**Remark.** As the above proof shows, the existence of a nearest point  $p(A, x)$  is guaranteed for all closed sets  $A$ . The convexity of  $A$  is responsible for the uniqueness of  $p(A, x)$ . A more general class of sets consists of closed sets  $A$ , for which the uniqueness of  $p(A, x)$  holds at least in a neighborhood of  $A$ , for all  $x \in A + rB^n$ ,  $r > 0$ . Such sets are called of *positive reach*, and the largest  $r$  for which uniqueness of the metric projection holds is called the *reach* of  $A$ . Convex sets thus have reach  $\infty$ .

**Definition.** Let  $A \subset \mathbb{R}^n$  be closed and convex and  $E = \{f = \alpha\}$  a hyperplane.  $E$  is called *supporting hyperplane* of  $A$ , if  $A \cap E \neq \emptyset$  and  $A$  is contained in one of the two closed half-spaces  $\{f \leq \alpha\}$ ,  $\{f \geq \alpha\}$  (or in both, but this implies  $A \subset \{f = \alpha\}$ , hence it is only possible for lower dimensional sets  $A$ ). The corresponding half-space containing  $A$  is called *supporting half-space*, the set  $A \cap E$  is called *support set* and any  $x \in A \cap E$  is called *supporting point*.

If  $E$  is a supporting hyperplane of  $A$ , we also say shortly that the hyperplane  $E$  supports  $A$ .

**Example.** The set

$$A := \{(x^{(1)}, x^{(2)}) \in \mathbb{R}^2 : x^{(2)} \geq \frac{1}{x^{(1)}}, x^{(1)} > 0\}$$

is closed and convex. The line  $g := \{x^{(1)} + x^{(2)} = 2\}$  is a supporting line, since  $(1, 1) \in A \cap g$  and  $A \subset \{x^{(1)} + x^{(2)} \geq 2\}$ . The lines  $h := \{x^{(1)} = 0\}$  and  $k := \{x^{(2)} = 0\}$  bound the set  $A$ , but are not supporting lines since they do not have a point in common with  $A$ .

**Theorem 1.4.2.** *Let  $A \subset \mathbb{R}^n$  be nonempty, closed and convex and let  $x \in \mathbb{R}^n \setminus A$ . Then, the hyperplane  $E$  through  $p(A, x)$ , orthogonal to  $x - p(A, x)$ , supports  $A$ . Moreover, the half-space bounded by  $E$  and not containing  $x$  is a supporting half-space.*

*Proof.* Obviously  $x \notin E$ . Let  $H$  be the half-space generated by  $E$  which fulfills  $x \notin H$ . Since  $p(A, x) \in E \cap A$ , it remains to show that  $A \subset H$ . Assume that there is  $y \in A, y \notin H$ . Then  $\langle y - p(A, x), x - p(A, x) \rangle > 0$ .

We make use of the following projection property:

(\*) Let  $x, y, z \in \mathbb{R}^n$  be different points with  $\langle y - z, x - z \rangle > 0$ . Then there exists  $y' \in (z, y]$  with  $\|y' - x\| < \|z - x\|$ .

In order to show (\*) we consider the orthogonal projection  $\bar{y}$  of  $x$  onto the line through  $z$  and  $y$ . By Pythagoras' theorem,  $\|\bar{y} - x\| < \|z - x\|$ . If  $\bar{y} \in (z, y]$ , we put  $y' := \bar{y}$ . Otherwise, we have  $y \in (z, \bar{y}]$  and put  $y' := y$ .

We apply (\*) with  $z = p(A, x)$  and obtain a point  $y' \in (p(A, x), y] \subset A$  with  $\|y' - x\| < \|p(A, x) - x\|$ . This is a contradiction, hence we conclude  $A \subset H$ .  $\square$

**Corollary 1.4.3.** *Every nonempty, closed convex set  $A \subset \mathbb{R}^n$ ,  $A \neq \mathbb{R}^n$ , is the intersection of all closed half-spaces which contain  $A$ . More specifically,  $A$  is the intersection of all its supporting half-spaces.*

*Proof.* Obviously,  $A$  lies in the intersection  $B$  of its supporting half-spaces. For  $x \notin A$ , Theorem 1.4.2 implies the existence of a supporting half-space  $H$  of  $A$  with  $x \notin H$ . Hence  $x \notin B$ .  $\square$

Theorem 1.4.2 and Corollary 1.4.3 do not imply that every boundary point of  $A$  is a support point. In order to show such a result, we approximate  $x \in \text{bd } A$  by points  $x_k$  from  $\mathbb{R}^n \setminus A$  and consider the corresponding supporting hyperplanes  $E_k$  which exist by Theorem 1.4.2. For  $x_k \rightarrow x$ , we want to define a supporting hyperplane in  $x$  as the limit of the  $E_k$ . A first step in this direction is to show that  $p(A, x_k) \rightarrow p(A, x)$  (where  $p(A, x) = x$ ), hence to show that  $p(A, \cdot)$  is continuous. We even show now that  $p(A, \cdot)$  is Lipschitz continuous.

**Theorem 1.4.4.** *Let  $A \subset \mathbb{R}^n$  be nonempty, closed and convex. Then,*

$$\|p(A, x) - p(A, y)\| \leq \|x - y\|,$$

for all  $x, y \in \mathbb{R}^n$ .

*Proof.* The assertion is trivial for  $p(A, x) = p(A, y)$ . Hence, we now assume  $p(A, x) \neq p(A, y)$ . Let  $E_x, E_y$  be the hyperplanes through  $p(A, x)$ , respectively  $p(A, y)$ , and orthogonal to  $p(A, x) - p(A, y)$ . Let  $H_x, H_y$  be the corresponding disjoint closed half-spaces bounded by  $E_x$ , respectively  $E_y$ . If  $x \notin H_x$ , then (\*) implies the existence of a point in  $(p(A, x), p(A, y)] \subset A$ , which has smaller distance to  $x$  than  $p(A, x)$ . This is a contradiction, therefore  $x \in H_x$ . Analogously, we obtain  $y \in H_y$  and therefore  $\|x - y\| \geq d(E_x, E_y) = \|p(A, x) - p(A, y)\|$ .  $\square$

**Theorem 1.4.5 (Support Theorem).** *Let  $A \subset \mathbb{R}^n$  be closed and convex. Then through each boundary point of  $A$  there exists a supporting hyperplane.*

*Proof.* For given  $x \in \text{bd } A$ , we consider the closed unit ball  $x + B(1)$  around  $x$ . For each  $k \in \mathbb{N}$ , we choose  $x_k \in x + B(1)$ ,  $x_k \notin A$ , and such that  $\|x - x_k\| < \frac{1}{k}$ . Then

$$\|x - p(A, x_k)\| \leq \|x - x_k\| < \frac{1}{k},$$

by Theorem 1.4.4. Since  $x_k, p(A, x_k)$  are interior points of  $x + B(1)$ , there is a (unique) boundary point  $y_k$  in  $x + B(1)$  such that  $x_k \in (p(A, x_k), y_k)$ . Theorem 1.4.2 then implies  $p(A, y_k) = p(A, x_k)$ . In view of the compactness of  $x + B(1)$ , we may choose a converging subsequence  $y_{k_r} \rightarrow y$ . By Theorem 1.4.4,  $p(A, y_{k_r}) \rightarrow p(A, y)$  and  $p(A, y_{k_r}) = p(A, x_{k_r}) \rightarrow p(A, x) = x$ , hence  $p(A, y) = x$ . Since  $y \in \text{bd } (x + B(1))$ , we also know that  $x \neq y$ . The assertion now follows from Theorem 1.4.2.  $\square$

**Remarks.** (1) Supporting hyperplanes, half-spaces and points can be defined for nonconvex sets  $A$  as well; they only exist however, if  $\text{conv } A$  is closed and not all of  $\mathbb{R}^n$ . Then,  $\text{conv } A$  is the intersection of all supporting half-spaces of  $A$ .

(2) Some of the previous results can be interpreted as separation theorems. Theorem 1.4.2 says that a closed convex set  $A$  and a point  $x \notin A$  can be separated by a hyperplane (even strongly, since there is a separating hyperplane which has a positive distance to both,  $A$  and  $x$ ). Theorem 1.4.5 says that each boundary point of  $A$  can be separated from  $A$  by a hyperplane. Both results can be extended to compact convex sets  $B$  (instead of the point  $x$ ).

(3) In topological vector spaces  $V$  of infinite dimensions similar support and separation theorems hold true, however there are some important differences, mainly due to the fact that convex sets  $A$  in  $V$  need not have relative interior points. Therefore a common assumption is that  $\text{int } A \neq \emptyset$ . Otherwise it is possible that  $A$  is closed but does not have any support points, or, in the other direction, that every point of  $A$  is a support point (although  $A$  does not lie in a hyperplane).

(4) Some of the properties which we derived are characteristic for convexity. For example, a closed set  $A \subset \mathbb{R}^n$  such that each  $x \notin A$  has a unique metric projection onto  $A$ , must be convex (Motzkin's Theorem). Also the Support Theorem has a converse. A closed set  $A \subset \mathbb{R}^n$ ,  $\text{int } A \neq \emptyset$ , such that each boundary point is a support point, must also be convex.

(5) Let  $A \subset \mathbb{R}^n$  be nonempty, closed and convex. Then we have, for each direction  $u \in S^{n-1}$ , a supporting hyperplane  $E(u)$  of  $A$  in direction  $u$  (i.e. with outer normal  $u$ ), if and only if  $A$  is compact.

For the rest of this section, we consider convex polytopes and show that for a polytope  $P$  finitely many supporting half-spaces suffice to generate  $P$  (as the intersection). In other words, we show that polytopes are polyhedral sets. First, we introduce the faces of a polytope.

**Definition.** The support sets of a polytope  $P$  are called *faces*. A face  $F$  of  $P$  is called a  $k$ -*face*, if  $\dim F = k$ ,  $k \in \{0, \dots, n-1\}$ .

**Theorem 1.4.6.** *The 0-faces of a polytope  $P \subset \mathbb{R}^n$  are given by the vertices of  $P$ , i.e. they are of the form  $\{x\}$ ,  $x \in \text{vert } P$ .*

*Proof.* Let  $\{x\}$  be a 0-face of  $P$  and assume  $x \notin \text{vert } P$ . Then, there exists a supporting hyperplane  $\{f = \alpha\}$  with  $f(x) = \alpha$ , but  $f(y) < \alpha$ , for all  $y \in \text{vert } P$ . Since  $\text{vert } P$  is finite, there is a  $\beta < \alpha$  with  $f(y) \leq \beta$ , for all  $y \in \text{vert } P$ . This implies  $f(z) \leq \beta$ , for all  $z \in \text{conv } \text{vert } P = P$ . Since  $x \in P$ , this is a contradiction, hence we have  $x \in \text{vert } P$ .

Conversely, let  $x \in \text{vert } P$  and let  $\text{vert } P \setminus \{x\} = \{x_1, \dots, x_k\}$ . Then,  $x \notin P' := \text{conv } \{x_1, \dots, x_k\}$ . By Theorem 1.4.2 there exists a supporting hyperplane  $\{f = \alpha\}$  of  $P'$  through  $p(P', x)$  with supporting half-space  $\{f \leq \alpha\}$  and such that  $f(x) > \alpha$ . Consequently, the parallel hyperplane  $\{f = \beta\}$ , where  $\beta := f(x) > \alpha$ , is supporting hyperplane to  $P$ . Consider  $y \in P \cap \{f = \beta\}$ . Since  $y = \alpha_1 x_1 + \dots + \alpha_k x_k + \alpha_{k+1} x$  with  $\alpha_i \geq 0$  and  $\sum \alpha_i = 1$ , we have

$$\beta = f(y) = \alpha_1 f(x_1) + \dots + \alpha_k f(x_k) + \alpha_{k+1} \beta$$

and hence

$$(1 - \alpha_{k+1})\beta \leq (1 - \alpha_{k+1})\alpha.$$

This is only possible if  $\alpha_{k+1} = 1$ , hence  $y = x$ . Therefore  $P \cap \{f = \beta\} = \{x\}$ .  $\square$

**Remark.** In Section 1.1, the vertices of a polytope  $P = \text{conv } \{x_1, \dots, x_k\}$  have been defined as those points in  $\{x_1, \dots, x_k\}$  which are not in the convex hull of the others. At that stage, it was not clear whether this definition may depend on the representation of  $P$ . Theorem 1.4.6 now shows that the set of vertices is in fact independent of the representation  $P = \text{conv } \{x_1, \dots, x_k\}$ .



**Definition.** The 1-faces of a polytope are called *edges*, and the  $(n - 1)$ -faces are called *facets*.

**Remark.** In the following, we shall not distinguish between 0-faces and vertices anymore, although one is a set and the other is a point.

**Theorem 1.4.7.** *Let  $P \subset \mathbb{R}^n$  be a polytope with  $\text{vert } P = \{x_1, \dots, x_k\}$  and let  $F$  be a face of  $P$ . Then,  $F = \text{conv} \{x_i : x_i \in F\}$ .*

*Proof.* Assume  $F = P \cap \{f = \alpha\}$  and, w.l.o.g.,  $x_1, \dots, x_m \in F$  and  $x_{m+1}, \dots, x_k \notin F$ . If  $\{f \leq \alpha\}$  is the supporting half-space, we have

$$x_{m+1}, \dots, x_k \in \{f < \alpha\},$$

i.e.  $f(x_j) = \alpha - \delta_j$ ,  $\delta_j > 0$ ,  $j = m + 1, \dots, k$ .

Let  $x \in P$ ,  $x = \alpha_1 x_1 + \dots + \alpha_k x_k$ ,  $\alpha_i \geq 0$ ,  $\sum \alpha_i = 1$ . Then,

$$\begin{aligned} f(x) &= \alpha_1 f(x_1) + \dots + \alpha_k f(x_k) \\ &= \alpha - \alpha_{m+1} \delta_{m+1} - \dots - \alpha_k \delta_k. \end{aligned}$$

Hence,  $x \in F$ , if and only if  $\alpha_{m+1} = \dots = \alpha_k = 0$ . □

**Remark.** Theorem 1.4.7 implies, in particular, that a face of a polytope is a polytope and that there are only finitely many faces.

**Corollary 1.4.8.** *A polytope  $P$  is polyhedral.*

*Proof.* If  $\dim P = k < n$ , we can assume w.l.o.g. that  $0 \in E := \text{aff } P$ . Also, it is possible to write  $E$  as an intersection of half-spaces  $\tilde{H}_1, \dots, \tilde{H}_r$  in  $\mathbb{R}^n$ ,  $E = \bigcap_{j=1}^r \tilde{H}_j$ . If  $P$  is polyhedral in  $E$ , i.e.

$$P = \bigcap_{i=1}^m H_i,$$

where  $H_i \subset E$  are  $k$ -dimensional half-spaces, then

$$P = \bigcap_{i=1}^m (H_i \oplus E^\perp) \cap \bigcap_{j=1}^r \tilde{H}_j,$$

hence  $P$  is polyhedral in  $\mathbb{R}^n$ . Therefore, it is sufficient to treat the case  $\dim P = n$ .

Let  $F_1, \dots, F_m$  be the faces of  $P$  and  $H_1, \dots, H_m$  corresponding supporting half-spaces (i.e. half-spaces with  $P \subset H_i$  and  $F_i = P \cap \text{bd } H_i$ ,  $i = 1, \dots, m$ ). Then we have

$$P \subset H_1 \cap \dots \cap H_m =: P'.$$

Assume, there is  $x \in P' \setminus P$ . We choose  $y \in \text{int } P$  and consider  $[y, x] \cap P$ . Since  $P$  is compact and convex (and  $x \notin P$ ), there is  $z \in (y, x)$  with  $\{z\} = [y, x] \cap \text{bd } P$ . By the support theorem there is a supporting hyperplane of  $P$  through  $z$ , and hence there is a face  $F_i$  of  $P$  with  $z \in F_i$ . Since each  $F_i$  lies in the boundary of  $P'$ , we have  $z \in \text{bd } P'$ . On the other hand, Proposition 1.3.4 shows that  $z \in \text{int } P'$ , a contradiction. □

**Exercises and problems**

1. Let  $A \subset \mathbb{R}^n$  be closed and  $\text{int } A \neq \emptyset$ . Show that  $A$  is convex, if and only if every boundary point of  $A$  is a support point.
- \* 2. Let  $A \subset \mathbb{R}^n$  be closed. Suppose that for each  $x \in \mathbb{R}^n$  the metric projection  $p(A, x)$  onto  $A$  is uniquely determined. Show that  $A$  is convex (MOTZKIN's theorem).
3. Let  $A \subset \mathbb{R}^n$  be non-empty, closed and convex. Show that  $A$  is compact, if and only if, for any direction  $u \in S^{n-1}$ , there is a supporting hyperplane  $E(u)$  of  $A$  in direction  $u$  (i.e. with outer normal  $u$ ).
4. Let  $A, B \subset \mathbb{R}^n$  be non-empty and convex, and assume  $\text{rel int } A \cap \text{rel int } B = \emptyset$ .  
Show that there is a hyperplane  $\{f = \alpha\}$ , separating  $A$  and  $B$  (i.e. such that  $A \subset \{f \leq \alpha\}$  and  $B \subset \{f \geq \alpha\}$ ).  
*Hint:* Use exercise 2(a) in 1.3 to show that the origin 0 can be separated from  $A - B$ .
5. Let  $A, K \subset \mathbb{R}^n$  be convex,  $A$  closed,  $K$  compact, and assume  $A \cap K = \emptyset$ .  
Show that there is a hyperplane  $\{f = \alpha\}$  with  $A \subset \{f < \alpha\}$  and  $B \subset \{f > \alpha\}$ . Show more generally that  $\alpha$  can be chosen such that there is an  $\epsilon > 0$  with  $A \subset \{f \leq \alpha - \epsilon\}$  and  $B \subset \{f \geq \alpha + \epsilon\}$  (*strong separation*).
6. A bavarian farmer is happy owner of a large herd of happy cows, consisting of totally black and totally white animals. One day he finds them sleeping in the sun on his largest meadow. Watching them, he notices that, for any four cows it would be possible to build a straight fence, separating the black cows from the white ones.  
Show that the farmer could build a straight fence, separating the whole herd into black and white animals.  
*Hint:* Cows are lazy. When they sleep, they sleep - even if you build a fence across the meadow.
7. Let  $F_1, \dots, F_m$  be the facets of the polytope  $P$  and  $H_1, \dots, H_m$  the corresponding supporting half-spaces. Show that

$$(*) \quad P = \bigcap_{i=1}^m H_i.$$

(This is a generalization of the representation shown in the proof of Corollary 1.4.8.) Show further, that the representation (\*) is minimal in the sense that, for each representation

$$P = \bigcap_{i \in I} \tilde{H}_i,$$

with a family of half-spaces  $\{\tilde{H}_i : i \in I\}$ , we have  $\{H_1, \dots, H_m\} \subset \{\tilde{H}_i : i \in I\}$ .

## 1.5 Extremal representations

In the previous section we have seen that the trivial representation of closed convex sets  $A \subset \mathbb{R}^n$  as intersection of all closed convex sets containing  $A$  can be improved to a nontrivial one, where  $A$  is represented as the intersection of the supporting half-spaces. On the other hand, we have the trivial representation of  $A$  as the set of all convex combinations of points of  $A$ . Therefore, we discuss now the similar nontrivial problem to find a subset  $B \subset A$ , as small as possible, for which  $A = \text{conv } B$  holds. Although there are some general results for closed convex sets  $A$ , we will concentrate on the compact case, where we can give a complete (and simple) solution for this problem.

**Definition.** Let  $A \subset \mathbb{R}^n$  be closed and convex. A point  $x \in A$  is called *extreme*, if  $x$  cannot be represented as a nontrivial convex combination of points of  $A$ , i.e. if  $x = \alpha y + (1 - \alpha)z$  with  $y, z \in A$ ,  $\alpha \in (0, 1)$ , implies that  $x = y = z$ . The set of all extreme points of  $A$  is denoted by  $\text{ext } A$ .

**Remarks.** (1) If  $A$  is a closed half-space,  $\text{ext } A = \emptyset$ . In general,  $\text{ext } A \neq \emptyset$ , if and only if  $A$  does not contain any lines.

(2) We have  $x \in \text{ext } A$ , if and only if  $A \setminus \{x\}$  is convex.

(3) If  $\{x\}$  is a support set of  $A$ , then  $x \in \text{ext } A$ . The converse is false.

**Definition.** Let  $A \subset \mathbb{R}^n$  be closed and convex. A point  $x \in A$  is called *exposed*, if  $\{x\}$  is a support set of  $A$ . The set of all exposed points of  $A$  is denoted by  $\text{exp } A$ .

**Remark.** In view of Remark (3) above, we have  $\text{exp } A \subset \text{ext } A$ .

**Theorem 1.5.1 (MINKOWSKI).** *Let  $K \subset \mathbb{R}^n$  be compact and convex and  $A \subset K$ . Then,  $K = \text{conv } A$ , if and only if  $\text{ext } K \subset A$ .*

*In particular,  $K = \text{conv ext } K$ .*

*Proof.* Suppose  $K = \text{conv } A$  and  $x \in \text{ext } K$ . By Theorem 1.1.2, there is a representation

$$x = \alpha_1 x_1 + \cdots + \alpha_k x_k$$

with  $k \in \mathbb{N}$ ,  $x_i \in A$ ,  $\alpha_i > 0$ , and  $\sum \alpha_i = 1$ . In case  $k = 1$ , we have  $x = x_1 \in A$ . If  $k \geq 2$ ,

$$x = \alpha_1 x_1 + (1 - \alpha_1) \frac{\alpha_2 x_2 + \cdots + \alpha_k x_k}{\alpha_2 + \cdots + \alpha_k}$$

and

$$\frac{\alpha_2 x_2 + \cdots + \alpha_k x_k}{\alpha_2 + \cdots + \alpha_k} \in K.$$

Since  $x$  is extreme, we obtain  $x = x_1 \in A$ . Thus, in both cases we have  $x \in A$ , therefore  $\text{ext } K \subset A$ .

In the other direction, we need only show that  $K = \text{conv ext } K$ . We prove this by induction on  $n$ . For  $n = 1$ , a compact convex subset of  $\mathbb{R}^1$  is a segment  $[a, b]$  and  $\text{ext } [a, b] = \{a, b\}$ .

Let  $n \geq 2$  and suppose the result holds in dimension  $n - 1$ . Since  $\text{ext } K \subset K$ , we obviously have  $\text{conv ext } K \subset K$ . We need to show the opposite inclusion. For that purpose, let  $x \in K$  and  $g$  an arbitrary line through  $x$ . Then  $g \cap K = [y, z]$  with  $x \in [y, z]$  and  $y, z \in \text{bd } K$ . By the support theorem,  $y, z$  are support points, i.e. there are supporting hyperplanes  $E_y, E_z$  of  $K$  with  $y \in K_1 := E_y \cap K$  and  $z \in K_2 := E_z \cap K$ . By the induction hypothesis,

$$K_1 = \text{conv ext } K_1; \quad K_2 = \text{conv ext } K_2.$$

We have  $\text{ext } K_1 \subset \text{ext } K$ . Namely, consider  $u \in \text{ext } K_1$  and  $u = \alpha v + (1 - \alpha)w$ ,  $v, w \in K$ ,  $\alpha \in (0, 1)$ . Since  $u$  lies in the supporting hyperplane  $E_y$ , the same must hold for  $v$  and  $w$ . Hence  $v, w \in K_1$  and since  $u \in \text{ext } K_1$ , we obtain  $u = v = w$ . Therefore,  $u \in \text{ext } K$ .

In the same way, we get  $\text{ext } K_2 \subset \text{ext } K$  and thus

$$\begin{aligned} x \in [y, z] &\subset \text{conv } \{\text{conv ext } K_1 \cup \text{conv ext } K_2\} \\ &\subset \text{conv ext } K. \end{aligned}$$

□

**Corollary 1.5.2.** *Let  $P \subset \mathbb{R}^n$  be compact and convex.*

- (a) *If  $P$  is a polytope, then  $\text{ext } P = \text{vert } P$ .*
- (b)  *$P$  is a polytope, if and only if  $\text{ext } P$  is finite.*

*Proof.* (a) Since the vertices are 0-faces, we have  $\text{vert } P \subset \text{ext } P$ . Since  $P = \text{conv vert } P$ , Theorem 1.5.1 implies the opposite inclusion  $\text{ext } P \subset \text{vert } P$ .

(b) One direction follows from (a). For the converse, assume that  $\text{ext } P$  is finite, hence  $\text{ext } P = \{x_1, \dots, x_k\}$ . Theorem 1.5.1 then shows  $P = \text{conv } \{x_1, \dots, x_k\}$ , hence  $P$  is a polytope. □

Now we are able to prove a converse of Corollary 1.4.8.

**Theorem 1.5.3.** *Let  $P \subset \mathbb{R}^n$  be a bounded polyhedral set. Then  $P$  is a polytope.*

*Proof.*  $P$  is compact. We show that  $\text{ext } P$  is finite.

Let  $x \in \text{ext } P$  and assume  $P = \bigcap_{i=1}^k H_i$  with half-spaces  $H_i$  bounded by the hyperplanes  $E_i$ ,  $i = 1, \dots, k$ . We consider the convex set

$$D := \bigcap_{i=1}^k A_i,$$

where

$$A_i = \begin{cases} E_i & \text{if } x \in E_i, \\ \text{int } H_i & \text{if } x \notin E_i. \end{cases}$$

We have  $x \in D \subset P$ . Since  $x$  is extreme and  $D$  is relatively open, we get  $\dim D = 0$ , hence  $D = \{x\}$ . Since there are only finitely many different sets  $D$  possible,  $\text{ext } P$  must be finite. The result now follows from Corollary 1.5.2(b).  $\square$

**Remark.** This result now shows that the intersection of finitely many polytopes is again a polytope.

If we replace, in Theorem 1.5.1 the set  $\text{ext } K$  by  $\text{exp } K$ , the corresponding result will be wrong (as follows from Theorem 1.5.1 and the remark preceding it). There is however a modified version which holds true for exposed points.

**Theorem 1.5.4.** *Let  $K \subset \mathbb{R}^n$  be compact and convex. Then*

$$K = \text{cl conv exp } K.$$

*Proof.* Since  $K$  is compact, there exists, for each  $x \in \mathbb{R}^n$ , a point  $y_x \in K$  farthest away, i.e. a point with

$$\|y_x - x\| = \max_{y \in K} \|y - x\|.$$

The hyperplane  $E$  through  $y_x$  orthogonal to  $y_x - x$  is then a supporting hyperplane of  $K$ .

We have  $E \cap K = \{y_x\}$ , hence  $y_x \in \text{exp } K$ . Let

$$\hat{K} := \text{cl conv } \{y_x : x \in \mathbb{R}^n\}.$$

Then  $\hat{K} \subset K$ , thus  $\hat{K}$  is compact.

Assume that there exists  $x \in K \setminus \hat{K}$ . Then, by Theorem 1.4.2 there is a hyperplane  $E' = \{f = \alpha\}$  with  $x \in \{f > \alpha\}$  and  $\hat{K} \subset \{f < \alpha\}$  ( $E'$  is obtained by a suitable translation of the supporting hyperplane through  $p(\hat{K}, x)$  in direction  $x - p(\hat{K}, x)$ ). Consider the half-line  $s$  starting in  $x$ , orthogonal to  $E'$  and in direction of that half-space of  $E'$ , which contains  $\hat{K}$ . On  $s$ , we can find a point  $z$  with  $\|x - z\| > \max_{y \in \hat{K}} \|y - z\|$  (e.g. we may choose a cube  $W$  large enough to contain  $\hat{K}$ , and such that the point  $\bar{x}$  with  $\{\bar{x}\} := s \cap E'$  is the center of a facet of  $W$ . Now we choose a ball  $B$  with center  $z \in s$  in such a way that  $W \subset B$ , but  $x \notin B$ . Then  $z$  is the required point).

By definition of  $\hat{K}$ , there exists  $y_z \in \hat{K}$  with

$$\|y_z - z\| = \max_{y \in \hat{K}} \|y - z\| \geq \|x - z\|,$$

a contradiction. Therefore,  $K = \hat{K}$ . Because of  $y_x \in \text{exp } K$ , for all  $x \in \mathbb{R}^n$ , we obtain

$$K = \hat{K} \subset \text{cl conv exp } K \subset K,$$

hence  $K = \text{cl conv exp } K$ .  $\square$

**Corollary 1.5.5 (STRASZEWICZ).** *Let  $K \subset \mathbb{R}^n$  be compact and convex. Then*

$$\text{cl exp } K \supset \text{ext } K.$$

*Proof.* By Theorems 1.5.4 and 1.3.7, we have

$$K = \text{cl conv exp } K \subset \text{cl conv cl exp } K = \text{conv cl exp } K \subset K,$$

hence

$$K = \text{conv cl exp } K.$$

By Theorem 1.5.1, this implies  $\text{ext } K \subset \text{cl exp } K$ . □

### Exercises and problems

1. Let  $A \subset \mathbb{R}^n$  be closed and convex. Show that  $\text{ext } A \neq \emptyset$ , if and only if  $A$  does not contain any line.
2. Let  $K \subset \mathbb{R}^n$  be compact and convex.
  - (a) If  $n = 2$ , show that  $\text{ext } K$  is closed.
  - (b) If  $n \geq 3$ , show by an example that  $\text{ext } K$  need not be closed.
3. Let  $A \subset \mathbb{R}^n$  be closed and convex. A subset  $M \subset A$  is called *extreme* (in  $A$ ), if  $M$  is convex and if  $x, y \in A$ ,  $(x, y) \cap M \neq \emptyset$  implies  $[x, y] \subset M$ .

Show that:

- (a) Extreme sets  $M$  are closed.
  - (b) Each support set of  $A$  is extreme.
  - (c) If  $M, N \subset A$  are extreme, then  $M \cap N$  is extreme.
  - (d) If  $M$  is extreme in  $A$  and  $N \subset M$  is extreme in  $M$ , then  $N$  is extreme in  $A$ .
  - (e) If  $M, N \subset A$  are extreme and  $M \neq N$ , then  $\text{rel int } M \cap \text{rel int } N = \emptyset$ .
  - (f) Let  $\mathcal{E}(A) := \{M \subset A : M \text{ extreme}\}$ . Then  $A = \bigcup_{M \in \mathcal{E}(A)} \text{rel int } M$ .
4. A real  $(n, n)$ -matrix  $A = ((\alpha_{ij}))$  is called *doubly stochastic*, if  $\alpha_{ij} \geq 0$  and

$$\sum_{k=1}^n \alpha_{kj} = \sum_{k=1}^n \alpha_{ik} = 1$$

for all  $i, j \in \{1, \dots, n\}$ . A doubly stochastic matrix with components in  $\{0, 1\}$  is called *permutation matrix*.

Show:

- (a) The set  $K \subset \mathbb{R}^{n^2}$  of doubly stochastic matrices is compact and convex.
- (b) The extreme points of  $K$  are precisely the permutation matrices.

*Hint for (b):* You may use the following simple combinatorial result (marriage theorem):

Given a finite set  $H$ , a nonempty set  $D$  and a function  $f : H \rightarrow \mathcal{P}(D)$  with

$$\left| \bigcup_{h \in \tilde{H}} f(h) \right| \geq |\tilde{H}|, \quad \text{for all } \tilde{H} \subset H,$$

then there exists an injective function  $g : H \rightarrow D$  with  $g(h) \in f(h)$ , for all  $h \in H$ .





# Chapter 2

## Convex functions

### 2.1 Properties and operations of convex functions

In the following, we consider functions

$$f : \mathbb{R}^n \rightarrow [-\infty, \infty].$$

We assume the usual rules for addition and multiplication with  $\infty$ , namely:

$$\begin{aligned} \alpha + \infty &:= \infty, & \text{for } \alpha \in (-\infty, \infty], \\ \alpha - \infty &:= -\infty, & \text{for } \alpha \in [-\infty, \infty), \\ \alpha \infty &:= \infty, \quad (-\alpha) \infty := -\infty, & \text{for } \alpha \in (0, \infty], \\ 0 \infty &:= 0. \end{aligned}$$

**Definition.** For a function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ , the set

$$\text{epi } f := \{(x, \alpha) : x \in \mathbb{R}^n, \alpha \in \mathbb{R}, f(x) \leq \alpha\} \subset \mathbb{R}^n \times \mathbb{R}$$

is called the *epigraph* of  $f$ .  $f$  is *convex*, if  $\text{epi } f$  is a convex subset of  $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ .

**Remarks.** (1) A function  $f : \mathbb{R}^n \rightarrow [-\infty, \infty)$  is *concave*, if  $-f$  is convex. Thus, for a convex function  $f$  we exclude the value  $-\infty$ , whereas for a concave function we exclude  $\infty$ .

(2) If  $A \subset \mathbb{R}^n$  is a subset, a function  $f : A \rightarrow (-\infty, \infty)$  is called *convex*, if the extended function  $\tilde{f} : \mathbb{R}^n \rightarrow (-\infty, \infty]$ , given by

$$\tilde{f} := \begin{cases} f & \text{on } A, \\ \infty & \text{on } \mathbb{R}^n \setminus A, \end{cases}$$

is convex. This automatically requires that  $A$  is a convex set. In view of this construction, we need not consider convex functions defined on subsets of  $\mathbb{R}^n$ , but we rather can assume that convex functions are always defined on all of  $\mathbb{R}^n$ .

(3) On the other hand, we often are only interested in convex functions  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  at points, where  $f$  is finite. We call

$$\text{dom } f := \{x \in \mathbb{R}^n : f(x) < \infty\}$$

the *effective domain* of the function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$ . For a convex function  $f$ , the effective domain  $\text{dom } f$  is convex.

(4) The function  $f \equiv \infty$  is convex, it is called the *improper* convex function; convex functions  $f$  with  $f \not\equiv \infty$  are called *proper*. The improper convex function  $f \equiv \infty$  has  $\text{epi } f = \emptyset$  and  $\text{dom } f = \emptyset$ .

**Theorem 2.1.1.** *A function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is convex, if and only if*

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),$$

for all  $x, y \in \mathbb{R}^n, \alpha \in [0, 1]$ .

*Proof.* By definition,  $f$  is convex, if and only if  $\text{epi } f = \{(x, \beta) : f(x) \leq \beta\}$  is convex. The latter condition means

$$\alpha(x_1, \beta_1) + (1 - \alpha)(x_2, \beta_2) = (\alpha x_1 + (1 - \alpha)x_2, \alpha \beta_1 + (1 - \alpha)\beta_2) \in \text{epi } f,$$

for all  $\alpha \in [0, 1]$  and whenever  $(x_1, \beta_1), (x_2, \beta_2) \in \text{epi } f$ , i.e. whenever  $f(x_1) \leq \beta_1, f(x_2) \leq \beta_2$ .

Hence,  $f$  is convex, if and only if

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha \beta_1 + (1 - \alpha)\beta_2,$$

for all  $x_1, x_2 \in \mathbb{R}^n, \alpha \in [0, 1]$  and all  $\beta_1 \geq f(x_1), \beta_2 \geq f(x_2)$ . Then, it is necessary and sufficient that this inequality is satisfied for  $\beta_1 = f(x_1), \beta_2 = f(x_2)$ , and we obtain the assertion.  $\square$

**Remarks.** (1) A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is affine, if and only if  $f$  is convex and concave. If  $f$  is affine, then  $\text{epi } f$  is a half-space in  $\mathbb{R}^{n+1}$  (and  $\text{dom } f = \mathbb{R}^n$ ).

(2) For a convex function  $f$ , the sublevel sets  $\{f < \alpha\}$  and  $\{f \leq \alpha\}$  are convex.

(3) If  $f, g$  are convex and  $\alpha, \beta \geq 0$ , then  $\alpha f + \beta g$  is convex.

(4) If  $(f_i)_{i \in I}$  is a family of convex functions, the (pointwise) supremum  $\sup_{i \in I} f_i$  is convex. This follows since

$$\text{epi} \left( \sup_{i \in I} f_i \right) = \bigcap_{i \in I} \text{epi } f_i.$$

(5) As a generalization of Theorem 2.1.1, we obtain that  $f$  is convex, if and only if

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \leq \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k),$$

for all  $k \in \mathbb{N}$ ,  $x_i \in \mathbb{R}^n$ , and  $\alpha_i \in [0, 1]$  with  $\sum \alpha_i = 1$ .

(6) A function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is *positively homogeneous* (of degree 1), if

$$f(\alpha x) = \alpha f(x), \quad \text{for all } x \in \mathbb{R}^n, \alpha \geq 0.$$

If  $f$  is positively homogeneous,  $f$  is convex if and only if it is *subadditive*, i.e. if

$$f(x + y) \leq f(x) + f(y),$$

for all  $x, y \in \mathbb{R}^n$ .

The following simple result is useful to generate convex functions from convex sets in  $\mathbb{R}^n \times \mathbb{R}$ .

**Theorem 2.1.2.** *Let  $A \subset \mathbb{R}^n \times \mathbb{R}$  be convex and suppose that*

$$f_A(x) := \inf \{ \alpha \in \mathbb{R} : (x, \alpha) \in A \} > -\infty,$$

for all  $x \in \mathbb{R}^n$ . Then,  $f_A$  is a convex function.

*Proof.* The definition of  $f_A(x)$  implies that

$$\text{epi } f_A = \{ (x, \beta) : \exists \alpha \in \mathbb{R}, \alpha \leq \beta, \text{ and a sequence } \alpha_i \searrow \alpha \text{ with } (x, \alpha_i) \in A \}.$$

It is easy to see that  $\text{epi } f_A$  is convex. □

**Remarks.** (1) The condition  $f_A > -\infty$  is fulfilled, if and only if  $A$  does not contain a vertical half-line which is unbounded from below.

(2) For  $x \in \mathbb{R}^n$ , let

$$x \times \mathbb{R} := \{ (x, \alpha) : \alpha \in \mathbb{R} \}$$

be the *vertical line* in  $\mathbb{R}^n \times \mathbb{R}$  through  $x$ . Then, we have  $A = \text{epi } f_A$ , if and only if

$$A \cap (x \times \mathbb{R}) = x \times [f_A, \infty), \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 2.1.2 allows us to define operations of convex functions by applying corresponding operations of convex sets to the epigraphs of the functions. We give two examples of that kind.

**Definition and Remark.** A convex function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is *closed*, if  $\text{epi } f$  is closed. If  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is convex, then  $\text{cl epi } f$  is the epigraph of a closed convex function, which we denote by  $\text{cl } f$ .

*Proof.* We have to show that  $A := \text{cl epi } f$  fulfills  $f_A > -\infty$ . The case  $f \equiv \infty$  is trivial, then  $f$  is closed and  $f_A = f$ .

Let  $f$  be proper, then  $\text{epi } f \neq \emptyset$ . W.l.o.g. we may assume that  $\dim \text{dom } f = n$ . We choose a point  $x \in \text{int dom } f$ . Then,  $(x, f(x)) \in \text{bd epi } f$ . Hence, there is a supporting hyperplane  $E \subset \mathbb{R}^n \times \mathbb{R}$  of  $\text{cl epi } f$  at  $(x, f(x))$ . The corresponding supporting half-space is the epigraph of an affine function  $h \leq f$ . Thus,  $f_A \geq h > -\infty$ . □

**Remark.**  $\text{cl } f$  is the largest closed convex function below  $f$ .

Our second example is the convex hull operator. If  $(f_i)_{i \in I}$  is a family of (arbitrary) functions  $f_i : \mathbb{R}^n \rightarrow (-\infty, \infty]$ , we consider  $A := \bigcup_{i \in I} \text{epi } f_i$ . Suppose  $\text{conv } A$  does not contain any vertical line, then, by Theorem 2.1.2,  $\text{conv } (f_i) := f_{\text{conv } A}$  is a convex function, which we call the *convex hull* of the functions  $f_i, i \in I$ . It is easy to see, that  $\text{conv } (f_i)$  is the largest convex function below all  $f_i$ , i.e.

$$\text{conv } (f_i) = \sup\{g : g \text{ convex, } g \leq f_i \forall i \in I\}.$$

$\text{conv } (f_i)$  exists, if and only if there is an affine function  $h$  with  $h \leq f_i$ , for all  $i \in I$ .

Further applications of Theorem 2.1.2 are listed in the exercises.

The following representation of convex functions is a counterpart to the support theorem for convex sets.

**Theorem 2.1.3.** *Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be closed and convex. Then,*

$$f = \sup\{h : h \leq f, h \text{ affine}\}.$$

*Proof.* By assumption,  $\text{epi } f$  is closed and convex. By Corollary 1.4.3,  $\text{epi } f$  is the intersection of all closed half-spaces  $H \subset \mathbb{R}^n \times \mathbb{R}$  which contain  $\text{epi } f$ .

There are three types of closed half-spaces in  $\mathbb{R}^n \times \mathbb{R}$ :

$$\begin{aligned} H_1 &= \{(x, r) : r \geq l(x)\}, & l : \mathbb{R}^n &\rightarrow \mathbb{R} \text{ affine,} \\ H_2 &= \{(x, r) : r \leq l(x)\}, & l : \mathbb{R}^n &\rightarrow \mathbb{R} \text{ affine,} \\ H_3 &= \tilde{H} \times \mathbb{R}, & \tilde{H} &\text{ half-space in } \mathbb{R}^n. \end{aligned}$$

Half-spaces of type  $H_2$  cannot occur, due to the definition of  $\text{epi } f$ . Half-spaces of type  $H_3$  can occur, hence we have to show that these ‘vertical’ half-spaces can be neglected, i.e.  $\text{epi } f$  is the intersection of all half-spaces of type  $H_1$  containing  $\text{epi } f$ . Then we are finished since the intersection of half-spaces of type  $H_1$  is the epigraph of the supremum of the corresponding affine functions  $l$ .

For the result just explained it is sufficient to show that any point  $(x_0, r_0) \notin \text{epi } f$  can be separated by a non-vertical hyperplane  $E$  from  $\text{epi } f$ . Hence, let  $E_3$  be a vertical hyperplane separating  $(x_0, r_0)$  and  $\text{epi } f$  and let  $H_3$  be the corresponding vertical half-space containing  $\text{epi } f$ . Since  $f > -\infty$ , there is at least one affine function  $l_1$  with  $l_1 \leq f$ . We may represent  $H_3$  as

$$H_3 = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : l_0(x) \leq 0\}$$

with some affine function  $l_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ .

For  $x \in \text{dom } f$ , we then have

$$l_0(x) \leq 0, \quad l_1(x) \leq f(x),$$

hence

$$\alpha l_0(x) + l_1(x) \leq f(x), \quad \text{for all } \alpha \geq 0.$$

For  $x \notin \text{dom } f$ , this inequality holds trivially since then  $f(x) = \infty$ . Hence

$$m_\alpha := \alpha l_0 + l_1$$

is an affine function fulfilling  $m_\alpha \leq f$ . Since  $l_0(x_0) > 0$ , we have  $m_\alpha(x_0) > r_0$  for sufficiently large  $\alpha$ .  $\square$

We now come to a further important operation on convex functions, the construction of the conjugate function.

**Definition.** Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be proper and convex, then the function  $f^*$  defined by

$$f^*(y) := \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - f(x)), \quad y \in \mathbb{R}^n,$$

is called the *conjugate* of  $f$ .

**Theorem 2.1.4.** *The conjugate  $f^*$  of a proper convex function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  fulfills:*

- (a)  $f^*$  is proper, closed and convex.
- (b)  $f^{**} := (f^*)^* = \text{cl } f$ .

*Proof.* (a) For  $x \notin \text{dom } f$ , we have  $\langle x, y \rangle - f(x) = -\infty$  (for all  $y \in \mathbb{R}^n$ ), hence

$$f^* = \sup_{x \in \text{dom } f} (\langle x, \cdot \rangle - f).$$

For  $x \in \text{dom } f$ , the function

$$g_x : y \mapsto \langle x, y \rangle - f(x)$$

is affine, therefore  $f^*$  is convex (as the supremum of affine functions).

Because of

$$\text{epi } f^* = \text{epi} \left( \sup_{x \in \text{dom } f} g_x \right) = \bigcap_{x \in \text{dom } f} \text{epi } g_x$$

and since  $\text{epi } g_x$  is a closed half-space,  $\text{epi } f^*$  is closed, and hence  $f^*$  is closed.

In order to show that  $f^*$  is proper, we consider an affine function  $h \leq f$  ( $h$  exists by Theorem 2.1.3).  $h$  has a representation

$$h = \langle \cdot, y \rangle - \alpha, \quad \text{with suitable } y \in \mathbb{R}^n, \alpha \in \mathbb{R}.$$

This implies

$$\langle \cdot, y \rangle - \alpha \leq f,$$

hence

$$\langle \cdot, y \rangle - f \leq \alpha,$$

and therefore  $f^*(y) \leq \alpha$ .

(b) By Theorem 2.1.3,

$$\text{cl } f = \sup\{h : h \leq \text{cl } f, h \text{ affine}\}.$$

Writing  $h$  again as

$$h = \langle \cdot, y \rangle - \alpha, \quad y \in \mathbb{R}^n, \alpha \in \mathbb{R},$$

we obtain

$$\text{cl } f = \sup_{(y, \alpha)} (\langle \cdot, y \rangle - \alpha),$$

where the supremum is taken over all  $(y, \alpha)$  with

$$\langle \cdot, y \rangle - \alpha \leq \text{cl } f.$$

The latter holds, if and only if

$$\alpha \geq \sup_x (\langle x, y \rangle - \text{cl } f(x)) = (\text{cl } f)^*(y).$$

Consequently, we have

$$\text{cl } f(x) \leq \sup_y (\langle x, y \rangle - (\text{cl } f)^*(y)) = (\text{cl } f)^{**}(x),$$

for all  $x$ . Since  $\text{cl } f \leq f$ , the definition of the conjugate function implies

$$(\text{cl } f)^* \geq f^*,$$

and therefore

$$\text{cl } f \leq (\text{cl } f)^{**} \leq f^{**}.$$

On the other hand,

$$\begin{aligned} f^{**}(x) &= (f^*)^*(x) \\ &= \sup_y (\langle x, y \rangle - f^*(y)), \end{aligned}$$

where

$$\begin{aligned} f^*(y) &= \sup_z (\langle z, y \rangle - f(z)) \\ &\geq \langle x, y \rangle - f(x). \end{aligned}$$

Therefore,

$$\begin{aligned} f^{**}(x) &\leq \sup_y (\langle x, y \rangle - \langle x, y \rangle + f(x)) \\ &= f(x), \end{aligned}$$

which gives us  $f^{**} \leq f$ . By part (a),  $f^{**}$  is closed, hence  $f^{**} \leq \text{cl } f$ . □

Finally, we mention a canonical possibility to describe convex sets  $A \subset \mathbb{R}^n$  by convex functions. The common way to describe a set  $A$  is by the function

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

however,  $\mathbf{1}_A$  is neither convex nor concave. Therefore, we here define the *indicator function*  $\delta_A$  of a (arbitrary) set  $A \subset \mathbb{R}^n$  by

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

**Remark.**  $A$  is convex, if and only if  $\delta_A$  is convex.

### Exercises and problems

1. Let  $A \subset \mathbb{R}^n$  be nonempty, closed and convex and containing no line. Let further  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be convex and assume there is a point  $y \in A$  with

$$f(y) = \max_{x \in A} f(x).$$

Show that there is also a  $z \in \text{ext } A$  with

$$f(z) = \max_{x \in A} f(x).$$

2. Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be convex. Show that the following assertions are equivalent:

- (i)  $f$  is closed.
- (ii)  $f$  is semi-continuous from above, i.e. for all  $x \in \mathbb{R}^n$  we have

$$f(x) \leq \liminf_{y \rightarrow x} f(y).$$

- (iii) All the sublevel sets  $\{f \leq \alpha\}$ ,  $\alpha \in \mathbb{R}$ , are closed.

3. Let  $f, f_1, \dots, f_m : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be convex functions and  $\alpha \geq 0$ . Show that:

- (a) The function  $\alpha \circ f : x \mapsto \inf\{\beta \in \mathbb{R} : (x, \beta) \in \alpha \cdot \text{epi } f\}$  is convex.
- (b) The function  $f_1 \square \dots \square f_m : x \mapsto \inf\{\beta \in \mathbb{R} : (x, \beta) \in \text{epi } f_1 + \dots + \text{epi } f_m\}$  is convex, and we have

$$f_1 \square \dots \square f_m(x) = \inf\{f_1(x_1) + \dots + f_m(x_m) : x_1, \dots, x_m \in \mathbb{R}^n, x_1 + \dots + x_m = x\}.$$

( $f_1 \square \dots \square f_m$  is called the *infimal convolution* of  $f_1, \dots, f_m$ .)

- (c) Let  $\{f_i : i \in I\}$  ( $I \neq \emptyset$ ) be a family of convex functions on  $\mathbb{R}^n$ , such that  $\text{conv}(f_i)$  exists. Show that

$$\text{conv}(f_i) = \inf\{\alpha_1 \circ f_{i_1} \square \cdots \square \alpha_m \circ f_{i_m} : \alpha_j \geq 0, \sum \alpha_j = 1, i_j \in I, m \in \mathbb{N}\}.$$

4. Let  $A \subset \mathbb{R}^n$  be convex and  $0 \in A$ . The *distance function*  $d_A : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is defined as

$$d_A(x) = \inf\{\alpha \geq 0 : x \in \alpha A\}, \quad x \in \mathbb{R}^n.$$

Show that  $d_A$  has the following properties:

- (a)  $d_A$  is positively homogeneous,  $\geq 0$  and convex.
- (b)  $d_A$  is finite, if and only if  $0 \in \text{int } A$ .
- (c)  $\{d_A < 1\} \subset A \subset \{d_A \leq 1\} \subset \text{cl } A$ .
- (d) If  $0 \in \text{int } A$ , then  $\text{int } A = \{d_A < 1\}$  and  $\text{cl } A = \{d_A \leq 1\}$ .
- (e)  $d_A(x) > 0$ , if and only if  $x \neq 0$  and  $\beta x \notin A$  for some  $\beta > 0$ .
- (f) Let  $A$  be closed. Then  $d_A$  is even (i.e.  $d_A(x) = d_A(-x) \forall x \in \mathbb{R}^n$ ), if and only if  $A$  is symmetric with respect to 0 (i.e.  $A = -A$ ).
- (g) Let  $A$  be closed. Then  $d_A$  is a norm on  $\mathbb{R}^n$ , if and only if  $A$  is symmetric, compact and contains 0 in its interior.
- (h) If  $A$  is closed, then  $d_A$  is closed.



## 2.2 Regularity of convex functions

We start with a continuity property of convex functions.

**Theorem 2.2.1.** *A convex function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is continuous in  $\text{int dom } f$ .*

*Proof.* Let  $x \in \text{int dom } f$ . There exists a  $n$ -simplex  $P$  with  $P \subset \text{int dom } f$  and  $x \in \text{int } P$ . If  $x_0, \dots, x_n$  are the vertices of  $P$  and  $y \in P$ , we have

$$y = \alpha_0 x_0 + \dots + \alpha_n x_n,$$

with  $\alpha_i \in [0, 1]$ ,  $\sum \alpha_i = 1$ , and hence

$$f(y) \leq \alpha_0 f(x_0) + \dots + \alpha_n f(x_n) \leq \max_{i=0, \dots, n} f(x_i) =: c.$$

Therefore,  $f \leq c$  on  $P$ .

Let now  $\alpha \in (0, 1)$  and choose an open ball  $U$  centered at  $x$  such that  $x + U \subset P$ . Let  $z = x + \alpha u$ ,  $u \in U$ . Then,

$$z = (1 - \alpha)x + \alpha(x + u),$$

hence

$$f(z) \leq (1 - \alpha)f(x) + \alpha f(x + u) \leq (1 - \alpha)f(x) + \alpha c.$$

This gives us

$$f(z) - f(x) \leq \alpha(c - f(x)).$$

On the other hand,

$$x = \frac{1}{1 + \alpha}(x + \alpha u) + \left(1 - \frac{1}{1 + \alpha}\right)(x - u),$$

and hence

$$f(x) \leq \frac{1}{1 + \alpha}f(x + \alpha u) + \left(1 - \frac{1}{1 + \alpha}\right)f(x - u),$$

which implies

$$f(x) \leq \frac{1}{1 + \alpha}f(z) + \frac{\alpha}{1 + \alpha}c.$$

We obtain

$$\alpha(f(x) - c) \leq f(z) - f(x).$$

Together, the two inequalities give

$$|f(z) - f(x)| \leq \alpha(c - f(x)),$$

for all  $z \in x + \alpha U$ . □

Now we discuss differentiability properties of convex functions. We first consider the case  $f : \mathbb{R}^1 \rightarrow (-\infty, \infty]$ .

**Theorem 2.2.2.** *Let  $f : \mathbb{R}^1 \rightarrow (-\infty, \infty]$  be convex.*

(a) *In each point  $x \in \text{int dom } f$ , the right derivative  $f^+(x)$  and the left derivative  $f^-(x)$  exist and fulfill  $f^-(x) \leq f^+(x)$ .*

(b) *On  $\text{int dom } f$ , the functions  $f^+$  and  $f^-$  are monotonically increasing and, for almost all  $x \in \text{int dom } f$  (with respect to the Lebesgue measure  $\lambda_1$  on  $\mathbb{R}^1$ ), we have  $f^-(x) = f^+(x)$ , hence  $f$  is almost everywhere differentiable on  $\text{cl dom } f$ .*

(c) *Moreover,  $f^+$  is continuous from the right and  $f^-$  continuous from the left, and  $f$  is the indefinite integral of  $f^+$  respectively  $f^-$  respectively  $f'$  in  $\text{int dom } f$ .*

*Proof.* W.l.o.g. we concentrate on the case  $\text{dom } f = \mathbb{R}^1$ .

(a) If  $0 < m \leq l$  and  $0 < h \leq k$ , the convexity of  $f$  implies

$$f(x - m) = f\left(\left(1 - \frac{m}{l}\right)x + \frac{m}{l}(x - l)\right) \leq \left(1 - \frac{m}{l}\right)f(x) + \frac{m}{l}f(x - l),$$

hence

$$\frac{f(x) - f(x - l)}{l} \leq \frac{f(x) - f(x - m)}{m}.$$

Similarly, we have

$$f(x) = f\left(\frac{h}{h+m}(x - m) + \frac{m}{h+m}(x + h)\right) \leq \frac{h}{h+m}f(x - m) + \frac{m}{h+m}f(x + h),$$

which gives us

$$\frac{f(x) - f(x - m)}{m} \leq \frac{f(x + h) - f(x)}{h}.$$

Finally,

$$f(x + h) = f\left(\left(1 - \frac{h}{k}\right)x + \frac{h}{k}(x + k)\right) \leq \left(1 - \frac{h}{k}\right)f(x) + \frac{h}{k}f(x + k),$$

and therefore

$$\frac{f(x + h) - f(x)}{h} \leq \frac{f(x + k) - f(x)}{k}.$$

We obtain that the left difference quotients in  $x$  increase monotonically and are bounded above by the right difference quotients, which decrease monotonically. Therefore, the limits

$$f^+(x) = \lim_{h \searrow 0} \frac{f(x + h) - f(x)}{h}$$

and

$$f^-(x) = \lim_{k \searrow 0} \frac{f(x) - f(x - k)}{k} \quad \left( = \lim_{t \nearrow 0} \frac{f(x + t) - f(x)}{t} \right)$$

exist and fulfill  $f^-(x) \leq f^+(x)$ .

(b) For  $x' > x$ , we have just seen that

$$(*) \quad f^-(x) \leq f^+(x) \leq \frac{f(x') - f(x)}{x' - x} \leq f^-(x') \leq f^+(x').$$

Therefore, the functions  $f^-$  and  $f^+$  are monotonically increasing. As is well-known, a monotonically increasing function has only countably many points of discontinuity (namely jumps), and therefore it is continuous almost everywhere. In the points  $x$  of continuity of  $f^-$ ,  $(*)$  implies  $f^-(x) = f^+(x)$ .

(c) Assume now  $x < y$ . From

$$\frac{f(y) - f(x)}{y - x} = \lim_{z \searrow x} \frac{f(y) - f(z)}{y - z} \geq \lim_{z \searrow x} f^+(z)$$

we obtain  $\lim_{z \searrow x} f^+(z) \leq f^+(x)$ , hence  $\lim_{z \searrow x} f^+(z) = f^+(x)$ . Similarly, we get  $\lim_{z \nearrow x} f^-(z) = f^-(x)$ .

Finally we define, for arbitrary  $a$ , a function  $g$  by

$$g(x) := f(a) + \int_a^x f^-(u) dy.$$

We first show that  $g$  is convex, and then  $g = f$ .

For  $z := \alpha x + (1 - \alpha)y$ ,  $\alpha \in [0, 1]$ ,  $x < y$ , we have

$$g(z) - g(x) = \int_x^z f^-(s) ds \leq (z - x)f^-(z),$$

$$g(y) - g(z) = \int_z^y f^-(s) ds \geq (y - z)f^-(z).$$

It follows that

$$\begin{aligned} \alpha(g(z) - g(x)) + (1 - \alpha)(g(z) - g(y)) &\leq \alpha(z - x)f^-(z) + (1 - \alpha)(z - y)f^-(z) \\ &= f^-(z)(\alpha z + (1 - \alpha)z - [\alpha x + (1 - \alpha)x]) = 0, \end{aligned}$$

therefore

$$g(z) \leq \alpha g(x) + (1 - \alpha)g(y),$$

i.e.  $g$  is convex.

As a consequence,  $g^+$  und  $g^-$  exist. Since

$$\frac{g(y) - g(x)}{y - x} = \frac{1}{y - x} \int_x^y f^-(s) ds = \frac{1}{y - x} \int_x^y f^+(s) ds \geq f^+(x),$$

we obtain  $g^+(x) \geq f^+(x)$ . Analogously, we get  $g^-(x) \leq f^-(x)$ . By the continuity from the left of  $g^-$  and  $f^-$ , and the continuity from the right of  $g^+$  and  $f^+$ , it follows that  $g^+ = f^+$  and  $g^- = f^-$ . Hence,  $h := g - f$  is differentiable everywhere and  $h' \equiv 0$ . Therefore,  $h \equiv c = 0$  because we have  $g(a) = f(a)$ .  $\square$

Now we consider the  $n$ -dimensional case. If  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is convex and  $x \in \text{int dom } f$ , then, for each  $u \in \mathbb{R}^n, u \neq 0$ , the equation

$$g_{(u)}(t) := f(x + tu), \quad t \in \mathbb{R},$$

defines a convex function  $g_{(u)} : \mathbb{R}^1 \rightarrow (-\infty, \infty]$  and we have  $x \in \text{int dom } g_{(u)}$ . By Theorem 2.2.2, the right derivative  $g_{(u)}^+(0)$  exists. This is precisely the *directional derivative*

$$(+) \quad f'(x; u) := \lim_{t \searrow 0} \frac{f(x + tu) - f(x)}{t}$$

of  $f$  in direction  $u$ . Therefore, we obtain the following corollary to Theorem 2.2.2.

**Corollary 2.2.3.** *Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be convex and  $x \in \text{int dom } f$ . Then, for each  $u \in \mathbb{R}^n, u \neq 0$ , the directional derivative  $f'(x; u)$  of  $f$  exists.*

The corollary does not imply that  $f'(x; u) = -f'(x; -u)$  holds (in fact, the latter equation is only true if  $g_{(u)}^-(0) = g_{(u)}^+(0)$ ). Also, the partial derivatives  $f_1(x), \dots, f_n(x)$  of  $f$  need not exist in each point  $x$ . However, in analogy to Theorem 2.2.2, one can show that  $f_1, \dots, f_n$  exist almost everywhere (with respect to the Lebesgue measure  $\lambda_n$  in  $\mathbb{R}^n$ ) and that in points  $x$ , where the partial derivatives  $f_1(x), \dots, f_n(x)$  exist, the function  $f$  is even differentiable. Even more, a convex function  $f$  on  $\mathbb{R}^n$  is twice differentiable almost everywhere (in a suitable sense). We refer to the exercises, for these and a number of further results on derivatives of convex functions.

The right-hand side of (+) also makes sense for  $u = 0$  and yields the value 0. We therefore define  $f'(x; 0) := 0$ . Then  $u \mapsto f'(x; u)$  is a positively homogeneous function on  $\mathbb{R}^n$  and if  $f$  is convex,  $f'(x; \cdot)$  is also convex. For support functions, we will continue the discussion of directional derivatives in the next section.

For a function  $f$  which is differentiable or twice differentiable, the first or second derivatives can be used to characterize convexity of  $f$ .

**Remarks. (1)** (see Exercise 3) Let  $A \subset \mathbb{R}$  be open and convex and let  $f : A \rightarrow \mathbb{R}$  be a real function.

If  $f$  is differentiable, then  $f$  is convex, if and only if  $f'$  is monotone increasing (on  $A$ ).

If  $f$  is twice differentiable, then  $f$  is convex, if and only if  $f'' \geq 0$  (on  $A$ ).

**(2)** (see Exercise 4) Let  $A \subset \mathbb{R}^n$  be open and convex and let  $f : A \rightarrow \mathbb{R}$  be a real function.

If  $f$  is continuously differentiable, then  $f$  is convex, if and only if

$$\langle \text{grad } f(x) - \text{grad } f(y), x - y \rangle \geq 0, \quad \text{for all } x, y \in A.$$

(Here,  $\text{grad } f(x) := (f_1(x), \dots, f_n(x))$  is the *gradient* of  $f$  at  $x$ .)

If  $f$  is twice continuously differentiable, then  $f$  is convex, if and only if the *Hesse matrix*

$$\partial^2 f := ((f_{ij}(x)))_{n \times n}$$

of  $f$  is positively semidefinite, for all  $x \in A$ .

### Exercises and problems

1. (a) Give an example of two convex functions  $f, g : \mathbb{R}^n \rightarrow (-\infty, \infty]$ , such that  $f$  and  $g$  both have minimal points (i.e. points in  $\mathbb{R}^n$ , where the infimum of the function is attained), but  $f + g$  does not have a minimal point.
- (b) Suppose  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  are convex functions, which both have a unique minimal point in  $\mathbb{R}^n$ . Show that  $f + g$  has a minimal point.

*Hint:* Show first that the sets

$$\{x \in \mathbb{R}^n : f(x) \leq \alpha\} \quad \text{resp.} \quad \{x \in \mathbb{R}^n : g(x) \leq \alpha\}$$

are compact, for each  $\alpha \in \mathbb{R}$ .

2. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function. Show that

$$f(x) - f(0) = \int_0^x f^+(t) dt = \int_0^x f^-(t) dt,$$

for all  $x \in \mathbb{R}$ .

3. Let  $A \subset \mathbb{R}$  be open and convex and  $f : A \rightarrow \mathbb{R}$  a real function.
  - (a) Assume  $f$  is differentiable. Show that  $f$  is convex, if and only if  $f'$  is monotone increasing (on  $A$ ).
  - (b) Assume  $f$  is twice differentiable. Show that  $f$  is convex, if and only if  $f'' \geq 0$  (on  $A$ ).
4. Let  $A \subset \mathbb{R}^n$  be open and convex and  $f : A \rightarrow \mathbb{R}$  a real function.

- (a) Assume  $f$  is continuously differentiable. Show that  $f$  is convex, if and only if

$$\langle \text{grad } f(x) - \text{grad } f(y), x - y \rangle \geq 0, \quad \text{for all } x, y \in A.$$

(Here,  $\text{grad } f(x) := (f_1(x), \dots, f_n(x))$  is the *gradient* of  $f$  at  $x$ .)

- (b) Assume  $f$  is twice continuously differentiable. Show that  $f$  is convex, if and only if the *Hesse matrix*

$$\partial^2 f := ((f_{ij}(x)))_{n \times n}$$

of  $f$  is positively semidefinite, for all  $x \in A$ .

5. For a convex function  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  and  $x \in \text{int dom } f$ , we define the *subgradient* of  $f$  at  $x$  by

$$\partial f(x) := \{v \in \mathbb{R}^n : f(y) \geq f(x) + \langle v, y - x \rangle \forall y \in \mathbb{R}^n\}.$$

Show that:

- (a)  $\partial f(x)$  is nonempty, compact and convex.  
 (b) We have

$$\partial f(x) = \{v \in \mathbb{R}^n : \langle v, u \rangle \leq f'(x; u) \forall u \in \mathbb{R}^n, u \neq 0\}.$$

- (c) If  $f$  is differentiable in  $x$ , then

$$\partial f(x) = \{\text{grad } f(x)\}.$$

- \* 6. Let  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be convex and  $x \in \text{int dom } f$ . Suppose that all partial derivatives  $f_1(x), \dots, f_n(x)$  at  $x$  exist. Show that  $f$  is differentiable at  $x$ .
7. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be convex. Show that  $f$  is differentiable almost everywhere.  
*Hint:* Use Exercise 6.

## 2.3 The support function

The most useful analytic description of compact convex sets is by the support function. It is one of the basic tools in the following chapter. The support function of a set  $A \subset \mathbb{R}^n$  with  $0 \in A$  is in a certain sense dual to the distance function, which was discussed in Exercise 2.1.3.

**Definition.** Let  $A \subset \mathbb{R}^n$  be nonempty and convex. The *support function*  $h_A : \mathbb{R}^n \rightarrow (-\infty, \infty]$  of  $A$  is defined as

$$h_A(x) := \sup_{y \in A} \langle x, y \rangle, \quad x \in \mathbb{R}^n.$$

**Theorem 2.3.1.** For nonempty, convex sets  $A, B \subset \mathbb{R}^n$ , we have:

- (a)  $h_A$  is positively homogeneous, closed and convex (and hence subadditive).  
 (b)  $h_A = h_{\text{cl}A}$  and

$$\text{cl}A = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_A(u) \quad \forall u \in \mathbb{R}^n\}.$$

(c)  $A \subset B$  implies  $h_A \leq h_B$ ; conversely,  $h_A \leq h_B$  implies  $\text{cl}A \subset \text{cl}B$ .

(d)  $h_A$  is finite, if and only if  $A$  is bounded.

(e)  $h_{\alpha A + \beta B} = \alpha h_A + \beta h_B$ , for all  $\alpha, \beta \geq 0$ .

(f)  $h_{-A}(x) = h_A(-x)$ , for all  $x \in \mathbb{R}^n$ .

(g) If  $A_i, i \in I$ , are nonempty and convex and  $A := \text{conv}(\bigcup_{i \in I} A_i)$ , then

$$h_A = \sup_{i \in I} h_{A_i}.$$

(h) If  $A_i, i \in I$ , are nonempty, convex and closed and if  $A := \bigcap_{i \in I} A_i$  is nonempty, then

$$h_A = \text{cl} \text{conv}(h_{A_i})_{i \in I}.$$

(i)  $\delta_A^* = h_A$ .

*Proof.* (a) For  $\alpha \geq 0$  and  $x, z \in \mathbb{R}^n$ , we have

$$h_A(\alpha x) = \sup_{y \in A} \langle \alpha x, y \rangle = \alpha \sup_{y \in A} \langle x, y \rangle = \alpha h_A(x)$$

and

$$h_A(x + z) = \sup_{y \in A} \langle x + z, y \rangle \leq \sup_{y \in A} \langle x, y \rangle + \sup_{y \in A} \langle z, y \rangle = h_A(x) + h_A(z).$$

Furthermore, as a supremum of closed functions,  $h_A$  is closed.

(b) The first part follows from

$$\sup_{x \in A} \langle x, u \rangle = \sup_{x \in \text{cl} A} \langle x, u \rangle, \quad u \in \mathbb{R}^n.$$

For  $x \in \text{cl} A$ , we therefore have  $\langle x, u \rangle \leq h_A(u)$ , for all  $u \in \mathbb{R}^n$ . Conversely, suppose  $x \in \mathbb{R}^n$  fulfills  $\langle x, \cdot \rangle \leq h_A(\cdot)$ , and assume  $x \notin \text{cl} A$ . Then, by Theorem 1.4.2, there exists a (supporting) hyperplane separating  $x$  and  $\text{cl} A$ , i.e. a direction  $y \in S^{n-1}$  and  $\alpha \in \mathbb{R}$  such that

$$\langle x, y \rangle > \alpha \text{ and } \langle z, y \rangle \leq \alpha, \text{ for all } z \in \text{cl} A.$$

This implies

$$h_{\text{cl} A}(y) = h_A(y) \leq \alpha < \langle x, y \rangle,$$

a contradiction.

(c) The first part is obvious, the second follows from (b).

(d) If  $A$  is bounded, we have  $A \subset B(r)$ , for some  $r > 0$ . Then, (c) implies  $h_A \leq h_{B(r)} = r\|\cdot\|$ , hence  $h_A < \infty$ . Conversely,  $h_A < \infty$  and Theorem 2.2.1 imply that  $h_A$  is continuous on  $\mathbb{R}^n$ . Therefore,  $h_A$  is bounded on  $S^{n-1}$ , i.e.  $h_A \leq r = h_{B(r)}$  on  $S^{n-1}$ , for some  $r > 0$ . The positive homogeneity, proved in (a), implies that  $h_A \leq h_{B(r)}$  on all of  $\mathbb{R}^n$ , hence (c) shows that  $\text{cl} A \subset B(r)$ , i.e.  $A$  is bounded.

(e) For any  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} h_{\alpha A + \beta B}(x) &= \sup_{y \in \alpha A + \beta B} \langle x, y \rangle = \sup_{u \in A, v \in B} \langle x, \alpha u + \beta v \rangle \\ &= \sup_{u \in A} \langle x, \alpha u \rangle + \sup_{v \in B} \langle x, \beta v \rangle = \alpha h_A(x) + \beta h_B(x). \end{aligned}$$

(f) For any  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} h_{-A}(x) &= \sup_{y \in -A} \langle x, y \rangle = \sup_{u \in A} \langle x, -u \rangle \\ &= \sup_{u \in A} \langle -x, u \rangle = h_A(-x). \end{aligned}$$

(g) Since  $A_i \subset A$ , we have  $h_{A_i} \leq h_A$  (from (c)), hence

$$\sup_{i \in I} h_{A_i} \leq h_A.$$

Conversely, any  $y \in A$  has a representation

$$y = \alpha_1 y_{i_1} + \cdots + \alpha_k y_{i_k},$$



with  $k \in \mathbb{N}$ ,  $y_{i_j} \in A_{i_j}$ ,  $\alpha_j \geq 0$ ,  $\sum \alpha_j = 1$ . Therefore, we get

$$\begin{aligned} h_A(x) &= \sup_{y \in A} \langle x, y \rangle = \sup_{y_{i_j} \in A_{i_j}, \alpha_j \geq 0, \sum \alpha_j = 1, i_j \in I, k \in \mathbb{N}} \langle x, \alpha_1 y_{i_1} + \cdots + \alpha_k y_{i_k} \rangle \\ &= \sup_{\alpha_j \geq 0, \sum \alpha_j = 1, i_j \in I, k \in \mathbb{N}} (\alpha_1 h_{A_{i_1}}(x) + \cdots + \alpha_k h_{A_{i_k}}(x)) \leq \sup_{i \in I} h_{A_i}(x). \end{aligned}$$

(h) Since  $A \subset A_i$ , we have  $h_A \leq h_{A_i}$  (from (c)), for all  $i \in I$ . Using the inclusion of the epigraphs, the definition of  $\text{cl}$  and  $\text{conv}$  for functions and (a), we obtain

$$h_A \leq \text{cl conv} (h_{A_i})_{i \in I}.$$

On the other hand, Theorem 2.1.3 shows that

$$g := \text{cl conv} (h_{A_i})_{i \in I}$$

is the supremum of all affine functions below  $g$ . Since  $g$  is positively homogeneous, we can concentrate on linear functions. Therefore, assume  $\langle \cdot, y \rangle \leq g$ ,  $y \in \mathbb{R}^n$ , is such a function. Then,

$$\langle \cdot, y \rangle \leq h_{A_i}, \quad \text{for all } i \in I.$$

(c) implies that  $y \in A_i$ ,  $i \in I$ , hence  $y \in \bigcap_{i \in I} A_i = A$ . Therefore,

$$\langle \cdot, y \rangle \leq h_A,$$

from which we get

$$g = \text{cl conv} (h_{A_i})_{i \in I} \leq h_A.$$

(i) For  $x \in \mathbb{R}^n$ , we have

$$\delta_A^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - \delta_A(y)) = \sup_{y \in A} \langle x, y \rangle = h_A(x),$$

hence  $\delta_A^* = h_A$ . □

The following result is crucial for the later considerations.

**Theorem 2.3.2.** *Let  $h : \mathbb{R}^n \rightarrow (-\infty, \infty]$  be positively homogeneous, closed and convex. Then there exists a unique nonempty, closed and convex set  $A \subset \mathbb{R}^n$  such that*

$$h_A = h.$$

*Proof.* If  $h \equiv \infty$ , then  $h = h_{\mathbb{R}^n}$  and, by the support theorem,  $\mathbb{R}^n$  is the only closed convex set  $A$  with  $h_A \equiv \infty$ . Hence, we may now assume that  $h$  is proper.

We consider  $h^*$ . For  $\alpha > 0$ , we obtain from the positive homogeneity

$$\begin{aligned} h^*(x) &= \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - h(y)) = \sup_{y \in \mathbb{R}^n} (\langle x, \alpha y \rangle - h(\alpha y)) \\ &= \alpha \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - h(y)) = \alpha h^*(x). \end{aligned}$$

Therefore,  $h^*$  can only obtain the values 0 and  $\infty$ . We put  $A := \text{dom } h^*$ . By Theorem 2.1.4(a),  $A$  is nonempty, closed and convex, and

$$h^* = \delta_A.$$

Theorem 2.3.1(i) implies

$$h^{**} = \delta_A^* = h_A.$$

By Theorem 2.1.4(b), we have  $h^{**} = h$ , hence  $h_A = h$ .

The uniqueness of  $A$  follows from Theorem 2.3.1(b).  $\square$

We mention without proof a couple of further properties of support functions, which are mostly simple consequences of the definition or the last two theorems. In the following remarks,  $A$  is always a nonempty closed convex subset of  $\mathbb{R}^n$ .

**Remarks.** (1) We have  $A = \{x\}$ , if and only if  $h_A = \langle x, \cdot \rangle$ .

(2) We have  $h_{A+x} = h_A + \langle x, \cdot \rangle$ .

(3)  $A$  is origin-symmetric (i.e.  $A = -A$ ), if and only if  $h_A$  is *even*, i.e.  $h_A(x) = h_A(-x)$ , for all  $x \in \mathbb{R}^n$ .

(4) We have  $0 \in A$ , if and only if  $h_A \geq 0$ .

Let  $A \subset \mathbb{R}^n$  be nonempty, closed and convex. For  $u \in S^{n-1}$ , we consider the sets

$$E(u) := \{x \in \mathbb{R}^n : \langle x, u \rangle = h_A(u)\}$$

and

$$A(u) := A \cap E(u) = \{x \in A : \langle x, u \rangle = h_A(u)\}.$$

If  $h_A(u) = \infty$ , both sets are empty. If  $h_A(u) < \infty$ , then  $E(u)$  is a hyperplane, which bounds  $A$ , but need not be a supporting hyperplane (see the example in Section 1.4), namely if  $A(u) = \emptyset$ . If  $A(u) \neq \emptyset$ , then  $E(u)$  is a supporting hyperplane of  $A$  (at each point  $x \in A(u)$ ) and  $A(u)$  is the corresponding support set. We discuss now the support function of  $A(u)$ . In order to simplify the considerations, we concentrate on the case, where  $A$  is compact (then  $A(u)$  is nonempty and compact, for all  $u \in S^{n-1}$ ). A compact convex set  $K \neq \emptyset$  is called a *convex body*. We denote by  $\mathcal{K}^n$  the set of all convex bodies in  $\mathbb{R}^n$ .

**Theorem 2.3.3.** *Let  $K \in \mathcal{K}^n$  and  $u \in S^{n-1}$ . Then,*

$$h_{K(u)}(x) = h'_K(u; x), \quad x \in \mathbb{R}^n,$$

*i.e. the support function of  $K(u)$  is given by the directional derivatives of  $h_K$  at the point  $u$ .*

*Proof.* For  $y \in K(u)$  and  $v \in \mathbb{R}^n$ , we have

$$\langle y, v \rangle \leq h_K(v),$$

since  $y$  belongs to  $K$ . In particular, for  $v := u + tx$ ,  $x \in \mathbb{R}^n$ ,  $t > 0$ , we thus get

$$\langle y, u \rangle + t\langle y, x \rangle \leq h_K(u + tx),$$

and hence

$$\langle y, x \rangle \leq \frac{h_K(u + tx) - h_K(u)}{t}$$

(because of  $h_K(u) = \langle y, u \rangle$ ). For  $t \searrow 0$ , we obtain

$$\langle y, x \rangle \leq h'_K(u; x).$$

Since this holds for all  $y \in K(u)$ , we arrive at

$$h_{K(u)}(x) \leq h'_K(u; x).$$

Conversely, we obtain from the subadditivity of  $h_K$

$$\frac{h_K(u + tx) - h_K(u)}{t} \leq \frac{h_K(tx)}{t} = h_K(x),$$

and thus

$$h'_K(u; x) \leq h_K(x).$$

We show that the function  $x \mapsto h'_K(u; x)$  is convex, positively homogeneous and finite. Namely,

$$\begin{aligned} h'_K(u; x + x') &= \lim_{t \searrow 0} \frac{h_K(u + tx + tx') - h_K(u)}{t} \\ &\leq \lim_{t \searrow 0} \frac{h_K(\frac{u}{2} + tx) - h_K(\frac{u}{2})}{t} + \lim_{t \searrow 0} \frac{h_K(\frac{u}{2} + tx') - h_K(\frac{u}{2})}{t} \\ &= h'_K(u; x) + h'_K(u; x') \end{aligned}$$

and

$$\begin{aligned} h'_K(u; \alpha x) &= \lim_{t \searrow 0} \frac{h_K(u + t\alpha x) - h_K(u)}{t} \\ &= \alpha h'_K(u; x), \end{aligned}$$

for  $x, x' \in \mathbb{R}^n$  and  $\alpha \geq 0$ . By Theorem 2.3.2 (in connection with Theorem 2.3.1(d)), there exists a nonempty, compact convex set  $L \subset \mathbb{R}^n$  with

$$h_L(x) = h'_K(u; x), \quad x \in \mathbb{R}^n.$$

For  $y \in L$ , we have

$$\langle y, x \rangle \leq h'_K(u; x) \leq h_K(x), \quad x \in \mathbb{R}^n,$$

hence  $y \in K$ . Furthermore,

$$\langle y, u \rangle \leq h'_K(u; u) = h_K(u)$$

and

$$\langle y, -u \rangle \leq h'_K(u; -u) = -h_K(u),$$

from which we obtain

$$\langle y, u \rangle = h_K(u),$$

and thus  $y \in K \cap E(u) = K(u)$ . It follows that  $L \subset K(u)$ , and therefore (again by Theorem 2.3.1)

$$h'_K(u; x) \leq h_{K(u)}(x).$$

Combining both inequalities, we obtain the assertion.  $\square$

**Remark.** As a consequence, we obtain that  $K(u)$  consists of one point, if and only if  $h'_K(u; \cdot)$  is linear. In view of Exercise 2.2.5 and Exercise 2.2.6, the latter is equivalent to the differentiability of  $h_K$  at  $u$ . If all the support sets  $K(u)$ ,  $u \in S^{n-1}$ , of a nonempty, compact convex set  $K$  consist of points, the boundary  $\text{bd } K$  does not contain any segments. Such sets  $K$  are called *strictly convex*. Hence,  $K$  is strictly convex, if and only if  $h_K$  is differentiable on  $\mathbb{R}^n \setminus \{0\}$ .

We finally consider the support functions of polytopes. We call a function  $h$  on  $\mathbb{R}^n$  *piecewise linear*, if there are finitely many convex cones  $A_1, \dots, A_m \subset \mathbb{R}^n$ , such that  $\mathbb{R}^n = \bigcup_{i=1}^m A_i$  and  $h$  is linear on  $A_i$ ,  $i = 1, \dots, m$ .

**Theorem 2.3.4.** *Let  $K \in \mathcal{K}^n$ . Then  $K$  is a polytope, if and only if  $h_K$  is piecewise linear.*

*Proof.*  $K$  is a polytope, if and only if

$$K = \text{conv} \{x_1, \dots, x_k\},$$

for some  $x_1, \dots, x_k \in \mathbb{R}^n$ . In view of Theorem 2.3.1, the latter is equivalent to

$$h_K = \max_{i=1, \dots, k} \langle x_i, \cdot \rangle,$$

which holds, if and only if  $h_K$  is piecewise linear.

To be more precise, if  $h_K$  has the above form, the cones  $A_i$  of linearity are given by

$$A_i := \{x \in \mathbb{R}^n : \max_{j=1, \dots, k} \langle x_j, \cdot \rangle = \langle x_i, \cdot \rangle\}, \quad i = 1, \dots, k.$$

Conversely, if  $h_K$  is linear on the cone  $A_i$ , we may assume that  $A_i$  is closed and has interior points. Then  $x_i$  is determined by

$$\langle x_i, \cdot \rangle = h_K$$

on  $A_i$ . The convexity of  $h_K$  implies that  $h_K \geq \langle x_i, \cdot \rangle$  on  $\mathbb{R}^n$ , hence

$$h_K = \max_{i=1, \dots, k} \langle x_i, \cdot \rangle$$

follows. □

### Exercises and problems

1. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be positively homogeneous and twice continuously partially differentiable on  $\mathbb{R}^n \setminus \{0\}$ . Show that there are nonempty, compact convex sets  $K, L \subset \mathbb{R}^n$  such that

$$f = h_K - h_L.$$

*Hint:* Use Exercise 2.2.2(b).

2. Let  $K \subset \mathbb{R}^n$  be compact and convex with  $0 \in \text{int } K$  and let  $K^\circ$  be the polar of  $K$  (see Exercise 1.1.4). Show that
  - (a)  $K^\circ$  is compact and convex with  $0 \in \text{int } K^\circ$ ,
  - (b)  $K^{\circ\circ} := (K^\circ)^\circ = K$ ,
  - (c)  $K$  is a polytope, if and only if  $K^\circ$  is a polytope,
  - (d)  $h_K = d_{K^\circ}$ .



# Chapter 3

## Convex bodies

### 3.1 The space of convex bodies

In the following, we mostly concentrate on *convex bodies* (nonempty compact convex sets)  $K$  in  $\mathbb{R}^n$  and first discuss the space  $\mathcal{K}^n$  of convex bodies. We emphasize that we do not require that a convex body has interior points; hence lower-dimensional bodies are included in  $\mathcal{K}^n$ . The set  $\mathcal{K}^n$  is closed under addition,

$$K, L \in \mathcal{K}^n \implies K + L \in \mathcal{K}^n,$$

and multiplication with nonnegative scalars,

$$K \in \mathcal{K}^n, \alpha \geq 0 \implies \alpha K \in \mathcal{K}^n.$$

(In fact, we even have  $\alpha K \in \mathcal{K}^n$ , for all  $\alpha \in \mathbb{R}$ , since the reflection  $-K$  of a convex body  $K$  is again a convex body.) Thus,  $\mathcal{K}^n$  is a convex cone and the question arises, whether we can embed this cone into a suitable vector space. Since  $(\mathcal{K}^n, +)$  is a (commutative) semi-group, the problem reduces to the question, whether this semi-group can be embedded into a group. A simple algebraic criterion (which is necessary and sufficient) is that the cancellation rule must be valid. Although this can be checked directly for convex bodies (see the exercises), we use now the support function for a direct embedding which has a number of additional advantages.

For this purpose, we consider the support function  $h_K$  of a convex body as a function on the unit sphere  $S^{n-1}$  (because of the positive homogeneity of  $h_K$ , the values on  $S^{n-1}$  determine  $h_K$  completely). Let  $\mathbf{C}(S^{n-1})$  be the vector space of continuous functions on  $S^{n-1}$ . This is a Banach space with respect to the maximum norm

$$\|f\| := \max_{u \in S^{n-1}} |f(u)|, \quad f \in \mathbf{C}(S^{n-1}).$$

We call a function  $f : S^{n-1} \rightarrow \mathbb{R}$  *convex*, if the homogeneous extension

$$\tilde{f} := \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

is convex on  $\mathbb{R}^n$ . Let  $\mathcal{H}^n$  be the set of all convex functions on  $S^{n-1}$ . By Remark (3) (after Theorem 2.1.1) and Theorem 2.2.1,  $\mathcal{H}^n$  is a convex cone in  $\mathbf{C}(S^{n-1})$ .

**Theorem 3.1.1.** *The mapping*

$$T : K \mapsto h_K$$

*is linear on  $\mathcal{K}^n$  and maps the convex cone  $\mathcal{K}^n$  one-to-one onto the convex cone  $\mathcal{H}^n$ . Moreover,  $T$  is compatible with the inclusion order on  $\mathcal{K}^n$  and the pointwise order  $\leq$  on  $\mathcal{H}^n$ .*

*In particular,  $T$  embeds the (ordered) convex cone  $\mathcal{K}^n$  into the (ordered) vector space  $\mathbf{C}(S^{n-1})$ .*

*Proof.* The linearity of  $T$  follows from Theorem 2.3.1(e) and the injectivity from Theorem 2.3.1(b). The fact that  $T(\mathcal{K}^n) = \mathcal{H}^n$  is a consequence of Theorem 2.3.2. The compatibility with respect to the orderings follows from Theorem 2.3.1(c).  $\square$

**Remark.** Linearity of  $T$  on the convex cone  $\mathcal{K}^n$  means

$$T(\alpha K + \beta L) = \alpha T(K) + \beta T(L),$$

for  $K, L \in \mathcal{K}^n$  and  $\alpha, \beta \geq 0$ . This linearity does not extend to negative  $\alpha, \beta$ , in particular not to difference bodies  $K - L = K + (-L)$ . One reason is that the function  $h_K - h_L$  is in general not convex, but even if it is, hence if

$$h_K - h_L = h_M,$$

for some  $M \in \mathcal{K}^n$ , the body  $M$  is in general different from the difference body  $K - L$ . We write  $K \ominus L := M$  and call this body the *Minkowski difference* of  $K$  and  $L$ . Whereas the difference body  $K - L$  exists for all  $K, L \in \mathcal{K}^n$ , the Minkowski difference  $K \ominus L$  only exists in special cases, namely if  $K$  can be decomposed as  $K = M + L$  (then  $M = K \ominus L$ ).

With respect to the norm topology provided by the maximum norm in  $\mathbf{C}(S^{n-1})$ , the cone  $\mathcal{H}^n$  is closed. Our next goal is to define a natural metric on  $\mathcal{K}^n$ , such that  $T$  becomes even an isometry (hence, we then have an isometric embedding of  $\mathcal{K}^n$  into the Banach space  $\mathbf{C}(S^{n-1})$ ).

**Definition.** For  $K, L \in \mathcal{K}^n$ , let

$$d(K, L) := \inf \{ \varepsilon \geq 0 : K \subset L + B(\varepsilon), L \subset K + B(\varepsilon) \}.$$

It is easy to see that the infimum is attained, hence it is in fact a minimum.



**Theorem 3.1.2.** For  $K, L \in \mathcal{K}^n$ , we have

$$d(K, L) = \|h_K - h_L\|.$$

Therefore,  $d$  is a metric on  $\mathcal{K}^n$  and fulfills

$$d(K + M, L + M) = d(K, L),$$

for all  $K, L, M \in \mathcal{K}^n$ .

*Proof.* From Theorem 2.3.1 we obtain

$$K \subset L + B(\varepsilon) \Leftrightarrow h_L \leq h_K + \varepsilon h_{B(1)}$$

and

$$L \subset K + B(\varepsilon) \Leftrightarrow h_K \leq h_L + \varepsilon h_{B(1)}.$$

Since  $h_{B(1)} \equiv 1$  on  $S^{n-1}$ , this implies

$$K \subset L + B(\varepsilon), L \subset K + B(\varepsilon) \Leftrightarrow \|h_K - h_L\| \leq \varepsilon,$$

and the assertions follow. □

In an arbitrary metric space  $(X, d)$ , the class  $\mathcal{C}(X)$  of nonempty compact subsets of  $X$  can be supplied with the *Hausdorff metric*  $\tilde{d}$ , which is defined by

$$\tilde{d}(A, B) := \max \left( \max_{x \in A} d(x, B), \max_{y \in B} d(y, A) \right).$$

Here  $A, B \in \mathcal{C}(X)$ , and we have used the abbreviation

$$d(u, C) := \min_{v \in C} d(u, v), \quad u \in X, C \in \mathcal{C}(X),$$

(the minimal and maximal values exist due to the compactness of the sets and the continuity of the metric). We show now that, on  $\mathcal{K}^n \subset \mathcal{C}(\mathbb{R}^n)$ , the Hausdorff metric  $\tilde{d}$  coincides with the metric  $d$ .

**Theorem 3.1.3.** For  $K, L \in \mathcal{K}^n$ , we have

$$d(K, L) = \tilde{d}(K, L).$$

*Proof.* We have

$$d(K, L) = \max \left( \inf \{ \varepsilon \geq 0 : K \subset L + B(\varepsilon) \}, \inf \{ \varepsilon \geq 0 : L \subset K + B(\varepsilon) \} \right).$$

Now

$$\begin{aligned} K \subset L + B(\varepsilon) &\Leftrightarrow d(x, L) \leq \varepsilon, \quad \text{for all } x \in K, \\ &\Leftrightarrow \max_{x \in K} d(x, L) \leq \varepsilon, \end{aligned}$$

hence

$$\inf \{ \varepsilon \geq 0 : K \subset L + B(\varepsilon) \} = \max_{x \in K} d(x, L),$$

which yields the assertion.  $\square$

We now come to an important topological property of the metric space  $(\mathcal{K}^n, d)$ : Every bounded subset  $\mathcal{M} \subset \mathcal{K}^n$  is relative compact. This is a special property which holds also, for example, in the metric space  $(\mathbb{R}^n, d)$ , but it does not hold in general metric spaces.

In  $\mathcal{K}^n$ , a subset  $\mathcal{M}$  is bounded, if there exists  $c > 0$  such that

$$d(K, L) \leq c, \quad \text{for all } K, L \in \mathcal{M}.$$

This is equivalent to

$$K \subset B(c'), \quad \text{for all } K \in \mathcal{M},$$

for some constant  $c' > 0$ . Here, we can replace the ball  $B(c')$  by any compact set, in particular by a cube  $W \subset \mathbb{R}^n$ . The subset  $\mathcal{M}$  is relative compact, if every sequence  $K_1, K_2, \dots$ , with  $K_k \in \mathcal{M}$ , has a converging subsequence. Therefore, the mentioned topological property is a consequence of the following theorem.

**Theorem 3.1.4 (BLASCHKE'S Selection Theorem).** *Let  $\mathcal{M} \subset \mathcal{K}^n$  be an infinite collection of convex bodies, all lying in a cube  $W$ . Then, there exists a sequence  $K_1, K_2, \dots$ , with  $K_k \in \mathcal{M}$  (pairwise different), and a body  $K_0 \in \mathcal{K}^n$  such that*

$$K_k \rightarrow K_0, \quad \text{as } k \rightarrow \infty.$$

*Proof.* W.l.o.g. we assume that  $W$  is the unit cube.

For each  $i \in \mathbb{N}$ , we divide  $W$  into  $2^{in}$  cubes of edge length  $1/2^i$ . For  $K \in \mathcal{M}$ , let  $W_i(K)$  be the union of all cubes in the  $i$ th dissection, which intersect  $K$ . Since there are only finitely many different sets  $W_i(K)$ ,  $K \in \mathcal{M}$ , but infinitely many bodies  $K \in \mathcal{M}$ , we first get a sequence (in  $\mathcal{M}$ )

$$K_1^{(1)}, K_2^{(1)}, \dots$$

with

$$W_1(K_1^{(1)}) = W_1(K_2^{(1)}) = \dots,$$

then a subsequence (of  $K_1^{(1)}, K_2^{(1)}, \dots$ )

$$K_1^{(2)}, K_2^{(2)}, \dots$$

with

$$W_2(K_1^{(2)}) = W_2(K_2^{(2)}) = \dots,$$

and in general a subsequence

$$K_1^{(j)}, K_2^{(j)}, \dots$$

of  $K_1^{(j-1)}, K_2^{(j-1)}, \dots$  with

$$W_j(K_1^{(j)}) = W_j(K_2^{(j)}) = \dots,$$

for all  $j \in \mathbb{N}$  ( $j \geq 2$ ).

Since

$$\min_{y \in K_l^{(j)}} d(x, y) \leq \frac{\sqrt{n}}{2^j},$$

for all  $x \in K_k^{(j)}$ , we have

$$d(K_k^{(j)}, K_l^{(j)}) \leq \frac{\sqrt{n}}{2^j}, \quad \text{for all } k, l \in \mathbb{N}, \text{ and all } j.$$

By the subsequence property we deduce

$$d(K_k^{(j)}, K_l^{(i)}) \leq \frac{\sqrt{n}}{2^i}, \quad \text{for all } k, l \in \mathbb{N}, \text{ and all } j \geq i.$$

In particular, if we choose the 'diagonal sequence'  $K_k := K_k^{(k)}$ ,  $k = 1, 2, \dots$ , then

$$d(K_k, K_l) \leq \frac{\sqrt{n}}{2^l}, \quad \text{for all } k \geq l.$$

Hence  $(K_k)_{k \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{M}$ .

Let

$$\tilde{K}_k := \text{cl conv} (K_k \cup K_{k+1} \cup \dots)$$

and

$$K_0 := \bigcap_{k=1}^{\infty} \tilde{K}_k.$$

We claim that

$$(*) \quad K_k \rightarrow K_0, \text{ as } k \rightarrow \infty, \quad \text{and} \quad K_0 \in \mathcal{K}^n.$$

First, by construction we have  $\tilde{K}_k \in \mathcal{K}^n$  and  $\tilde{K}_{k+1} \subset \tilde{K}_k$ ,  $k = 1, 2, \dots$ . Therefore,  $K_0 \in \mathcal{K}^n$ .

Since  $(K_k)_{k \in \mathbb{N}}$  is a Cauchy sequence, for each  $\varepsilon > 0$  there exists  $m \in \mathbb{N}$  such that

$$d(K_k, K_l) < \varepsilon, \quad \text{for all } k, l \geq m.$$

This implies

$$K_l \subset K_k + B(\varepsilon), \quad \text{for all } k, l \geq m,$$

therefore

$$\tilde{K}_{k'} \subset K_k + B(\varepsilon), \quad \text{for all } k, k' \geq m,$$

and thus

$$K_0 \subset K_k + B(\varepsilon), \quad \text{for all } k \geq m.$$

Conversely, for each  $\varepsilon > 0$ , there is  $\bar{m} \in \mathbb{N}$  such that

$$\tilde{K}_k \subset K_0 + B(\varepsilon), \quad \text{for all } k \geq \bar{m}.$$

Namely, assume on the contrary that

$$\tilde{K}_k \not\subset K_0 + B(\varepsilon), \quad \text{for infinitely many } k.$$

Then

$$\tilde{K}_{k_i} \cap [W \setminus \text{int}(K_0 + B(\varepsilon))] \neq \emptyset,$$

for a suitable sequence  $k_1, k_2, \dots$ . Since  $K_{k_i}$  and  $W \setminus \text{int}(K_0 + B(\varepsilon))$  are compact, this would imply

$$\bigcap_{i=1}^{\infty} (K_{k_i} \cap [W \setminus \text{int}(K_0 + B(\varepsilon))]) = K_0 \cap [W \setminus \text{int}(K_0 + B(\varepsilon))] \neq \emptyset,$$

a contradiction.

Since  $\tilde{K}_k \subset K_0 + B(\varepsilon)$  implies  $K_k \subset K_0 + B(\varepsilon)$ , we obtain

$$d(K_0, K_k) \leq \varepsilon, \quad \text{for all } k \geq \max(m, \bar{m}).$$

□

The topology on  $\mathcal{K}^n$  given by the Hausdorff metric allows us to introduce and study geometric functionals on convex bodies by first defining them for a special subclass, for example the class  $\mathcal{P}^n$  of polytopes. Such a program requires that the geometric functionals under consideration have a continuity or monotonicity property and also that the class  $\mathcal{P}^n$  of polytopes is dense in  $\mathcal{K}^n$ . We now discuss the latter aspect; geometric functionals will be investigated in the next section.

**Theorem 3.1.5.** *Let  $K \in \mathcal{K}^n$  and  $\varepsilon > 0$ .*

- (a) *There exists a polytope  $P \in \mathcal{P}^n$  with  $P \subset K$  and  $d(K, P) \leq \varepsilon$ .*
- (b) *There exists a polytope  $P \in \mathcal{P}^n$  with  $K \subset P$  and  $d(K, P) \leq \varepsilon$ .*
- (c) *If  $0 \in \text{rel int } K$ , then there exists a polytope  $P \in \mathcal{P}^n$  with  $P \subset K \subset (1 + \varepsilon)P$ .  
There is even a polytope  $\tilde{P} \in \mathcal{P}^n$  with  $\tilde{P} \subset \text{rel int } K$  and  $K \subset \text{rel int } ((1 + \varepsilon)\tilde{P})$ .*

*Proof.* (a) The family

$$\{x + \text{int } B(\varepsilon) : x \in \text{bd } K\}$$

is an open covering of the compact set  $\text{bd } K$ , therefore there exist  $x_1, \dots, x_m \in \text{bd } K$  with

$$\text{bd } K \subset \bigcup_{i=1}^m (x_i + \text{int } B(\varepsilon)).$$

Let

$$P := \text{conv } \{x_1, \dots, x_m\},$$

then

$$P \subset K \quad \text{and} \quad \text{bd } K \subset P + B(\varepsilon).$$

The latter implies  $K \subset P + B(\varepsilon)$  and therefore  $d(K, P) \leq \varepsilon$ .

(b) For each  $u \in S^{n-1}$ , there is a supporting hyperplane  $E(u)$  of  $K$  (in direction  $u$ ). Let  $A(u)$  be the open half-space of  $E(u)$  which fulfills  $A(u) \cap K = \emptyset$  ( $A(u)$  has the form  $\{\langle \cdot, u \rangle > h_K(u)\}$ ). Then, the family

$$\{A(u) : u \in S^{n-1}\}$$

is an open covering of the compact set  $\text{bd } (K + B(\varepsilon))$ , since every  $y \in \text{bd } (K + B(\varepsilon))$  fulfills  $y \notin K$  and is therefore separated from  $K$  by a supporting hyperplane  $E = E(u)$  of  $K$ . Therefore there exist  $u_1, \dots, u_m \in S^{n-1}$  with

$$\text{bd } (K + B(\varepsilon)) \subset \bigcup_{i=1}^m A(u_i).$$

Let  $H(u_i) := \mathbb{R}^n \setminus A(u_i)$  be the corresponding supporting half-space and

$$P := \bigcap_{i=1}^m H(u_i),$$

then

$$K \subset P.$$

Since  $\mathbb{R}^n \setminus P = \bigcup_{i=1}^m A(u_i)$ , we also have

$$P \subset K + B(\varepsilon),$$

and therefore  $d(K, P) \leq \varepsilon$ .

(c) W.l.o.g. we may assume that  $\dim K = n$ , hence  $0 \in \text{int } K$ . If we copy the proof of (b) with  $B(\varepsilon) = \varepsilon B(1)$  replaced by  $\varepsilon K$ , we obtain a polytope  $P'$  with

$$K \subset P' \subset (1 + \varepsilon)K.$$

The polytope  $P := \frac{1}{1+\varepsilon}P'$  then fulfills

$$P \subset K \subset (1 + \varepsilon)P.$$

In particular, we get a polytope  $\bar{P}$  with

$$\bar{P} \subset K \subset (1 + \frac{\varepsilon}{2})\bar{P}.$$

We choose  $\tilde{P} := \delta\bar{P}$  with  $0 < \delta < 1$ . Then

$$\tilde{P} \subset \text{rel int } \bar{P} \subset \text{rel int } K.$$

If  $\delta$  is close to 1, such that  $(1 + \frac{\varepsilon}{2})\frac{1}{\delta} < 1 + \varepsilon$ , then

$$K \subset (1 + \frac{\varepsilon}{2})\frac{1}{\delta}\tilde{P} \subset \text{rel int } ((1 + \varepsilon)\tilde{P}).$$

□

**Remark.** The theorem shows that  $\text{cl } \mathcal{P}^n = \mathcal{K}^n$ . One can even show that the metric space  $\mathcal{K}^n$  is separable, since there is a countable dense set  $\tilde{\mathcal{P}}^n$  of polytopes. For this purpose, the above proofs have to be modified such that the polytopes involved have vertices with rational coordinates.

### Exercises and problems

1. Let  $K, L, M \in \mathcal{K}^n$ . Without using support functions, show that:

(a) For  $u \in S^{n-1}$ , we have

$$K(u) + M(u) = (K + M)(u).$$

(b) If  $K + L \subset M + L$ , then  $K \subset M$  (generalized cancellation rule).

2. Let  $(K_i)_{i \in \mathbb{N}}$  be a sequence in  $\mathcal{K}^n$  and  $K \in \mathcal{K}^n$ . Show that  $K_i \rightarrow K$  (in the Hausdorff metric), if and only if the following two conditions are fulfilled:

(a) Each  $x \in K$  is a limit point of a suitable sequence  $(x_i)_{i \in \mathbb{N}}$  with  $x_i \in K_i$ , for all  $i \in \mathbb{N}$ .

(b) For each sequence  $(x_i)_{i \in \mathbb{N}}$  with  $x_i \in K_i$ , for all  $i \in \mathbb{N}$ , every culmination point lies in  $K$ .

3. (a) Let  $K, M \in \mathcal{K}^n$  be convex bodies, which cannot be separated by a hyperplane (i.e., there is no hyperplane  $\{f = \alpha\}$  with  $K \subset \{f \leq \alpha\}$  and  $M \subset \{f \geq \alpha\}$ ). Further, let  $(K_i)_{i \in \mathbb{N}}$  and  $(M_i)_{i \in \mathbb{N}}$  be sequences in  $\mathcal{K}^n$ . Show that

$$K_i \rightarrow K, M_i \rightarrow M \implies K_i \cap M_i \rightarrow K \cap M.$$

- (b) Let  $K \in \mathcal{K}^n$  be a convex body and  $E \subset \mathbb{R}^n$  an affine subspace with  $E \cap \text{int } K \neq \emptyset$ . Further, let  $(K_i)_{i \in \mathbb{N}}$  be a sequence in  $\mathcal{K}^n$ . Show that

$$K_i \rightarrow K \implies (E \cap K_i) \rightarrow E \cap K.$$

*Hint:* Use Exercise 2 above.

4. Let  $K \subset \mathbb{R}^n$  be compact. Show that:
- There is a unique ball  $K_a$  of smallest diameter with  $K \subset K_a$  (*circumball*).
  - If  $\text{int } K \neq \emptyset$ , then there exists a ball  $K_i$  of maximal diameter with  $K_i \subset K$  (*inball*).
5. A body  $K \in \mathcal{K}^n$  is *strictly convex*, if it does not contain any segments in the boundary.
- Show that the set of all strictly convex bodies in  $\mathbb{R}^n$  is a  $G_\delta$ -set in  $\mathcal{K}^n$ , i.e. intersection of countably many open sets in  $\mathcal{K}^n$ .
  - \* Show that the set of all strictly convex bodies in  $\mathbb{R}^n$  is dense in  $\mathcal{K}^n$ .
6. Let  $(K_i)_{i \in \mathbb{N}}$  be a sequence in  $\mathcal{K}^n$ , for which the support functions  $h_{K_i}(u)$  converge to the values  $h(u)$  of a function  $h : S^{n-1} \rightarrow \mathbb{R}$ , for each  $u \in S^{n-1}$ . Show that  $h$  is the support function of a convex body and that  $h_{K_i} \rightarrow h$  uniformly on  $S^{n-1}$ .
7. Let  $P$  be a convex polygon in  $\mathbb{R}^2$  with  $\text{int } P \neq \emptyset$ . Show that:
- There is a polygon  $P_1$  and a triangle (or a segment)  $\Delta$  with  $P = P_1 + \Delta$ .
  - $P$  has a representation  $P = \Delta_1 + \cdots + \Delta_m$ , with triangles (segments)  $\Delta_j$  which are pairwise not homothetic.
  - $P$  is a triangle, if and only if  $m = 1$ .
- \* 8. A body  $K \in \mathcal{K}^n, n \geq 2$ , is *indecomposable*, if  $K = M + L$  implies  $M = \alpha K + x$  and  $L = \beta K + y$ , for some  $\alpha, \beta \geq 0$  and  $x, y \in \mathbb{R}^n$ . Show that:
- If  $P \in \mathcal{K}^n$  is a polytope and all 2-faces of  $P$  are triangles,  $P$  is indecomposable.
  - For  $n \geq 3$ , the set of indecomposable convex bodies is a dense  $G_\delta$ -set in  $\mathcal{K}^n$ .
9. Let  $\mathcal{I}^n$  be the set of convex bodies  $K \in \mathcal{K}^n$ , which are strictly convex **and** indecomposable.
- Show that  $\mathcal{I}^n$  is dense in  $\mathcal{K}^n$ .
  - P Find one element of  $\mathcal{I}^n$ .

### 3.2 Volume and surface area

The volume of a convex body  $K \in \mathcal{K}^n$  can be defined as the Lebesgue measure  $\lambda_n(K)$  of  $K$ . However, the convexity of  $K$  implies that the volume also exists in an elementary sense and, moreover, that also the surface area of  $K$  exists. Therefore, we now introduce both notions in an elementary way, first for polytopes and then for arbitrary convex bodies by approximation.

Since we will use a recursive definition on the dimension  $n$ , we first remark that the support set  $K(u)$ ,  $u \in S^{n-1}$ , of a convex body  $K$  lies in a hyperplane parallel to  $u^\perp$ . Therefore, the orthogonal projection  $K(u)|_{u^\perp}$  is a translate of  $K(u)$ , and we can consider  $K(u)|_{u^\perp}$  as a convex body in  $\mathbb{R}^{n-1}$  (if we identify  $u^\perp$  with  $\mathbb{R}^{n-1}$ ). Assuming that the volume is already defined in  $\mathbb{R}^{n-1}$ , we then denote by  $V^{(n-1)}(K(u)|_{u^\perp})$  the  $(n-1)$ -dimensional volume of this projection. In principle, the identification of  $u^\perp$  with  $\mathbb{R}^{n-1}$  requires that we have given an orthonormal basis in  $u^\perp$ . However, it will be apparent that the quantities we define depend only on the Euclidean metric in  $u^\perp$ , hence they are independent of the choice of a basis.

**Definition.** Let  $P \in \mathcal{P}^n$  be a polytope.

For  $n = 1$ , hence  $P = [a, b]$ , we define  $V^{(1)}(P) := b - a$  and  $F^{(1)}(P) := 2$ .

For  $n \geq 2$ , let

$$V^{(n)}(P) := \begin{cases} \frac{1}{n} \sum_{(*)} h_P(u) V^{(n-1)}(P(u)|_{u^\perp}) & \text{if } \dim P \geq n-1, \\ 0 & \dim P \leq n-2, \end{cases}$$

and

$$F^{(n)}(P) := \begin{cases} \sum_{(*)} V^{(n-1)}(P(u)|_{u^\perp}) & \text{if } \dim P \geq n-1, \\ 0 & \dim P \leq n-2, \end{cases}$$

where the summation  $(*)$  is over all  $u \in S^{n-1}$ , for which  $P(u)$  is a facet of  $P$ . We shortly write  $V(P)$  for  $V^{(n)}(P)$  and call this the *volume* of  $P$ . Similarly, we write  $F(P)$  instead of  $F^{(n)}(P)$  and call this the *surface area* of  $P$ .

For  $\dim P = n-1$ , there are two support sets of  $P$  which are facets, namely  $P = P(u_0)$  and  $P = P(-u_0)$ , where  $u_0$  is a normal vector to  $P$ . Since then  $V^{(n-1)}(P(u_0)|_{u_0^\perp}) = V^{(n-1)}(P(-u_0)|_{u_0^\perp})$  and  $h_P(u_0) = -h_P(-u_0)$ , we obtain  $V(P) = 0$ , in coincidence with the Lebesgue measure of  $P$ . Also, in this case,  $F(P) = 2V^{(n-1)}(P(u_0)|_{u_0^\perp})$ . For  $\dim P \leq n-2$ , the polytope  $P$  does not have any facets, hence  $V(P) = 0$  and  $F(P) = 0$ .



**Proposition 3.2.1.** *The volume  $V$  and surface area  $F$  of polytopes  $P, Q$  have the following properties:*

- (1)  $V(P) = \lambda_n(P)$ ,
- (2)  $V$  and  $F$  are invariant with respect to rigid motions,
- (3)  $V(\alpha P) = \alpha^n V(P)$ ,  $F(\alpha P) = \alpha^{n-1} F(P)$ , for  $\alpha \geq 0$ ,
- (4)  $V(P) = 0$ , if and only if  $\dim P \leq n - 1$ ,
- (5) if  $P \subset Q$ , then  $V(P) \leq V(Q)$  and  $F(P) \leq F(Q)$ .

*Proof.* (1) We proceed by induction on  $n$ . The result is clear for  $n = 1$ . Let  $n \geq 2$ . As we have already mentioned,  $V(P) = 0 = \lambda_n(P)$  if  $\dim P \leq n - 1$ . For  $\dim P = n$ , let  $P(u_1), \dots, P(u_k)$  be the facets of  $P$ . Then, we have

$$V(P) = \frac{1}{n} \sum_{i=1}^k h_P(u_i) V^{(n-1)}(P(u_i)|u_i^\perp),$$

where, by the inductive assumption,  $V^{(n-1)}(P(u_i)|u_i^\perp)$  equals the  $(n - 1)$ -dimensional Lebesgue measure (in  $u_i^\perp$ ) of  $P(u_i)|u_i^\perp$ . We assume w.l.o.g. that  $h_P(u_1), \dots, h_P(u_m) \geq 0$  and  $h_P(u_{m+1}), \dots, h_P(u_k) < 0$ , and consider the pyramid-shaped polytopes  $P_i := \text{conv}(P(u_i) \cup \{0\})$ ,  $i = 1, \dots, k$ . Then  $V(P_i) = \frac{1}{n} h_P(u_i) V^{(n-1)}(P(u_i)|u_i^\perp)$ ,  $i = 1, \dots, m$ , and  $V(P_i) = -\frac{1}{n} h_P(u_i) V^{(n-1)}(P(u_i)|u_i^\perp)$ ,  $i = m + 1, \dots, k$ . Hence,

$$\begin{aligned} V(P) &= \sum_{i=1}^m V(P_i) - \sum_{i=m+1}^k V(P_i) \\ &= \sum_{i=1}^m \lambda_n(P_i) - \sum_{i=m+1}^k \lambda_n(P_i) \\ &= \lambda_n(P). \end{aligned}$$

Here, we have used that the Lebesgue measure of the pyramid  $P_i$  is  $\frac{1}{n}$  times the content of the base (here  $V^{(n-1)}(P(u_i)|u_i^\perp)$ ) times the height (here  $h_P(u_i)$ ). Moreover the Lebesgue measure of the pyramid parts outside  $P$  cancel out, and the pyramid parts inside  $P$  yield a dissection of  $P$  (into sets with disjoint interior).

(2), (3), (4) and the first part of (5) follow now directly from (1) (and the corresponding properties of the Lebesgue measure). It remains to show  $F(P) \leq F(Q)$ , for polytopes  $P \subset Q$ . We may assume  $\dim Q = n$ . Again, we denote the facets of  $P$  by  $P(u_1), \dots, P(u_k)$ . We make use of the following elementary inequality (a generalization of the triangle inequality),

$$(*) \quad V^{(n-1)}(P(u_i)|u_i^\perp) \leq \sum_{j \neq i} V^{(n-1)}(P(u_j)|u_j^\perp), \quad i = 1, \dots, k.$$

In order to motivate (\*), we project  $P(u_j), j \neq i$ , orthogonally onto the hyperplane  $u_i^\perp$ . The projections then cover  $P(u_i)|u_i^\perp$ . Since the projection does not increase the  $(n-1)$ -dimensional Lebesgue measure, (\*) follows. (\*) implies that

$$F(Q \cap H) \leq F(Q),$$

for any closed half-space  $H \subset \mathbb{R}^n$ . Since  $P \subset Q$  is a finite intersection of half-spaces, we obtain  $F(P) \leq F(Q)$  by successive truncation.  $\square$

**Remark.** We can now simplify our formulas for the volume  $V(P)$  and the surface area  $F(P)$  of a polytope  $P$ . First, since we have shown that our elementarily defined volume equals the Lebesgue measure and is thus translation invariant, we do not need the orthogonal projection of the facets anymore. Second, since  $V^{(n-1)}(P(u)) = 0$ , for  $\dim P(u) \leq n-2$ , we can sum over all  $u \in S^{n-1}$ . If we write, in addition,  $v$  instead of  $V^{(n-1)}$ , we obtain

$$V(P) = \frac{1}{n} \sum_{u \in S^{n-1}} h_P(u) v(P(u))$$

and

$$F(P) = \sum_{u \in S^{n-1}} v(P(u)).$$

These are the formulas which we will use, in the following.

For a convex body  $K \in \mathcal{K}^n$ , we define

$$V_+(K) := \inf_{P \supset K} V(P), \quad V_-(K) := \sup_{P \subset K} V(P),$$

and

$$F_+(K) := \inf_{P \supset K} F(P), \quad F_-(K) := \sup_{P \subset K} F(P),$$

(here  $P \in \mathcal{P}^n$ ).

**Theorem 3.2.2 (and Definition).** *Let  $K \in \mathcal{K}^n$ .*

(a) *We have*

$$V_+(K) = V_-(K) =: V(K)$$

and

$$F_+(K) = F_-(K) =: F(K).$$

$V(K)$  is called the volume and  $F(K)$  is called the surface area of  $K$ .

(b) *Volume and surface area have the following properties:*

(b1)  $V(K) = \lambda_n(K)$ ,

(b2)  $V$  and  $F$  are invariant with respect to rigid motions,

(b3)  $V(\alpha K) = \alpha^n V(K)$ ,  $F(\alpha K) = \alpha^{n-1} F(K)$ , for  $\alpha \geq 0$ ,

(b4)  $V(K) = 0$ , if and only if  $\dim K \leq n - 1$ ,

(b5) if  $K \subset L$ , then  $V(K) \leq V(L)$  and  $F(K) \leq F(L)$ ,

(b6)  $K \mapsto V(K)$  is continuous.

*Proof.* (a) We first remark, that for a polytope  $P$  the monotonicity of  $V$  and  $F$  (which was proved in Proposition 3.2.1(5)) shows that  $V^+(P) = V(P) = V^-(P)$  and  $F^+(P) = F(P) = F^-(P)$ . Hence, the new definition of  $V(P)$  and  $F(P)$  is consistent with the old one.

For an arbitrary body  $K \in \mathcal{K}^n$ , we get from Proposition 3.2.1(5)

$$V_-(K) \leq V_+(K) \quad \text{and} \quad F_-(K) \leq F_+(K),$$

and by Proposition 3.2.1(2),  $V_-(K)$ ,  $V_+(K)$ ,  $F_-(K)$  and  $F_+(K)$  are motion invariant. After a suitable translation, we may therefore assume  $0 \in \text{relint } K$ . For  $\varepsilon > 0$ , we then use Theorem 3.1.5(c) and find a polytope  $P$  with  $P \subset K \subset (1 + \varepsilon)P$ . From Proposition 3.2.1(3), we get

$$V(P) \leq V_-(K) \leq V_+(K) \leq V((1 + \varepsilon)P) = (1 + \varepsilon)^n V(P)$$

and

$$F(P) \leq F_-(K) \leq F_+(K) \leq F((1 + \varepsilon)P) = (1 + \varepsilon)^{n-1} F(P).$$

For  $\varepsilon \rightarrow 0$ , this proves (a).

(b1) – (b5) follow now directly for bodies  $K \in \mathcal{K}^n$  ((b1) by approximation with polytopes; (b2) – (b5) partially by approximation or from the corresponding properties of the Lebesgue measure).

It remains to prove (b6). Consider a converging sequence  $K_i \rightarrow K$ ,  $K_i, K \in \mathcal{K}^n$ . In view of (b2), we can assume  $0 \in \text{relint } K$ . Using again Theorem 3.1.5(c), we find a polytope  $P$  with  $P \subset \text{relint } K$ ,  $K \subset \text{relint } (1 + \varepsilon)P$ . If  $\dim K = n$ , we can choose a ball  $B(r)$ ,  $r > 0$ , with  $K + B(r) \subset (1 + \varepsilon)P$  (choose  $r = \min_{u \in S^{n-1}} (h_{(1+\varepsilon)P}(u) - h_K(u))$ .) Then  $K_i \subset (1 + \varepsilon)P$ , for  $i \geq i_0$ . Analogously, we get  $P \subset K_i$ , for  $i \geq i_1$ . For  $i \geq \max(i_0, i_1)$ , we therefore obtain

$$\begin{aligned} |V(K_i) - V(K)| &\leq (1 + \varepsilon)^n V(P) - V(P) \\ &\leq [(1 + \varepsilon)^n - 1] V(K) \rightarrow 0, \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . If  $\dim K = j \leq n - 1$ , hence  $V(K) = 0$ , we have

$$K \subset \text{int } ((1 + \varepsilon)P + \varepsilon W),$$

where  $W$  is a cube, centred at 0, with edge length 1 and dimension  $n - j$ , lying in the orthogonal space  $(\text{aff } K)^\perp$ . As above, we obtain  $K_i \subset (1 + \varepsilon)P + \varepsilon W$ , for  $i \geq i_0$ . Since

$$V((1 + \varepsilon)P + \varepsilon W) \leq \varepsilon^{n-j}(1 + \varepsilon)^j C$$

(where we can choose the constant  $C$  to be the  $j$ -dimensional Lebesgue measure of  $K$ ), this gives us  $V(K_i) \rightarrow 0 = V(K)$ , as  $\varepsilon \rightarrow 0$ .  $\square$

**Remark.** We shall see in the next section that the surface area  $F$  is also continuous.

### Exercises and problems

1. A convex body  $K \in \mathcal{K}^2$  is called *universal cover*, if for each  $L \in \mathcal{K}^2$  with diameter  $\leq 1$  there is a rigid motion  $g_L$  with  $L \subset g_L K$ .
  - (a) Show that there is a universal cover  $K_0$  with minimal area.
  - P (b) Find the shape and the area of  $K_0$ .

### 3.3 Mixed volumes

There is another, commonly used definition of the surface area of a set  $K \subset \mathbb{R}^n$ , namely as the derivative of the volume functional of the outer parallel sets of  $K$ , i.e.

$$F(K) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} (V(K + B(\varepsilon)) - V(K)).$$

We will see now, that our notion of surface area of a convex body  $K$  fulfills also this limit relation. In fact, we will show that  $V(K + B(\varepsilon))$  is a polynomial in  $\varepsilon$  (this is the famous *STEINER formula*) and thereby get a whole family of geometric functionals. We start with an even more general problem and investigate how the volume

$$V(\alpha_1 K_1 + \cdots + \alpha_m K_m),$$

for  $K_i \in \mathcal{K}^n, \alpha_i > 0$ , depends on the variables  $\alpha_1, \dots, \alpha_m$ . This will lead us to a family of mixed functionals of convex bodies, the *mixed volumes*.

Again, we first consider the case of polytopes. Since the recursive representation of the volume of a polytope  $P$  was based on the support sets (facets) of  $P$ , we discuss how support sets behave under linear combinations.

**Proposition 3.3.1.** *Let  $m \in \mathbb{N}, \alpha_1, \dots, \alpha_m > 0, P_1, \dots, P_m \in \mathcal{P}^n$  be polytopes, and  $u, v \in S^{n-1}$ . Then,*

(a)  $(\alpha_1 P_1 + \cdots + \alpha_m P_m)(u) = \alpha_1 P_1(u) + \cdots + \alpha_m P_m(u),$

(b)  $\dim (\alpha_1 P_1 + \cdots + \alpha_m P_m)(u) = \dim (P_1 + \cdots + P_m)(u),$

(c) *if  $(P_1 + \cdots + P_m)(u) \cap (P_1 + \cdots + P_m)(v) \neq \emptyset$ , then*

$$(P_1 + \cdots + P_m)(u) \cap (P_1 + \cdots + P_m)(v) = (P_1(u) \cap P_1(v)) + \cdots + (P_m(u) \cap P_m(v)).$$

*Proof.* (a) By Theorem 2.3.3, we have

$$\begin{aligned} h_{(\alpha_1 P_1 + \cdots + \alpha_m P_m)(u)}(x) &= h'_{\alpha_1 P_1 + \cdots + \alpha_m P_m}(u; x) \\ &= \alpha_1 h'_{P_1}(u; x) + \cdots + \alpha_m h'_{P_m}(u; x) \\ &= \alpha_1 h_{P_1(u)}(x) + \cdots + \alpha_m h_{P_m(u)}(x) \\ &= h_{\alpha_1 P_1(u) + \cdots + \alpha_m P_m(u)}(x). \end{aligned}$$

Theorem 2.3.1 now yields the assertion.

(b) Let  $P := P_1 + \cdots + P_m$  and  $\tilde{P} := \alpha_1 P_1 + \cdots + \alpha_m P_m$ . W.l.o.g. we may assume  $0 \in \text{rel int } P_i(u), i = 1, \dots, m$ . Then,  $0 \in \text{rel int } P(u)$ . We put

$$\alpha := \min_{i=1, \dots, m} \alpha_i, \quad \beta := \max_{i=1, \dots, m} \alpha_i.$$

Then,  $0 < \alpha < \beta$  and (in view of (a))

$$\alpha P(u) \subset \tilde{P}(u) \subset \beta P(u),$$

that is,  $\dim P(u) = \dim \tilde{P}(u)$ .

(c) Using the notation introduced above, we assume  $P(u) \cap P(v) \neq \emptyset$ . Consider  $x \in P(u) \cap P(v)$ . Since  $x \in P$ , it has a representation  $x = x_1 + \cdots + x_m$ , with  $x_i \in P_i$ . Because of

$$h_P(u) = \langle x, u \rangle = \sum_{i=1}^m \langle x_i, u \rangle \leq \sum_{i=1}^m h_{P_i}(u) = h_P(u),$$

we get that  $\langle x_i, u \rangle = h_{P_i}(u)$  and thus  $x_i \in P_i(u)$ , for  $i = 1, \dots, m$ . In the same way, we obtain  $x_i \in P_i(v)$ ,  $i = 1, \dots, m$ .

Conversely, it is clear that any  $x \in (P_1(u) \cap P_1(v)) + \cdots + (P_m(u) \cap P_m(v))$  fulfills  $x \in P_1(u) + \cdots + P_m(u) = P(u)$  and  $x \in P_1(v) + \cdots + P_m(v) = P(v)$ , by (a).  $\square$

In analogy to the recursive definition of the volume of a polytope, we now define the mixed volume of polytopes. Again, we use projections of the support sets (faces) in order to make the definition rigorous. After we have shown translation invariance of the functionals, the corresponding formulas will become simpler.

For polytopes  $P_1, \dots, P_k \in \mathcal{P}_n$ , let  $N(P_1, \dots, P_k)$  denote the set of all facet normals of  $P_1 + \cdots + P_k$ .

**Definition.** For polytopes  $P_1, \dots, P_n \in \mathcal{P}^n$ , we define the *mixed volume*  $V^{(n)}(P_1, \dots, P_n)$  of  $P_1, \dots, P_n$  recursively:

$$V^{(1)}(P_1) := V(P_1) = h_{P_1}(1) + h_{P_1}(-1) (= b - a, \text{ if } P_1 = [a, b]), \quad \text{for } n = 1,$$

$$V^{(n)}(P_1, \dots, P_n)$$

$$:= \frac{1}{n} \sum_{u \in N(P_1, \dots, P_{n-1})} h_{P_n}(u) V^{(n-1)}(P_1(u)|u^\perp, \dots, P_{n-1}(u)|u^\perp), \quad \text{for } n \geq 2.$$

**Theorem 3.3.2.** *The mixed volume  $V^{(n)}(P_1, \dots, P_n)$  of polytopes  $P_1, \dots, P_n \in \mathcal{P}^n$  is symmetric in the indices  $1, \dots, n$ , independent of individual translations of the polytopes  $P_1, \dots, P_n$ , and for  $\dim(P_1 + \cdots + P_n) \leq n - 1$ , we have  $V^{(n)}(P_1, \dots, P_n) = 0$ .*

*Furthermore, for  $m \in \mathbb{N}$ ,  $P_1, \dots, P_m \in \mathcal{P}^n$ , and  $\alpha_1, \dots, \alpha_m \geq 0$ , we have*

$$(*) \quad V(\alpha_1 P_1 + \cdots + \alpha_m P_m) = \sum_{i_1=1}^m \cdots \sum_{i_n=1}^m \alpha_{i_1} \cdots \alpha_{i_n} V^{(n)}(P_{i_1}, \dots, P_{i_n}).$$

For the proof, it is convenient to extend the  $k$ -dimensional mixed volume  $V^{(k)}(Q_1, \dots, Q_k)$  (which is defined for polytopes  $Q_1, \dots, Q_k$  in a  $k$ -dimensional linear subspace  $E \subset \mathbb{R}^d$ ) to polytopes  $Q_1, \dots, Q_k \in \mathcal{P}^n$ , which fulfill  $\dim(Q_1 + \cdots + Q_k) \leq k$ , namely by

$$V^{(k)}(Q_1, \dots, Q_k) := V^{(k)}(Q_1|E, \dots, Q_k|E),$$

where  $E$  is a  $k$ -dimensional subspace parallel to  $Q_1 + \cdots + Q_k$ ,  $1 \leq k \leq n-1$ . The translation invariance and the dimensional condition, which we will prove, show that this extension is consistent (and independent of  $E$  in case  $\dim(Q_1 + \cdots + Q_k) < k$ ). In the following inductive proof, we already make use of this extension in order to simplify the presentation. In particular, in the induction step, we use the mixed volume  $V^{(n-1)}(P_1(u), \dots, P_{n-1}(u))$ .

In addition, we extend the mixed volume to the empty set, namely as  $V^{(n)}(P_1, \dots, P_n) := 0$ , if one of the sets  $P_i$  is empty.

*Proof.* We use induction on the dimension  $n$ .

For  $n = 1$ , the polytopes  $P_i$  are intervals and the mixed volume equals the (one-dimensional) volume  $V^{(1)}$  (the length of the intervals), which is linear

$$V^{(1)}(\alpha_1 P_1 + \cdots + \alpha_m P_m) = \sum_{i=1}^m \alpha_i V^{(1)}(P_i).$$

Hence, (\*) holds as well as the remaining assertions.

Now we assume that the assertions of the theorem are true for all dimensions  $\leq n-1$ , and we consider dimension  $n \geq 2$ . We first discuss the dimensional statement. If  $\dim(P_1 + \cdots + P_n) \leq n-1$ , then either  $N(P_1, \dots, P_{n-1}) = \emptyset$  or  $N(P_1, \dots, P_{n-1}) = \{-u, u\}$ , where  $u$  is the normal on  $\text{aff}(P_1 + \cdots + P_n)$ . In the first case, we have  $V^{(n)}(P_1, \dots, P_n) = 0$ , by definition; in the second case, we have

$$\begin{aligned} & V^{(n)}(P_1, \dots, P_n) \\ &= \frac{1}{n} (h_{P_n}(u) V^{(n-1)}(P_1(u), \dots, P_{n-1}(u)) + h_{P_n}(-u) V^{(n-1)}(P_1(-u), \dots, P_{n-1}(-u))) \\ &= \frac{1}{n} (h_{P_n}(u) V^{(n-1)}(P_1(u), \dots, P_{n-1}(u)) - h_{P_n}(u) V^{(n-1)}(P_1(u), \dots, P_{n-1}(u))) \\ &= 0. \end{aligned}$$

Next, we show (\*). If  $\alpha_i = 0$ , for a certain index  $i$ , the correspondent summand  $\alpha_i P_i$  in the left-hand sum can be deleted, as well as all summands in the right-hand sum which contain the index  $i$ . Therefore, it is sufficient to consider the case  $\alpha_1 > 0, \dots, \alpha_m > 0$ . By the definition of volume and Proposition 3.3.1,

$$\begin{aligned} V(\alpha_1 P_1 + \cdots + \alpha_m P_m) &= \frac{1}{n} \sum_{u \in N(P_1, \dots, P_m)} h_{\sum_{i=1}^m \alpha_i P_i}(u) v \left( \sum_{i=1}^m \alpha_i P_i(u) \right) \\ &= \sum_{i_n=1}^m \alpha_{i_n} \frac{1}{n} \sum_{u \in N(P_1, \dots, P_m)} h_{P_{i_n}}(u) v \left( \sum_{i=1}^m \alpha_i (P_i(u) | u^\perp) \right). \end{aligned}$$

The inductive assumption implies

$$v \left( \sum_{i=1}^m \alpha_i (P_i(u) | u^\perp) \right) = \sum_{i_1=1}^m \cdots \sum_{i_{n-1}=1}^m \alpha_{i_1} \cdots \alpha_{i_{n-1}} V^{(n-1)}(P_{i_1}(u), \dots, P_{i_{n-1}}(u)).$$

Hence, we obtain

$$\begin{aligned}
& V(\alpha_1 P_1 + \cdots + \alpha_m P_m) \\
&= \sum_{i_1=1}^m \cdots \sum_{i_{n-1}=1}^m \sum_{i_n=1}^m \alpha_{i_1} \cdots \alpha_{i_{n-1}} \alpha_{i_n} \frac{1}{n} \sum_{u \in N(P_1, \dots, P_m)} h_{P_{i_n}}(u) V^{(n-1)}(P_{i_1}(u), \dots, P_{i_{n-1}}(u)) \\
&= \sum_{i_1=1}^m \cdots \sum_{i_n=1}^m \alpha_{i_1} \cdots \alpha_{i_n} V^{(n)}(P_{i_1}, \dots, P_{i_n}).
\end{aligned}$$

Here, we have used that, for a given set of indices  $\{i_1, \dots, i_n\}$ , the summation over  $N(P_1, \dots, P_m)$  can be replaced by the summation over  $N(P_{i_1}, \dots, P_{i_{n-1}})$ . Namely, for  $u \notin N(P_{i_1}, \dots, P_{i_{n-1}})$  the support set  $P_{i_1}(u) + \cdots + P_{i_{n-1}}(u) = (P_{i_1} + \cdots + P_{i_{n-1}})(u)$  has dimension  $\leq n - 2$  and hence  $V^{(n-1)}(P_{i_1}(u), \dots, P_{i_{n-1}}(u)) = 0$ . We will use this fact also in the following parts of the proof.

We next show the symmetry. Since  $V^{(n-1)}(P_1(u), \dots, P_{n-1}(u))$  is symmetric (in the indices) by the inductive assumption, it suffices to show

$$V^{(n)}(P_1, \dots, P_{n-2}, P_{n-1}, P_n) = V^{(n)}(P_1, \dots, P_{n-2}, P_n, P_{n-1}).$$

Moreover, we may assume that  $P := P_1 + \cdots + P_n$  fulfills  $\dim P = n$ . We first assume  $n \geq 3$ . By definition,

$$\begin{aligned}
& V^{(n-1)}(P_1(u), \dots, P_{n-1}(u)) \\
&= \frac{1}{n-1} \sum_{\tilde{v} \in \tilde{N}} h_{P_{n-1}(u)}(\tilde{v}) V^{(n-2)}((P_1(u))(\tilde{v}), \dots, (P_{n-2}(u))(\tilde{v})),
\end{aligned}$$

where we have to sum over the set  $\tilde{N}$  of facet normals of  $P(u)$  (in  $u^\perp$ ). Formally, we would have to work with the projections (the shifted support sets)  $P_1(u)|_{u^\perp}, \dots, P_{n-1}(u)|_{u^\perp}$ , but here we make use of our extended definition of the  $(n-2)$ -dimensional mixed volume and of the fact that

$$h_{P_{n-1}(u)|_{u^\perp}}(\tilde{v}) = h_{P_{n-1}(u)}(\tilde{v}),$$

for all  $\tilde{v} \perp u$ . The facets of  $P(u)$  are  $(n-2)$ -dimensional faces of  $P$ , thus they arise (because of  $\dim P = n$ ) as intersections  $P(u) \cap P(v)$  of the facet  $P(u)$  with another facet  $P(v)$  of  $P$ . Since  $\dim P = n$ , the case  $v = -u$  cannot occur. If  $P(u) \cap P(v)$  is a  $(n-2)$ -face of  $P$ , hence a facet of  $P(u)$ , the corresponding normal (in  $u^\perp$ ) is given by  $\tilde{v} = \frac{v|_{u^\perp}}{\|v|_{u^\perp}\|}$ .

Now we have

$$\begin{aligned}
P(u) \cap P(v) &= (P_1(u) + \cdots + P_n(u)) \cap (P_1(v) + \cdots + P_n(v)) \\
&= (P_1(u) \cap P_1(v)) + \cdots + (P_n(u) \cap P_n(v)),
\end{aligned}$$



from Proposition 3.3.1(c). For a  $(n-2)$ -face  $P(u) \cap P(v)$  of  $P$ , we therefore obtain

$$(P_i(u))(\tilde{v}) = P_i(u) \cap P_i(v), \quad i = 1, \dots, n-2,$$

which implies

$$\begin{aligned} & V^{(n-1)}(P_1(u), \dots, P_{n-1}(u)) \\ &= \frac{1}{n-1} \sum_{v \in N(P_1, \dots, P_n), v \neq \pm u} h_{P_{n-1}(u)}\left(\frac{v|u^\perp}{\|v|u^\perp\|}\right) V^{(n-2)}(P_1(u) \cap P_1(v), \dots, P_{n-2}(u) \cap P_{n-2}(v)). \end{aligned}$$

Here, we may sum again over all  $v \in N(P_1, \dots, P_n)$ ,  $v \neq \pm u$ , since for those  $v$ , for which  $P(u) \cap P(v)$  is not a  $(n-2)$ -face of  $P$ , the mixed volume  $V^{(n-2)}(P_1(u) \cap P_1(v), \dots, P_{n-2}(u) \cap P_{n-2}(v))$  vanishes by the inductive hypothesis (resp. by definition, if the intersection is empty).

Let  $\gamma(u, v)$  denote the (outer) angle between  $u$  and  $v$ , then

$$\|v|u^\perp\| = \sin \gamma(u, v), \quad \langle u, v \rangle = \cos \gamma(u, v),$$

and hence

$$\frac{v|u^\perp}{\|v|u^\perp\|} = \frac{1}{\sin \gamma(u, v)} v - \frac{1}{\tan \gamma(u, v)} u.$$

For  $x \in P_{n-1}(u) \cap P_{n-1}(v)$ , we have

$$\begin{aligned} h_{P_{n-1}(u)}(\tilde{v}) &= \langle x, \tilde{v} \rangle \\ &= \frac{1}{\sin \gamma(u, v)} \langle x, v \rangle - \frac{1}{\tan \gamma(u, v)} \langle x, u \rangle \\ &= \frac{1}{\sin \gamma(u, v)} h_{P_{n-1}(v)} - \frac{1}{\tan \gamma(u, v)} h_{P_{n-1}(u)}. \end{aligned}$$

Hence, altogether we obtain

$$\begin{aligned} & V^{(n)}(P_1, \dots, P_{n-2}, P_{n-1}, P_n) \\ &= \frac{1}{n} \sum_{u \in N(P_1, \dots, P_n)} h_{P_n}(u) V^{(n-1)}(P_1(u), \dots, P_{n-1}(u)) \\ &= \frac{1}{n(n-1)} \sum_{u, v \in N(P_1, \dots, P_n), v \neq \pm u} \left[ \frac{1}{\sin \gamma(u, v)} h_{P_n}(u) h_{P_{n-1}(v)} \right. \\ &\quad \left. - \frac{1}{\tan \gamma(u, v)} h_{P_n}(u) h_{P_{n-1}(u)} \right] V^{(n-2)}(P_1(u) \cap P_1(v), \dots, P_{n-2}(u) \cap P_{n-2}(v)) \\ &= V^{(n)}(P_1, \dots, P_{n-2}, P_n, P_{n-1}), \end{aligned}$$

and the symmetry is proved.

The case  $n = 2$  is similar, but since here the  $(n - 2)$ -dimensional mixed volume in the above consideration disappears, one has to replace the summation over all  $u, v \in N(P_1, P_2)$  by the one over all pairs  $(u, v)$  of normals of **adjacent** edges of  $P_1 + P_2$ .

For the remaining assertion, we put  $m = n$  in (\*). Since the left-hand side of (\*) is invariant with respect to individual translations of the polytopes  $P_i$ , the same holds true for the coefficients of the polynomial on the right-hand side, in particular for the coefficient  $V^{(n)}(P_1, \dots, P_n)$ . Here we need the symmetry of the coefficients and we make use of the fact that the coefficients of a polynomial in several variables are uniquely determined, if they are chosen to be symmetric.  $\square$

**Remark.** In the following, we use similar abbreviations as in the case of volume,

$$V(P_1, \dots, P_n) := V^{(n)}(P_1, \dots, P_n)$$

and

$$v(P_1(u), \dots, P_{n-1}(u)) := V^{(n-1)}(P_1(u), \dots, P_{n-1}(u)).$$

As a special case of the polynomial expansion of volumes, we obtain

$$V(P_1 + \dots + P_m) = \sum_{i_1=1}^m \dots \sum_{i_n=1}^m V(P_{i_1}, \dots, P_{i_n}).$$

The question arises, whether this expansion can be inverted.

**Corollary 3.3.3 (Inversion Formula).** *For  $P_1, \dots, P_n \in \mathcal{P}^n$ , we have*

$$V(P_1, \dots, P_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{1 \leq r_1 < \dots < r_k \leq n} V(P_{r_1} + \dots + P_{r_k}).$$

*Proof.* We denote the right-hand side by  $f(P_1, \dots, P_n)$ , then formula (\*) in Theorem 3.3.2 implies that  $f(\alpha_1 P_1, \dots, \alpha_n P_n)$  is a homogeneous polynomial of degree  $n$  in the variables  $\alpha_1 \geq 0, \dots, \alpha_n \geq 0$ . Replacing  $P_1$  by  $\{0\}$ , we have

$$\begin{aligned} & (-1)^{n+1} n! f(\{0\}, P_2, \dots, P_n) \\ &= \sum_{2 \leq r \leq n} V(P_r) - \left[ \sum_{2 \leq r \leq n} V(\{0\} + P_r) + \sum_{2 \leq r < s \leq n} V(P_r + P_s) \right] \\ &+ \left[ \sum_{2 \leq r < s \leq n} V(\{0\} + P_r + P_s) + \sum_{2 \leq r < s < t \leq n} V(P_r + P_s + P_t) \right] \\ &- \dots \\ &= 0, \end{aligned}$$

which means that  $f(\{0\}, \alpha_2 P_2, \dots, \alpha_n P_n) = f(0 \cdot P_1, \alpha_2 P_2, \dots, \alpha_n P_n)$  is the zero polynomial. Consequently, in the polynomial  $f(\alpha_1 P_1, \dots, \alpha_n P_n)$ , only those coefficients can be non-vanishing which contain the index 1. Replacing 1 subsequently by  $2, \dots, n$ , we obtain that only the coefficient of  $\alpha_1 \cdots \alpha_n$  can be non-zero. This coefficient occurs only once in the representation of  $f$ , namely for  $k = n$  with  $(r_1, \dots, r_n) = (1, \dots, n)$ . Therefore, by Theorem 3.3.2, this coefficient must coincide with  $V(P_1, \dots, P_n)$ .  $\square$

**Theorem 3.3.4 (and definition).** For  $K_1, \dots, K_n \in \mathcal{K}^n$  and arbitrary sequences  $(P_1^{(j)})_{j \in \mathbb{N}}, \dots, (P_n^{(j)})_{j \in \mathbb{N}}$  of polytopes, such that  $P_i^{(j)} \rightarrow K_i, i = 1, \dots, n$ , as  $j \rightarrow \infty$ , the limit

$$V(K_1, \dots, K_n) = \lim_{j \rightarrow \infty} V(P_1^{(j)}, \dots, P_n^{(j)})$$

exists and is independent of the choice of the approximating sequences  $(P_i^{(j)})_{j \in \mathbb{N}}$ .  $V(K_1, \dots, K_n)$  is called the mixed volume of  $K_1, \dots, K_n$ . The mapping  $V : (\mathcal{K}^n)^n \rightarrow \mathbb{R}$  defined by

$$(K_1, \dots, K_n) \mapsto V(K_1, \dots, K_n)$$

is also called the mixed volume. We have

$$(*) \quad V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n+k} \sum_{1 \leq r_1 < \dots < r_k \leq n} V(K_{r_1} + \dots + K_{r_k}).$$

For  $m \in \mathbb{N}$ ,  $K_1, \dots, K_m \in \mathcal{K}^n$  und  $\alpha_1, \dots, \alpha_m \geq 0$  we obtain

$$(*) \quad V(\alpha_1 K_1 + \dots + \alpha_m K_m) = \sum_{i_1=1}^m \dots \sum_{i_n=1}^m \alpha_{i_1} \cdots \alpha_{i_n} V(K_{i_1}, \dots, K_{i_n}).$$

Furthermore:

(a)  $V(K, \dots, K) = V(K)$  and  $nV(K, \dots, K, B(1)) = F(K)$ , for all  $K \in \mathcal{K}^n$ .

(b)  $V$  is symmetric.

(c)  $V$  is multilinear, i.e.

$$V(\alpha K + \beta L, K_2, \dots, K_n) = \alpha V(K, K_2, \dots, K_n) + \beta V(L, K_2, \dots, K_n),$$

for all  $\alpha, \beta \geq 0, K, L, K_2, \dots, K_n \in \mathcal{K}^n$ .

(d) We have  $V(K_1 + x_1, \dots, K_n + x_n) = V(K_1, \dots, K_n)$ , for all  $K_i \in \mathcal{K}^n$  and all  $x_i \in \mathbb{R}^n$ .

(e) We have  $V(gK_1, \dots, gK_n) = V(K_1, \dots, K_n)$ , for all  $K_i \in \mathcal{K}^n$  and all rigid motions  $g$ .

(f)  $V$  is continuous, i.e.

$$V(K_1^{(j)}, \dots, K_n^{(j)}) \rightarrow V(K_1, \dots, K_n),$$

provided  $K_i^{(j)} \rightarrow K_i, i = 1, \dots, n$ .

(g)  $V \geq 0$  and  $V$  is monotone in each argument.

*Proof.* The existence of the limit

$$V(K_1, \dots, K_n) = \lim_{j \rightarrow \infty} V(P_1^{(j)}, \dots, P_n^{(j)})$$

(including the independence from the approximating sequences) and formula  $(*_1)$  follow from Corollary 3.3.3 and the continuity of the addition of convex bodies and of the volume functional.  $(*_2)$  is a consequence of  $(*)$ .

(d), (e) and (f) follow now directly from  $(*_1)$ .

(a) The relation  $V(K, \dots, K) = V(K)$  follows for polytopes by induction and for general bodies  $K$  by approximation with polytopes; alternatively, one can obtain it from Corollary 3.3.3 and  $(*_1)$ . Concerning the relation  $nV(K, \dots, K, B(1)) = F(K)$ , we also first discuss the case  $K \in \mathcal{P}^n$ . Let  $(Q_j)_{j \in \mathbb{N}}$  be a sequence of polytopes with  $Q_j \rightarrow B(1)$ . Then,

$$nV(K, \dots, K, Q_j) \rightarrow nV(K, \dots, K, B(1))$$

and also

$$\begin{aligned} nV(K, \dots, K, Q_j) &= \sum_{u \in N(K)} h_{Q_j}(u) v(K(u)) \\ &\rightarrow \sum_{u \in N(K)} h_{B(1)}(u) v(K(u)) = \sum_{u \in N(K)} v(K(u)) = F(K). \end{aligned}$$

For the generalization to arbitrary bodies  $K$ , we approximate  $K$  from inside and outside by polytopes and use (f); here only the monotonicity of the surface area is needed, not the continuity (which we have not proved yet).

(b) follows from the corresponding property for polytopes.

(c) is a consequence of  $(*_2)$ , if we apply it to the linear combination

$$\alpha_1(\alpha K + \beta L) + \alpha_2 K_2 + \dots + \alpha_m K_m = \alpha_1 \alpha K + \alpha_1 \beta L + \alpha_2 K_2 + \dots + \alpha_m K_m$$

twice (once as a combination of  $m$  bodies and once as a combination of  $m+1$  bodies), and then compare the coefficients.

(g) Again it is sufficient to prove this for polytopes. Then  $V \geq 0$  follows by induction and the formula

$$V(P_1, \dots, P_n) = \frac{1}{n} \sum_{u \in N(P_1, \dots, P_{n-1})} h_{P_n}(u) v(P_1(u), \dots, P_{n-1}(u)),$$

where we may assume, in view of (d), that  $0 \in \text{rel int } P_n$ , hence  $h_{P_n} \geq 0$ . If  $P_n \subset Q_n$ , then  $h_{P_n} \leq h_{Q_n}$ , hence

$$V(P_1, \dots, P_n) \leq V(P_1, \dots, P_{n-1}, Q_n),$$

by the same formula (and  $V \geq 0$ ). □

**Remarks.** (1) In addition to  $V \geq 0$ , one can show that  $V(K_1, \dots, K_n) > 0$ , if and only if there exist segments  $s_1 \subset K_1, \dots, s_n \subset K_n$  with linearly independent directions (see the exercises below).

(2) Theorem 3.3.4 (a) and (f) now imply the continuity of the surface area  $F$ .

Now we consider the parallel body  $K + B(\alpha)$ ,  $\alpha \geq 0$ , of a body  $K \in \mathcal{K}^n$ . Theorem 3.3.4 implies

$$\begin{aligned}
 V(K + B(\alpha)) &= V(K + \alpha B(1)) = V(\alpha_1 K_1 + \alpha_2 K_2) \\
 (SF) \qquad &= \sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 \alpha_{i_1} \cdots \alpha_{i_n} V(K_{i_1}, \dots, K_{i_n}) \\
 &= \sum_{i=0}^n \alpha^i \binom{n}{i} V(\underbrace{K, \dots, K}_{n-i}, \underbrace{B(1), \dots, B(1)}_i).
 \end{aligned}$$

**Definition.** For  $K \in \mathcal{K}^n$ ,

$$W_i(K) := V(\underbrace{K, \dots, K}_{n-i}, \underbrace{B(1), \dots, B(1)}_i)$$

is called the  $i$ -th *quermassintegral* of  $K$ ,  $i = 0, \dots, n$ , and

$$V_j(K) = V_j^{(n)}(K) := \frac{\binom{n}{j}}{\kappa_{n-j}} W_{n-j}(K) = \frac{\binom{n}{j}}{\kappa_{n-j}} V(\underbrace{K, \dots, K}_j, \underbrace{B(1), \dots, B(1)}_{n-j})$$

is called the  $j$ -th *intrinsic volume* of  $K$ ,  $j = 0, \dots, n$ . Here,  $\kappa_k$  is the volume of the  $k$ -dimensional unit ball. Since we have extended the mixed volume to the empty set, we also define

$$W_i(\emptyset) := V_j(\emptyset) := 0, \quad i, j = 0, \dots, n.$$

Formula (SF) directly implies the following result.

**Theorem 3.3.5 (STEINER formula).** For  $K \in \mathcal{K}^n$  and  $\alpha \geq 0$ , we have

$$V(K + B(\alpha)) = \sum_{i=0}^n \alpha^i \binom{n}{i} W_i(K),$$

respectively

$$V(K + B(\alpha)) = \sum_{j=0}^n \alpha^{n-j} \kappa_{n-j} V_j(K).$$

**Remarks.** (1) We get in particular

$$F(K) = nW_1(K) = \lim_{\alpha \searrow 0} \frac{1}{\alpha} (V(K + B(\alpha)) - V(K)),$$

hence the surface area is the "derivative" of the volume functional.

(2) As a generalization of the STEINER formula (SF), one can show

$$V_k(K + B(\alpha)) = \sum_{j=0}^k \alpha^{k-j} \binom{n-j}{n-k} \frac{\kappa_{n-j}}{\kappa_{n-k}} V_j(K),$$

for  $k = 0, \dots, n-1$  (see the exercises).

The quermassintegrals are the classical notation used in most of the older books. The name quermassintegral will become clear in chapter 4 where we discuss some projection formulas. The intrinsic volumes follow the more modern terminology. Their advantages are that the index  $j$  of  $V_j$  corresponds to the degree of homogeneity,

$$V_j(\alpha K) = \alpha^j V_j(K), \quad K \in \mathcal{K}^n, \alpha \geq 0,$$

and that they are independent of the surrounding dimension, i.e. for a body  $K \in \mathcal{K}^n$  with  $\dim K = k < n$ , we have

$$V_j^{(n)}(K) = V_j^{(k)}(K), \quad j = 0, \dots, k.$$

The intrinsic volumes are important geometric functionals of a convex body. First, by definition,

$$V_n(K) = V(K, \dots, K) = V(K)$$

is the volume of  $K$ . Then,

$$2V_{n-1}(K) = nV(K, \dots, K, B(1)) = F(K)$$

is the surface area of  $K$  (such that, for a body  $K$  of dimension  $n-1$ ,  $V_{n-1}(K)$  is the  $(n-1)$ -dimensional content of  $K$ ). At the other end,  $V_1(K)$  is proportional to the *mean width* of  $K$ . Namely,

$$\frac{\kappa_{n-1}}{n} V_1(K) = V(K, B(1), \dots, B(1)).$$

Approximating the unit ball by polytopes, one can show that

$$V(K, B(1), \dots, B(1)) = \frac{1}{n} \int_{S^{n-1}} h_K(u) du,$$

where the integration is with respect to the spherical Lebesgue measure. A rigorous proof of this fact will be given in Section 3.5. Since  $b_K(u) := h_K(u) + h_K(-u)$  gives

the *width* of  $K$  in direction  $u$  (the distance between the two parallel supporting hyperplanes), we obtain

$$\begin{aligned} \frac{1}{n} \int_{S^{n-1}} h_K(u) du &= \frac{1}{2n} \int_{S^{n-1}} b_K(u) du \\ &= \frac{\kappa_n}{2} \bar{B}(K), \end{aligned}$$

where

$$\bar{B}(K) := \frac{1}{n\kappa_n} \int_{S^{n-1}} b_K(u) du$$

denotes the mean width. Hence

$$V_1(K) = \frac{n\kappa_n}{2\kappa_{n-1}} \bar{B}(K).$$

Finally,

$$V_0(K) = \frac{1}{\kappa_n} W_n(K) = \begin{cases} 1 & \text{if } K \neq \emptyset, \\ 0 & \text{if } K = \emptyset. \end{cases}$$

is the EULER-POINCARÉ *characteristic* of  $K$ . It plays an important role in integral geometry (see Chapter 4). The other intrinsic volumes  $V_j(K)$ ,  $1 < j < n - 1$ , have interpretations as integrals of curvature functions, if the boundary of  $K$  is smooth, e.g.  $V_{n-2}(K)$  is proportional to the integral mean curvature of  $K$ .

**Remark.** From Theorem 3.3.4 we obtain the following additional properties of the intrinsic volumes  $V_j$ :

- $K \mapsto V_j(K)$  is continuous,
- $V_j$  is motion invariant,
- $V_j \geq 0$  and  $V_j$  is monotone.

Later, in Section 4.3, we shall discuss a further property of  $V_j$ , namely the additivity. The intrinsic volume  $V_j$  is *additive* in the sense that

$$V_j(K \cup M) + V_j(K \cap M) = V_j(K) + V_j(M),$$

for all  $K, M \in \mathcal{K}^n$  such that  $K \cup M \in \mathcal{K}^n$ .

**Exercises and problems**

1. (a) Let  $s_1, \dots, s_n \in \mathcal{K}^n$  be segments of the form  $s_i = [0, x_i]$ ,  $x_i \in \mathbb{R}^n$ . Show that

$$n! V(s_1, \dots, s_n) = |\det(x_1, \dots, x_n)|.$$

- (b) For  $K_1, \dots, K_n \in \mathcal{K}^n$ , show that  $V(K_1, \dots, K_n) > 0$  if and only if there exist segments  $s_i \subset K_i$ ,  $i = 1, \dots, n$ , with linearly independent directions.

2. (a) For  $K, M \in \mathcal{K}^2$  show the inequality

$$V(K, M) \leq \frac{1}{8} F(K) F(M).$$

*Hint:* Use Exercise 7 in Section 3.1.

- \* (b) Show that equality holds in the above inequality, if and only if  $K, M$  are orthogonal segments (or if one of the bodies is a point).

- 3.\* (a) For  $K \in \mathcal{K}^2$  show the inequality

$$V(K, -K) \leq \frac{\sqrt{3}}{18} F^2(K).$$

- P (b) Show that equality holds in the above inequality, if and only if  $K$  is an equilateral triangle (or a point).

4. For  $K, K' \in \mathcal{K}^n$ , show that

$$\int_{\{x: K \cap (K'+x) \neq \emptyset\}} dx = \sum_{j=0}^n \binom{n}{j} V(\underbrace{K, \dots, K}_{n-j}, \underbrace{-K', \dots, -K'}_j).$$

5. For  $K \in \mathcal{K}^n$ , show that the intrinsic volume  $V_j(K) = V_j^{(n)}(K)$  is independent of the dimension  $n$ , i.e. if  $\dim K = k < n$ , then

$$V_j^{(k)}(K) = V_j^{(n)}(K), \quad \text{for } 0 \leq j \leq k.$$

6. Suppose  $K \in \mathcal{K}^n$  and  $L$  is a  $q$ -dimensional linear subspace of  $\mathbb{R}^n$ ,  $q \in \{0, \dots, n-1\}$ . Let  $B_L$  denote the unit ball in  $L$ .

Show that:

(a)  $V(K + \alpha B_L) = \sum_{j=0}^q \alpha^{q-j} \kappa_{q-j} \int_{L^\perp} V_j(K \cap (L+x)) d\lambda_{n-q}(x)$ , for all  $\alpha \geq 0$ .

- (b) The  $(n-q)$ -dimensional volume of the projection  $K|L^\perp$  fulfills

$$V_{n-q}(K|L^\perp) = \frac{\binom{n}{q}}{\kappa_q} V(\underbrace{K, \dots, K}_{n-q}, \underbrace{B_L, \dots, B_L}_q).$$



*Hint for (a):* Use FUBINI's theorem in  $\mathbb{R}^n = L \times L^\perp$  for the left-hand side and apply Exercise 5.

7. For a convex body  $K \in \mathcal{K}^n$  and  $\alpha \geq 0$ , prove the following STEINER formula for the intrinsic volumes:

$$V_k(K + B(\alpha)) = \sum_{j=0}^k \alpha^{k-j} \binom{n-j}{n-k} \frac{\kappa_{n-j}}{\kappa_{n-k}} V_j(K) \quad (0 \leq k \leq n-1).$$

- P 8. Prove the following Theorem of HADWIGER:

Let  $f : \mathcal{K}^n \rightarrow \mathbb{R}$  be additive, motion invariant and continuous (resp. monotone). Then, there are constants  $\beta_j \in \mathbb{R}$  (resp.  $\beta_j \geq 0$ ), such that

$$f = \sum_{j=0}^n \beta_j V_j.$$

### 3.4 The BRUNN-MINKOWSKI Theorem

The BRUNN-MINKOWSKI Theorem was one of the first results on convex bodies (proved around 1890). It says that, for convex bodies  $K, L \in \mathcal{K}^n$ , the function

$$t \mapsto \sqrt[n]{V(tK + (1-t)L)}, \quad t \in [0, 1],$$

is concave. As consequences we will get inequalities for mixed volumes, in particular the celebrated isoperimetric inequality.

We first need an auxiliary result.

**Lemma 3.4.1.** *For  $\alpha \in (0, 1)$  and  $r, s, t > 0$ , we have*

$$\left(\frac{\alpha}{r} + \frac{1-\alpha}{s}\right)[\alpha r^t + (1-\alpha)s^t]^{\frac{1}{t}} \geq 1$$

with equality, if and only if  $r = s$ .

*Proof.* The function  $x \mapsto \ln x$  is strictly concave, therefore we have

$$\begin{aligned} & \ln\left\{\left(\frac{\alpha}{r} + \frac{1-\alpha}{s}\right)[\alpha r^t + (1-\alpha)s^t]^{\frac{1}{t}}\right\} \\ &= \frac{1}{t} \ln(\alpha r^t + (1-\alpha)s^t) + \ln\left(\frac{\alpha}{r} + \frac{1-\alpha}{s}\right) \\ &\geq \frac{1}{t}(\alpha \ln r^t + (1-\alpha) \ln s^t) + \alpha \ln \frac{1}{r} + (1-\alpha) \ln \frac{1}{s} \\ &= 0 \end{aligned}$$

with equality if and only if  $r = s$  (the use of the logarithm is possible since  $(\frac{\alpha}{r} + \frac{1-\alpha}{s})[\alpha r^t + (1-\alpha)s^t]^{\frac{1}{t}} > 0$ ). The strict monotonicity of the logarithm now proves the result.  $\square$

**Theorem 3.4.2 (BRUNN-MINKOWSKI).** *For convex bodies  $K, L \in \mathcal{K}^n$  and  $\alpha \in (0, 1)$ , we have*

$$\sqrt[n]{V(\alpha K + (1-\alpha)L)} \geq \alpha \sqrt[n]{V(K)} + (1-\alpha) \sqrt[n]{V(L)}$$

with equality, if and only if  $K$  and  $L$  lie in parallel hyperplanes or  $K$  and  $L$  are homothetic.

**Remark.**  $K$  and  $L$  are homothetic, if and only if  $K = \alpha L + x$  or  $L = \alpha K + x$ , for some  $x \in \mathbb{R}^n, \alpha \geq 0$ . This includes the case of points, i.e.  $K$  and  $L$  are always homothetic, if one of them is a point.

*Proof.* We distinguish four cases.

Case 1:  $K$  and  $L$  lie in parallel hyperplanes. Then  $V(K) = V(L) = 0$  and  $V(\alpha K + (1 - \alpha)L) = 0$ .

Case 2: We have  $\dim K \leq n - 1$  and  $\dim L \leq n - 1$ , but  $K$  and  $L$  do not lie in parallel hyperplanes, i.e.  $\dim(K + L) = n$ . Then  $\dim(\alpha K + (1 - \alpha)L) = n$ , for all  $\alpha \in (0, 1)$ , hence

$$\sqrt[n]{V(\alpha K + (1 - \alpha)L)} > 0 = \alpha \sqrt[n]{V(K)} + (1 - \alpha) \sqrt[n]{V(L)},$$

for all  $\alpha \in (0, 1)$ .

Case 3: We have  $\dim K \leq n - 1$  and  $\dim L = n$  (or vice versa). Then, for  $x \in K$ , we obtain

$$\alpha x + (1 - \alpha)L \subset \alpha K + (1 - \alpha)L$$

and thus

$$(1 - \alpha)^n V(L) = V(\alpha x + (1 - \alpha)L) \leq V(\alpha K + (1 - \alpha)L)$$

with equality, if and only if  $K = \{x\}$ .

Case 4: We have  $\dim K = \dim L = n$ . We may assume  $V(K) = V(L) = 1$ . Namely, for general  $K, L$ , we put

$$\bar{K} := \frac{1}{\sqrt[n]{V(K)}} K, \quad \bar{L} := \frac{1}{\sqrt[n]{V(L)}} L$$

and

$$\bar{\alpha} := \frac{\alpha \sqrt[n]{V(K)}}{\alpha \sqrt[n]{V(K)} + (1 - \alpha) \sqrt[n]{V(L)}}.$$

Then

$$\sqrt[n]{V(\bar{\alpha} \bar{K} + (1 - \bar{\alpha}) \bar{L})} \geq 1$$

implies the BRUNN-MINKOWSKI inequality, which we have to prove. Moreover,  $K$  and  $L$  are homothetic, if and only if  $\bar{K}$  and  $\bar{L}$  are homothetic.

Thus, we assume  $V(K) = V(L) = 1$  and we have to show that

$$V(\alpha K + (1 - \alpha)L) \geq 1$$

with equality if and only if  $K, L$  are homothetic. Because the volume is translation invariant, we can make the additional assumption that  $K$  and  $L$  have their center of gravity at 0 (the center of gravity of an  $n$ -dimensional convex body  $M$  is the point  $c \in \mathbb{R}^n$  fulfilling

$$\langle c, u \rangle = \frac{1}{V(M)} \int_M \langle x, u \rangle dx,$$

for all  $u \in S^{n-1}$ ). The equality case then reduces to the claim that  $K = L$ .

We now proceed by induction on  $n$ . For  $n = 1$  the above inequality follows from the linearity of the 1-dimensional volume and we even have equality which corresponds to the fact that in  $\mathbb{R}^1$  any two convex bodies (compact intervals) are homothetic. Now assume  $n \geq 2$  and the assertion is true in dimension  $n - 1$ . We choose a unit vector  $u \in S^{n-1}$  and denote by

$$E_\eta := \{\langle \cdot, u \rangle = \eta\}, \quad \eta \in \mathbb{R},$$

the hyperplane in direction  $u$  with (signed) distance  $\eta$  from the origin. For  $0 \leq \tau \leq 1$ , we define  $\beta(\tau)$  as the (unique) value for which the hyperplane  $E_{\beta(\tau)}$  intersects  $K$  and fulfills the condition

$$V(K \cap \{\langle \cdot, u \rangle \leq \beta(\tau)\}) = \tau.$$

Thus,  $\beta$  is a function on  $[0, 1]$  with  $\beta(1) = h_K(u)$ ,  $\beta(0) = -h_K(-u)$  and  $E_{\beta(1)}$  and  $E_{\beta(0)}$  are the supporting hyperplanes to  $K$  in direction  $u$ , resp.  $-u$ . Since  $\tau \mapsto \beta(\tau)$  is strictly increasing and does not have jumps,  $\beta$  is continuous. It is even differentiable on  $(0, 1)$ . Namely, the function

$$\xi \mapsto f(\xi) := \int_{\beta(0)}^{\xi} v(K \cap E_\eta) d\eta$$

is differentiable with  $f'(\eta) = v(K \cap E_\eta)$ . Since, by FUBINI's theorem (or the CAVALIERY principle),

$$\tau = \int_{\beta(0)}^{\beta(\tau)} v(K \cap E_\eta) d\eta,$$

we have  $f(\beta(\tau)) = \tau$ , hence  $\beta$  is the inverse function to  $f$ . Consequently,  $\beta$  is differentiable and

$$\beta'(\tau) = \frac{1}{f'(\beta(\tau))} = \frac{1}{v(K \cap E_{\beta(\tau)})}, \quad 0 < \tau < 1.$$

Analogously, we get for the body  $L$  a function  $\gamma$  on  $[0, 1]$  with

$$\tau = \int_{\gamma(0)}^{\gamma(\tau)} v(L \cap E_\eta) d\eta$$

and

$$\gamma'(\tau) = \frac{1}{v(L \cap E_{\gamma(\tau)})}, \quad 0 < \tau < 1.$$

Because of

$$\alpha(K \cap E_{\beta(\tau)}) + (1 - \alpha)(L \cap E_{\gamma(\tau)}) \subset (\alpha K + (1 - \alpha)L) \cap E_{\alpha\beta(\tau) + (1 - \alpha)\gamma(\tau)},$$

for  $\alpha, \tau \in [0, 1]$ , we obtain from the inductive assumption

$$\begin{aligned}
& V(\alpha K + (1 - \alpha)L) \\
&= \int_{-\infty}^{\infty} v((\alpha K + (1 - \alpha)L) \cap E_{\eta}) d\eta \\
&= \int_0^1 v((\alpha K + (1 - \alpha)L) \cap E_{\alpha\beta(\tau) + (1-\alpha)\gamma(\tau)}) (\alpha\beta'(\tau) + (1 - \alpha)\gamma'(\tau)) d\tau \\
&\geq \int_0^1 v(\alpha(K \cap E_{\beta(\tau)}) + (1 - \alpha)(L \cap E_{\gamma(\tau)})) \left( \frac{\alpha}{v(K \cap E_{\beta(\tau)})} + \frac{1 - \alpha}{v(L \cap E_{\gamma(\tau)})} \right) d\tau \\
&\geq \int_0^1 \left[ \alpha^{n-1} \sqrt[n]{v(K \cap E_{\beta(\tau)})} + (1 - \alpha)^{n-1} \sqrt[n]{v(L \cap E_{\gamma(\tau)})} \right]^{n-1} \\
&\quad \times \left( \frac{\alpha}{v(K \cap E_{\beta(\tau)})} + \frac{1 - \alpha}{v(L \cap E_{\gamma(\tau)})} \right) d\tau.
\end{aligned}$$

Choosing  $r := v(K \cap E_{\beta(\tau)})$ ,  $s := v(L \cap E_{\gamma(\tau)})$  and  $t := \frac{1}{n-1}$ , we obtain from Lemma 3.4.1 that the integrand is  $\geq 1$ , which yields the required inequality.

Now assume

$$V(\alpha K + (1 - \alpha)L) = 1.$$

Then we must have equality in our last estimation, which implies that the integrand equals 1, for all  $\tau$ . Again by Lemma 3.4.1, this yields that

$$v(K \cap E_{\beta(\tau)}) = v(L \cap E_{\gamma(\tau)}), \quad \text{for all } \tau \in [0, 1].$$

Therefore  $\beta' = \gamma'$ , hence the function  $\beta - \gamma$  is a constant. Because the center of gravity of  $K$  is at the origin, we obtain

$$\begin{aligned}
0 &= \int_K \langle x, u \rangle dx \\
&= \int_{\beta(0)}^{\beta(1)} \eta v(K \cap E_{\eta}) d\eta \\
&= \int_0^1 \beta(\tau) d\tau
\end{aligned}$$

and, in an analogous way,

$$0 = \int_0^1 \gamma(\tau) d\tau.$$

Consequently,

$$\int_0^1 (\beta(\tau) - \gamma(\tau)) d\tau = 0$$

and therefore  $\beta = \gamma$ . In particular, we obtain

$$h_K(u) = \beta(1) = \gamma(1) = h_L(u).$$

Since  $u$  was arbitrary,  $V(\alpha K + (1 - \alpha)L) = 1$  implies  $h_K = h_L$ , and hence  $K = L$ .

Conversely, it is clear that  $K = L$  implies  $V(\alpha K + (1 - \alpha)L) = 1$ .  $\square$

**Remark.** Theorem 3.4.2 implies that the function

$$f(t) := \sqrt[n]{V(tK + (1 - t)L)}$$

is concave on  $[0, 1]$ . Namely, let  $x, y, \alpha \in [0, 1]$ , then

$$\begin{aligned} f(\alpha x + (1 - \alpha)y) &= \sqrt[n]{V([\alpha x + (1 - \alpha)y]K + [1 - \alpha x - (1 - \alpha)y]L)} \\ &= \sqrt[n]{V(\alpha[xK + (1 - x)L] + (1 - \alpha)[yK + (1 - y)L])} \\ &\geq \alpha \sqrt[n]{V(xK + (1 - x)L)} + (1 - \alpha) \sqrt[n]{V(yK + (1 - y)L)} \\ &= \alpha f(x) + (1 - \alpha)f(y). \end{aligned}$$

As a consequence of Theorem 3.4.2, we obtain an inequality for mixed volumes which was first proved by MINKOWSKI.

**Theorem 3.4.3.** For  $K, L \in \mathcal{K}^n$ , we have

$$V(K, \dots, K, L)^n \geq V(K)^{n-1}V(L)$$

with equality, if and only if  $\dim K \leq n - 2$  or  $K$  and  $L$  lie in parallel hyperplanes or  $K$  and  $L$  are homothetic.

*Proof.* For  $\dim K \leq n - 1$ , the inequality holds since the right-hand side is zero. Moreover, we then have equality, if and only if either  $\dim K \leq n - 2$  or  $K$  and  $L$  lie in parallel hyperplanes (compare Exercise 3.3.1). Hence, we now assume  $\dim K = n$ .

By Theorem 3.4.2, the function

$$f(t) := \sqrt[n]{V(tK + (1 - t)L)} - t \sqrt[n]{V(K)} - (1 - t) \sqrt[n]{V(L)}, \quad t \in [0, 1],$$

is concave,  $\geq 0$  and fulfills  $f(0) = f(1) = 0$ . Moreover,  $f \equiv 0$ , if and only if  $K$  and  $L$  are homothetic. Theorem 3.3.4 yields

$$V(tK + (1 - t)L) = \sum_{j=0}^n \binom{n}{j} t^j (1 - t)^{n-j} V(\underbrace{K, \dots, K}_j, \underbrace{L, \dots, L}_{n-j}).$$

Since  $V(K) > 0$ , this representation allows us to extend  $f$  to a  $\delta$ -neighborhood of 1,  $\delta > 0$ , and also shows that this extended function  $f$  is differentiable in  $(0, 1 + \delta)$ . We obtain (for  $t \in (0, 1]$ )

$$f'(t) = \frac{1}{n} V(tK + (1 - t)L)^{\frac{1-n}{n}} \sum_{j=0}^n \binom{n}{j} [j t^{j-1} (1 - t)^{n-j} - (n - j) t^j (1 - t)^{n-j-1}]$$

$$\times V(\underbrace{K, \dots, K}_j, \underbrace{L, \dots, L}_{n-j}) - \sqrt[n]{V(K)} + \sqrt[n]{V(L)}.$$

Our conditions on  $f$  in  $[0, 1]$  imply that  $f'(1) \leq 0$  and  $f'(1) = 0$ , if and only if  $f \equiv 0$  in  $[0, 1]$ , hence if and only if  $K$  and  $L$  are homothetic. Since

$$\begin{aligned} f'(1) &= \frac{1}{n} V(K)^{\frac{1-n}{n}} [nV(K) - nV(K, \dots, K, L)] - \sqrt[n]{V(K)} + \sqrt[n]{V(L)} \\ &= V(L)^{\frac{1}{n}} - V(K)^{\frac{1-n}{n}} V(K, \dots, K, L), \end{aligned}$$

this completes the proof.  $\square$

**Corollary 3.4.4 (Isoperimetric inequality).** *Let  $K \in \mathcal{K}^n$  be a convex body of dimension  $n$ . Then,*

$$\left( \frac{F(K)}{F(B(1))} \right)^n \geq \left( \frac{V(K)}{V(B(1))} \right)^{n-1}.$$

*Equality holds, if and only if  $K$  is a ball.*

*Proof.* We put  $L := B(1)$  in Theorem 3.4.3 and get

$$V(K, \dots, K, B(1))^n \geq V(K)^{n-1} V(B(1))$$

or, equivalently,

$$\frac{n^n V(K, \dots, K, B(1))^n}{n^n V(B(1), \dots, B(1), B(1))^n} \geq \frac{V(K)^{n-1}}{V(B(1))^{n-1}}.$$

$\square$

The isoperimetric inequality states that, among all convex bodies of given volume (given surface area), the balls have the smallest surface area (the largest volume).

Using  $V(B(1)) =: \kappa_n$  and  $F(B(1)) = n\kappa_n$ , we can re-write the inequality in the form

$$V(K)^{n-1} \leq \frac{1}{n^n \kappa_n} F(K)^n.$$

For  $n = 2$  and using the common terminology  $A(K)$  for the area (the "volume" in  $\mathbb{R}^2$ ) and  $L(K)$  for the boundary length (the "surface area" in  $\mathbb{R}^2$ ), we obtain

$$A(K) \leq \frac{1}{4\pi} L(K)^2,$$

and for  $n = 3$ ,

$$V(K)^2 \leq \frac{1}{36\pi} F(K)^3.$$

Exchanging  $K$  and  $B(1)$  in the proof above yields a similar inequality for the mixed volume  $V(B(1), \dots, B(1), K)$ , hence we obtain the following corollary for the mean width  $\overline{B}(K)$ .

**Corollary 3.4.5.** *Let  $K \in \mathcal{K}^n$  be a convex body. Then,*

$$\left( \frac{\overline{B}(K)}{\overline{B}(B(1))} \right)^n \geq \frac{V(K)}{V(B(1))}.$$

*Equality holds, if and only if  $K$  is a ball.*

**Remark.** Since  $\overline{B}(K)$  is not greater than the diameter of  $K$ , the corollary implies an inequality for the diameter.

Using Theorem 3.4.2 and the second derivatives, we obtain in a similar manner inequalities of quadratic type.

**Theorem 3.4.6.** *For  $K, L \in \mathcal{K}^n$ , we have*

$$(*) \quad V(K, \dots, K, L)^2 \geq V(K, \dots, K, L, L)V(K).$$

The *proof* is left as an exercise. The case of equality is not known completely. Equality holds for homothetic bodies, but there are also non-homothetic bodies (with interior points) for which equality holds.

Replacing  $K$  or  $L$  in  $(*)$  by the unit ball, we obtain more special inequalities, for example (in  $\mathbb{R}^3$ )

$$\pi \overline{B}(K)^2 \geq F(K)$$

or

$$F(K)^2 \geq 6\pi \overline{B}(K)V(K).$$

### Exercises and problems

1. Give a proof of Theorem 3.4.6.
2. The *diameter*  $D(K)$  of a convex body  $K \in \mathcal{K}^n$  is defined as

$$D(K) := \sup\{\|x - y\| : x, y \in K\}.$$

- (a) Prove that

$$\overline{B}(K) \leq D(K) \leq \frac{n\kappa_n}{2\kappa_{n-1}} \cdot \overline{B}(K).$$

- (b) If there is equality in one of the two inequalities, what can be said about  $K$ ?



### 3.5 Surface area measures

In Section 3.3, we have shown that, for polytopes  $P_1, \dots, P_n \in \mathcal{P}^n$ , the mixed volume fulfills the formula

$$V(P_1, \dots, P_{n-1}, P_n) = \frac{1}{n} \sum_{u \in S^{n-1}} h_{P_n}(u) v(P_1(u), \dots, P_{n-1}(u)).$$

Here, the summation takes place over all unit vectors  $u$  for which  $v(P_1(u), \dots, P_{n-1}(u)) > 0$ , that is, over all facet normals of the polytope  $P_1 + \dots + P_{n-1}$ . By approximation (and using the continuity of mixed volumes and support functions), we therefore get the same formula for arbitrary bodies  $K_n \in \mathcal{K}^n$ ,

$$(*) \quad V(P_1, \dots, P_{n-1}, K_n) = \frac{1}{n} \sum_{u \in S^{n-1}} h_{K_n}(u) v(P_1(u), \dots, P_{n-1}(u)).$$

We define

$$(+ \quad S(P_1, \dots, P_{n-1}, \cdot) := \sum_{u \in S^{n-1}} v(P_1(u), \dots, P_{n-1}(u)) \varepsilon_u,$$

where  $\varepsilon_u$  denotes the Dirac measure in  $u \in S^{n-1}$ ,

$$\varepsilon_u(A) := \begin{cases} 1 & \text{if } u \in A, \\ 0 & \text{if } u \notin A, \end{cases}$$

(here,  $A$  runs through all Borel sets in  $S^{n-1}$ ). Then,  $S(P_1, \dots, P_{n-1}, \cdot)$  is a finite Borel measure on the unit sphere  $S^{n-1}$ , which is called the *mixed surface area measure* of the polytopes  $P_1, \dots, P_{n-1}$ . Equation (\*) is then equivalent to

$$(**) \quad V(P_1, \dots, P_{n-1}, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_n}(u) dS(P_1, \dots, P_{n-1}, u).$$

Our next goal is to extend this integral representation to arbitrary convex bodies  $K_1, \dots, K_{n-1}$  (and thus define mixed surface area measures for general convex bodies).

We first need an auxiliary result.

**Lemma 3.5.1.** *For convex bodies  $K_1, \dots, K_{n-1}, K_n, K'_n \in \mathcal{K}^n$ , we have*

$$\begin{aligned} |V(K_1, \dots, K_{n-1}, K_n) - V(K_1, \dots, K_{n-1}, K'_n)| \\ \leq \|h_{K_n} - h_{K'_n}\| V(K_1, \dots, K_{n-1}, B(1)). \end{aligned}$$

*Proof.* First, let  $K_1, \dots, K_{n-1}$  be polytopes. Since  $h_{B(1)} \equiv 1$  (on  $S^{n-1}$ ), we obtain from (\*), that

$$\begin{aligned}
& |V(K_1, \dots, K_{n-1}, K_n) - V(K_1, \dots, K_{n-1}, K'_n)| \\
&= \frac{1}{n} \left| \sum_{u \in S^{n-1}} (h_{K_n}(u) - h_{K'_n}(u)) v(K_1(u), \dots, K_{n-1}(u)) \right| \\
&\leq \frac{1}{n} \sum_{u \in S^{n-1}} |h_{K_n}(u) - h_{K'_n}(u)| v(K_1(u), \dots, K_{n-1}(u)) \\
&\leq \frac{1}{n} \sup_{v \in S^{n-1}} |h_{K_n}(v) - h_{K'_n}(v)| \sum_{u \in S^{n-1}} v(K_1(u), \dots, K_{n-1}(u)) \\
&= \frac{1}{n} \|h_{K_n} - h_{K'_n}\| \sum_{u \in S^{n-1}} h_{B(1)}(u) v(K_1(u), \dots, K_{n-1}(u)) \\
&= \|h_{K_n} - h_{K'_n}\| V(K_1, \dots, K_{n-1}, B(1)).
\end{aligned}$$

By Theorem 3.3.4 (continuity of the mixed volume), the inequality extends to arbitrary convex bodies.  $\square$

Now we can extend (\*\*) to arbitrary convex bodies.

**Theorem 3.5.2.** *For  $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ , there exists a uniquely determined finite Borel measure  $S(K_1, \dots, K_{n-1}, \cdot)$  on  $S^{n-1}$ , such that*

$$V(K_1, \dots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS(K_1, \dots, K_{n-1}, u),$$

for all  $K \in \mathcal{K}^n$ .

*Proof.* We consider the Banach space  $\mathbf{C}(S^{n-1})$  and the linear subspace  $\mathbf{C}^2(S^{n-1})$  of twice continuously differentiable functions. Here, a function  $f$  on  $S^{n-1}$  is called twice continuously differentiable, if the homogeneous extension  $\tilde{f}$  of  $f$ ,

$$\tilde{f}(x) := \begin{cases} \|x\| f\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \quad x \in \mathbb{R}^n,$$

is twice continuously differentiable on  $\mathbb{R}^n \setminus \{0\}$ . From analysis we use the fact that the subspace  $\mathbf{C}^2(S^{n-1})$  is dense in  $\mathbf{C}(S^{n-1})$ , that is, for each  $f \in \mathbf{C}(S^{n-1})$  there is a sequence of functions  $f_i \in \mathbf{C}^2(S^{n-1})$  with  $f_i \rightarrow f$  in the maximum norm, as  $i \rightarrow \infty$  (this can be proved either by a convolution argument or by using a result of STONE-WEIERSTRASS type).

Further, we consider the set  $\mathcal{L}^n$  of all functions  $f \in \mathbf{C}(S^{n-1})$  which have a representation  $f = h_K - h_{K'}$  with convex bodies  $K, K' \in \mathcal{K}^n$ . Obviously,  $\mathcal{L}^n$  is also a linear subspace. Exercise 3.2.1 shows that  $\mathbf{C}^2(S^{n-1}) \subset \mathcal{L}^n$ , therefore  $\mathcal{L}^n$  is dense in  $\mathbf{C}(S^{n-1})$ .

We now define a functional  $T_{K_1, \dots, K_{n-1}}$  on  $\mathcal{L}^n$  by

$$T_{K_1, \dots, K_{n-1}}(f) := nV(K_1, \dots, K_{n-1}, K) - nV(K_1, \dots, K_{n-1}, K'),$$

where  $f = h_K - h_{K'}$ . This definition is actually independent of the particular representation of  $f$ . Namely, if  $f = h_K - h_{K'} = h_L - h_{L'}$ , then  $K + L' = K' + L$  and hence

$$\begin{aligned} V(K_1, \dots, K_{n-1}, K) + V(K_1, \dots, K_{n-1}, L') \\ = V(K_1, \dots, K_{n-1}, K') + V(K_1, \dots, K_{n-1}, L), \end{aligned}$$

by the multilinearity of mixed volumes. This yields

$$\begin{aligned} nV(K_1, \dots, K_{n-1}, K) - nV(K_1, \dots, K_{n-1}, K') \\ = nV(K_1, \dots, K_{n-1}, L) - nV(K_1, \dots, K_{n-1}, L'). \end{aligned}$$

The argument just given also shows that  $T_{K_1, \dots, K_{n-1}}$  is linear. Moreover,  $T_{K_1, \dots, K_{n-1}}$  is a positive functional since  $f = h_K - h_{K'} \geq 0$  implies  $K \supset K'$ . Hence

$$V(K_1, \dots, K_{n-1}, K) \geq V(K_1, \dots, K_{n-1}, K')$$

and therefore  $T_{K_1, \dots, K_{n-1}}(f) \geq 0$ . Finally,  $T_{K_1, \dots, K_{n-1}}$  is continuous (with respect to the maximum norm), since Lemma 3.5.1 shows that

$$|T_{K_1, \dots, K_{n-1}}(f)| \leq c(K_1, \dots, K_{n-1}) \|f\|$$

with  $c(K_1, \dots, K_{n-1}) := nV(K_1, \dots, K_{n-1}, B(1))$ .

Since  $\mathcal{L}^n$  is dense in  $\mathbf{C}(S^{n-1})$ , the inequality just proven (or alternatively, the theorem of HAHN-BANACH) implies that there is a unique continuous extension of  $T_{K_1, \dots, K_{n-1}}$  to a positive linear functional on  $\mathbf{C}(S^{n-1})$ . The RIESZ representation theorem then shows that

$$T_{K_1, \dots, K_{n-1}}(f) = \int_{S^{n-1}} f(u) dS(K_1, \dots, K_{n-1}, u),$$

for  $f \in \mathbf{C}(S^{n-1})$ , with a finite (nonnegative) Borel measure  $S(K_1, \dots, K_{n-1}, \cdot)$  on  $S^{n-1}$ , which is uniquely determined by  $T_{K_1, \dots, K_{n-1}}$ . The assertion of the theorem now follows, if we put  $f = h_K$ .  $\square$

**Definition.** The measure  $S(K_1, \dots, K_{n-1}, \cdot)$  is called the *mixed surface area measure* of the bodies  $K_1, \dots, K_{n-1}$ . In particular,

$$S_j(K, \cdot) := S(\underbrace{K, \dots, K}_j, \underbrace{B(1), \dots, B(1)}_{n-1-j}, \cdot)$$

is called the  $j$ th order *surface area measure* of  $K$ ,  $j = 0, \dots, n - 1$ .

**Remarks.** (1) For polytopes  $K_1, \dots, K_{n-1}$ , the mixed surface area measure  $S(K_1, \dots, K_{n-1}, \cdot)$  equals the measure defined in (+).

(2) All surface area measures have centroid 0. Namely, since

$$V(K_1, \dots, K_{n-1}, \{x\}) = 0,$$

we have

$$\int_{S^{n-1}} \langle x, u \rangle dS(K_1, \dots, K_{n-1}, u) = 0,$$

for all  $x \in \mathbb{R}^n$ .

(3) We have

$$\begin{aligned} S_j(K, S^{n-1}) &= nV(\underbrace{K, \dots, K}_j, \underbrace{B(1), \dots, B(1)}_{n-j}) \\ &= \frac{n\kappa_{n-j}}{\binom{n}{j}} V_j(K), \end{aligned}$$

in particular

$$S_{n-1}(K, S^{n-1}) = 2V_{n-1}(K) = F(K),$$

which explains the name surface area measure.

(4) The measure  $S_0(K, \cdot) = S(B(1), \dots, B(1), \cdot) = S_j(B(1), \cdot)$  (for  $j = 0, \dots, n - 1$  and  $K \in \mathcal{K}^n$ ) equals the spherical Lebesgue measure  $\omega_{n-1}$  (this follows from part (d) of the following theorem), hence we obtain the equation

$$V(K, B(1), \dots, B(1)) = \frac{1}{n} \int_{S^{n-1}} h_K(u) du,$$

which we used already at the end of Section 3.3.

Further properties of mixed surface area measures follow, if we combine Theorem 3.5.2 with Theorem 3.3.4. In order to formulate a continuity result, we make use of the weak convergence of measures on  $S^{n-1}$  (since  $S^{n-1}$  is compact, weak and vague convergence are the same). A sequence of finite measures  $\mu_i, i = 1, 2, \dots$ , on  $S^{n-1}$  is said to *converge weakly* to a finite measure  $\mu$  on  $S^{n-1}$ , if and only if

$$\int_{S^{n-1}} f(u) d\mu_i(u) \rightarrow \int_{S^{n-1}} f(u) d\mu(u), \quad \text{as } i \rightarrow \infty,$$

for all  $f \in \mathbf{C}(S^{n-1})$ .

**Theorem 3.5.3.** *The mapping  $S : (K_1, \dots, K_{n-1}) \mapsto S(K_1, \dots, K_{n-1}, \cdot)$  has the following properties:*

(a)  *$S$  is symmetric, i.e.*

$$S(K_1, \dots, K_{n-1}, \cdot) = S(K_{\pi(1)}, \dots, K_{\pi(n-1)}, \cdot),$$

for all  $K_1, \dots, K_{n-1} \in \mathcal{K}^n$  and all permutations  $\pi$  of  $1, \dots, n-1$ .

(b)  *$S$  is multilinear, i.e.*

$$S(\alpha K + \beta L, K_2, \dots, K_{n-1}, \cdot) = \alpha S(K, K_2, \dots, K_{n-1}, \cdot) + \beta S(L, K_2, \dots, K_{n-1}, \cdot),$$

for all  $\alpha, \beta \geq 0, K, L, K_2, \dots, K_{n-1} \in \mathcal{K}^n$ .

(c)  *$S$  is translation invariant, i.e.*

$$S(K_1 + x_1, \dots, K_{n-1} + x_{n-1}, \cdot) = S(K_1, \dots, K_{n-1}, \cdot),$$

for all  $K_i \in \mathcal{K}^n$  and all  $x_i \in \mathbb{R}^n$ .

(d)  *$S$  is rotation covariant, i.e.*

$$S(\vartheta K_1, \dots, \vartheta K_{n-1}, \vartheta A) = S(K_1, \dots, K_{n-1}, A),$$

for all  $K_i \in \mathcal{K}^n$ , all Borel sets  $A \subset S^{n-1}$ , and all rotations  $\vartheta$ .

(e)  *$S$  is continuous, i.e.*

$$S(K_1^{(m)}, \dots, K_{n-1}^{(m)}, \cdot) \rightarrow S(K_1, \dots, K_{n-1}, \cdot)$$

weakly, as  $m \rightarrow \infty$ , provided  $K_i^{(m)} \rightarrow K_i, i = 1, \dots, n-1$ .

*Proof.* (a), (b) and (c) follow directly from the integral representation and the uniqueness in Theorem 3.5.2 together with the corresponding properties of mixed volumes in Theorem 3.3.4.

(d) If  $\rho \circ \mu$  denotes the image of a measure  $\mu$  on  $S^{n-1}$  under the rotation  $\rho$ , then the assertion follows from

$$\begin{aligned} & \int_{S^{n-1}} h_{K_n}(u) d[\vartheta^{-1} \circ S(\vartheta K_1, \dots, \vartheta K_{n-1}, \cdot)](u) \\ &= \int_{S^{n-1}} h_{K_n}(\vartheta^{-1}u) dS(\vartheta K_1, \dots, \vartheta K_{n-1}, u) \\ &= \int_{S^{n-1}} h_{\vartheta K_n}(u) dS(\vartheta K_1, \dots, \vartheta K_{n-1}, u) \\ &= nV(\vartheta K_1, \dots, \vartheta K_{n-1}, \vartheta K_n) \\ &= nV(K_1, \dots, K_{n-1}, K_n) \\ &= \int_{S^{n-1}} h_{K_n}(u) dS(K_1, \dots, K_{n-1}, u), \end{aligned}$$

where  $K_n \in \mathcal{K}^n$  is arbitrary.

(e) For  $\varepsilon > 0$  and  $f \in \mathbf{C}(S^{n-1})$ , choose  $K, L \in \mathcal{K}^n$  with

$$\|f - (h_K - h_L)\| \leq \varepsilon$$

and then  $m_0$  such that  $K_i^{(m)} \subset K_i + B(1)$ ,  $i = 1, \dots, n-1$ , and

$$|V(K_1^{(m)}, \dots, K_{n-1}^{(m)}, K) - V(K_1, \dots, K_{n-1}, K)| \leq \varepsilon,$$

as well as

$$|V(K_1^{(m)}, \dots, K_{n-1}^{(m)}, L) - V(K_1, \dots, K_{n-1}, L)| \leq \varepsilon,$$

for all  $m \geq m_0$ . Then,

$$\begin{aligned} & \left| \int_{S^{n-1}} f(u) dS(K_1^{(m)}, \dots, K_{n-1}^{(m)}, \cdot)(u) - \int_{S^{n-1}} f(u) dS(K_1, \dots, K_{n-1}, \cdot)(u) \right| \\ & \leq \left| \int_{S^{n-1}} (f - (h_K - h_L))(u) dS(K_1^{(m)}, \dots, K_{n-1}^{(m)}, \cdot)(u) \right| \\ & \quad + \left| \int_{S^{n-1}} (h_K - h_L)(u) dS(K_1^{(m)}, \dots, K_{n-1}^{(m)}, \cdot)(u) \right. \\ & \quad \left. - \int_{S^{n-1}} (h_K - h_L)(u) dS(K_1, \dots, K_{n-1}, \cdot)(u) \right| \\ & \quad + \left| \int_{S^{n-1}} (f - (h_K - h_L))(u) dS(K_1, \dots, K_{n-1}, \cdot)(u) \right| \\ & \leq \|f - (h_K - h_L)\| n V(K_1 + B(1), \dots, K_{n-1} + B(1), B(1)) \\ & \quad + n |V(K_1^{(m)}, \dots, K_{n-1}^{(m)}, K) - V(K_1, \dots, K_{n-1}, K)| \\ & \quad + n |V(K_1^{(m)}, \dots, K_{n-1}^{(m)}, L) - V(K_1, \dots, K_{n-1}, L)| \\ & \quad + \|f - (h_K - h_L)\| n V(K_1, \dots, K_{n-1}, B(1)) \\ & \leq c(K_1, \dots, K_{n-1}) \varepsilon, \end{aligned}$$

for  $m \geq m_0$ . □

**Corollary 3.5.4.** *For  $j = 0, \dots, n-1$ , the mapping  $K \mapsto S_j(K, \cdot)$  on  $\mathcal{K}^n$  is translation invariant, rotation covariant and continuous.*

Moreover,

$$S_{n-1}(K + B(\alpha), \cdot) = \sum_{j=0}^{n-1} \alpha^{n-1-j} \binom{n-1}{j} S_j(K, \cdot),$$

for  $\alpha \geq 0$  (local STEINER formula).

*Proof.* We only have to prove the local STEINER formula. The latter follows from Theorem 3.5.3(b). □

The interpretation of the surface area measure  $S_{n-1}(P, \cdot)$  for a polytope  $P$  is quite simple. For a Borel set  $A \subset S^{n-1}$ , the value of  $S_{n-1}(P, A)$  gives the total surface area of the set of all boundary points of  $P$  which have an outer normal in  $A$  (since this set is a union of facets, the surface area is defined). In an appropriate way (and using approximation by polytopes), this interpretation carries over to arbitrary bodies  $K$ :  $S_{n-1}(K, A)$  measures the total surface area of the set of all boundary points of  $K$  which have an outer normal in  $A$ .

Now we study the problem, how far a convex body  $K$  is determined by one of its surface area measures  $S_j(K, \cdot)$ ,  $j \in \{1, \dots, n-1\}$ . For  $j = n-1$  (and  $n$ -dimensional bodies), we can give a strong answer to this question.

**Theorem 3.5.5.** *Let  $K, L \in \mathcal{K}^n$  with  $\dim K = \dim L = n$ . Then*

$$S_{n-1}(K, \cdot) = S_{n-1}(L, \cdot),$$

*if and only if  $K$  and  $L$  are translates.*

*Proof.* For translates  $K, L$ , the equality of the surface area measures follows from Corollary 3.5.4.

Assume now  $S_{n-1}(K, \cdot) = S_{n-1}(L, \cdot)$ . Then, Theorem 3.5.2 implies

$$\begin{aligned} V(K, \dots, K, L) &= \frac{1}{n} \int_{S^{n-1}} h_L(u) dS_{n-1}(K, u) \\ &= \frac{1}{n} \int_{S^{n-1}} h_L(u) dS_{n-1}(L, u) \\ &= V(L). \end{aligned}$$

In the same way, we obtain  $V(L, \dots, L, K) = V(K)$ . The MINKOWSKI inequalities (Theorem 3.3.5) therefore show that

$$V(L)^n \geq V(K)^{n-1}V(L)$$

and

$$V(K)^n \geq V(L)^{n-1}V(K),$$

which implies  $V(K) = V(L)$ . Therefore we have equality in both inequalities and hence  $K$  and  $L$  are homothetic. Since they have the same volume, they must be translates.  $\square$

The uniqueness result holds more generally for the  $j$ -th order surface area measures ( $j \in \{1, \dots, n-1\}$ ), if the bodies have dimension at least  $j+1$  (for  $j=1$  even without a dimensional restriction). The proof uses a deep generalization of the MINKOWSKI inequalities (the ALEXANDROV-FENCHEL inequalities).

Theorem 3.5.5 can be used to express certain properties of convex bodies in terms of their surface area measures. We mention only one application of this type,

other results can be found in the exercises. We recall that a convex body  $K \in \mathcal{K}^n$  is *centrally symmetric*, if there is a point  $x \in \mathbb{R}^n$  such that  $K - x = -(K - x)$  (then  $x \in K$  and  $x$  is the center of symmetry). Also, a measure  $\mu$  on  $S^{n-1}$  is called *even*, if  $\mu$  is invariant under reflection, i.e.  $\mu(A) = \mu(-A)$ , for all Borel sets  $A \subset S^{n-1}$ .

**Corollary 3.5.6.** *Let  $K \in \mathcal{K}^n$  with  $\dim K = n$ . Then,  $K$  is centrally symmetric, if and only if  $S_{n-1}(K, \cdot)$  is an even measure.*

In the following, we study the problem which measures  $\mu$  on  $S^{n-1}$  arise as surface area measures  $S_{n-1}(K, \cdot)$  of convex bodies  $K$  (the *existence problem*). Obviously, a necessary condition is that  $\mu$  must have centroid 0. Another condition arises from a dimensional restriction. Namely, if  $\dim K \leq n - 2$ , then  $S_{n-1}(K, \cdot) = 0$ , whereas for  $\dim K = n - 1$ ,  $K \subset u^\perp$ ,  $u \in S^{n-1}$ , we have  $S_{n-1}(K, \cdot) = V_{n-1}(K)(\varepsilon_u + \varepsilon_{-u})$  (both results follow from Theorem 3.5.2). Hence, for  $\dim K \leq n - 1$ , the existence problem is not of any interest. Therefore, we now concentrate on bodies  $K \in \mathcal{K}^n$  with  $\dim K = n$ . Again, Theorem 3.5.2 shows that this implies  $\dim S_{n-1}(K, \cdot) = n$ , where the latter condition means that  $S_{n-1}(K, \cdot)$  is not supported by any lower dimensional sphere, i.e.  $S_{n-1}(K, S^{n-1} \setminus E) > 0$  for all hyperplanes  $E$  through 0. As we shall show now, these two conditions (the centroid condition and the dimensional condition) characterize  $(n - 1)$ -st surface area measures. We first prove the polytopal case.

**Theorem 3.5.7.** *For  $k \geq n + 1$ , let  $u_1, \dots, u_k \in S^{n-1}$  be unit vectors which span  $\mathbb{R}^n$  and let  $v^{(1)}, \dots, v^{(k)} > 0$  be numbers such that*

$$\sum_{i=1}^k v^{(i)} u_i = 0.$$

*Then, there exists a (up to a translation unique) polytope  $P \in \mathcal{P}^n$  with  $\dim P = n$ , for which*

$$S_{n-1}(P, \cdot) = \sum_{i=1}^k v^{(i)} \varepsilon_{u_i},$$

*i.e. the  $u_i$  are the facet normals of  $P$  and the  $v^{(i)}$  are the corresponding facet contents.*

*Proof.* The uniqueness follows from Theorem 3.5.5.

For the existence, we denote by  $\mathbb{R}_+^k$  the set of all vectors  $y = (y^{(1)}, \dots, y^{(k)})$  with  $y^{(i)} \geq 0$ ,  $i = 1, \dots, k$ . For  $y \in \mathbb{R}_+^k$ , let

$$P_{[y]} := \bigcap_{i=1}^k \{ \langle \cdot, u_i \rangle \leq y^{(i)} \}.$$



Since  $0 \in P_{[y]}$ , this set is nonempty and polyhedral. Moreover,  $P_{[y]}$  is bounded hence a convex polytope in  $\mathbb{R}^n$ . Namely, assume  $\alpha x \in P_{[y]}$ , for some  $x \in S^{n-1}$  and all  $\alpha \geq 0$ , then

$$\langle x, u_i \rangle \leq 0, \quad i = 1, \dots, k.$$

Since the centroid condition implies

$$u_1 = - \sum_{i=2}^k \frac{v^{(i)}}{v^{(1)}} u_i,$$

we would get

$$0 \geq \langle x, u_1 \rangle = - \sum_{i=2}^k \frac{v^{(i)}}{v^{(1)}} \langle x, u_i \rangle \geq 0,$$

and hence

$$\langle x, u_1 \rangle = \dots = \langle x, u_k \rangle = 0.$$

As a consequence  $\langle x, y \rangle = 0$ , for all  $y \in \mathbb{R}^n$ , since  $u_1, \dots, u_k$  span  $\mathbb{R}^n$ . Hence  $x = 0$ , a contradiction.

Therefore,  $P_{[y]}$  is a polytope. We next show that the mapping  $y \mapsto P_{[y]}$  is concave, i.e.

$$(*) \quad \gamma P_{[y]} + (1 - \gamma) P_{[z]} \subset P_{[\gamma y + (1 - \gamma) z]},$$

for  $y, z \in \mathbb{R}_+^k$  and  $\gamma \in [0, 1]$ . This follows since a point  $x \in \gamma P_{[y]} + (1 - \gamma) P_{[z]}$ ,  $x = \gamma x' + (1 - \gamma) x''$ ,  $x' \in P_{[y]}$ ,  $x'' \in P_{[z]}$ , fulfills

$$\begin{aligned} \langle x, u_i \rangle &= \gamma \langle x', u_i \rangle + (1 - \gamma) \langle x'', u_i \rangle \\ &\leq \gamma y^{(i)} + (1 - \gamma) z^{(i)}. \end{aligned}$$

Since the normal vectors  $u_i$  of the half spaces  $\{\langle \cdot, u_i \rangle \leq y^{(i)}\}$  are fixed and only their distances  $y^{(i)}$  from the origin vary, the mapping  $y \mapsto P_{[y]}$  is continuous (w.r.t. the Hausdorff metric). Therefore,  $y \mapsto V(P_{[y]})$  is continuous, which implies that the set

$$\mathcal{M} := \{y \in \mathbb{R}_+^k : V(P_{[y]}) = 1\}$$

is nonempty and closed. The linear function

$$\varphi := \frac{1}{n} \langle \cdot, v \rangle, \quad v := (v^{(1)}, \dots, v^{(k)}),$$

is nonnegative on  $\mathcal{M}$  (and continuous). Since  $v^{(i)} > 0, i = 1, \dots, k$ , there is a vector  $y_0$  such that  $\varphi(y_0) =: \alpha \geq 0$  is the minimum of  $\varphi$  on  $\mathcal{M}$ .

We consider the polytope  $Q := P_{[y_0]}$ . Since  $V(Q) = 1$ ,  $Q$  has interior points (and  $0 \in Q$ ). We may assume that  $0 \in \text{int } Q$ . Namely, for  $0 \in \text{bd } Q$ , we can choose a translation vector  $t \in \mathbb{R}^n$  such that  $0 \in \text{int } (Q + t)$ . Then

$$Q + t = \bigcap_{i=1}^k \{ \langle \cdot, u_i \rangle \leq \tilde{y}_0^{(i)} \}$$

with  $\tilde{y}_0^{(i)} := y_0^{(i)} + \langle t, u_i \rangle$ ,  $i = 1, \dots, k$ . Obviously,  $\tilde{y}_0^{(i)} > 0$  and  $Q + t = P_{[\tilde{y}_0]}$ . Moreover,  $V(Q + t) = V(Q) = 1$  and

$$\begin{aligned} \varphi(\tilde{y}_0) &= \frac{1}{n} \langle y_0, v \rangle + \frac{1}{n} \sum_{i=1}^k \langle t, u_i \rangle v^{(i)} \\ &= \varphi(y_0) + \frac{1}{n} \langle t, \sum_{i=1}^k u_i v^{(i)} \rangle \\ &= \alpha, \end{aligned}$$

since  $\sum_{i=1}^k u_i v^{(i)} = 0$ . Hence, we now assume  $0 \in \text{int } Q$ , which gives us  $y_0^{(i)} > 0$  and therefore  $\alpha > 0$ . We define a vector  $w = (w^{(1)}, \dots, w^{(k)})$ , where  $w^{(i)}$  is the content  $V_{n-1}(Q(u_i))$  of the support set of  $Q$  in direction  $u_i$ ,  $i = 1, \dots, k$ . Then,

$$\begin{aligned} 1 = V(Q) &= \frac{1}{n} \sum_{i=1}^k y_0^{(i)} w^{(i)} = \frac{1}{n} \langle y_0, w \rangle \\ &= \frac{1}{\alpha} \varphi(y_0) = \frac{1}{\alpha n} \langle y_0, v \rangle. \end{aligned}$$

Hence,

$$\langle y_0, w \rangle = \langle y_0, \frac{1}{\alpha} v \rangle = n.$$

Next, we define the hyperplanes

$$E := \{ \langle \cdot, w \rangle = n \}$$

and

$$F := \{ \langle \cdot, \frac{1}{\alpha} v \rangle = n \}$$

in  $\mathbb{R}^k$ . We want to show that  $E = F$ . First, we notice that  $y_0 \in E \cap F$ . Since  $y_0$  has positive components, we can find a neighborhood  $U$  of  $y_0$ , such that  $y \in U$  has the following two properties. First,  $y^{(i)} > 0$  for  $i = 1, \dots, k$  and second every facet normal of  $Q = P_{[y_0]}$  is also a facet normal of  $P_{[y]}$ . We now consider  $y \in F \cap U$ . Assume  $V(P_{[y]}) > 1$ , then there exists  $0 < \beta < 1$  with

$$V(P_{[\beta y]}) = 1.$$

Since  $y \in F$ ,

$$\varphi(\beta y) = \frac{1}{n} \langle \beta y, v \rangle = \beta \alpha < \alpha,$$

a contradiction. Therefore,  $V(P_{[y]}) \leq 1$ . For  $\vartheta \in [0, 1]$ , the point  $\vartheta y + (1 - \vartheta)y_0$  is also in  $F \cap U$ . Therefore the volume inequality just proven applies and we get from (\*)

$$V(\vartheta P_{[y]} + (1 - \vartheta)Q) \leq V(P_{[\vartheta y + (1 - \vartheta)y_0]}) \leq 1.$$

This yields

$$\begin{aligned} V(Q, \dots, Q, P_{[y]}) &= \frac{1}{n} \lim_{\vartheta \rightarrow 0} \frac{V(\vartheta P_{[y]} + (1 - \vartheta)Q) - (1 - \vartheta)^n}{\vartheta} \\ &\leq \frac{1}{n} \lim_{\vartheta \rightarrow 0} \frac{1 - (1 - \vartheta)^n}{\vartheta} \\ &= 1. \end{aligned}$$

Since by our assumption, each facet normal of  $Q$  is a facet normal of  $P_{[y]}$ , we have  $h_{P_{[y]}}(u_i) = y^{(i)}$ , for all  $i$  for which  $w^{(i)} > 0$ . Hence

$$1 \geq V(Q, \dots, Q, P_{[y]}) = \frac{1}{n} \sum_{i=1}^k h_{P_{[y]}}(u_i) w^{(i)} = \frac{1}{n} \langle y, w \rangle,$$

for all  $y \in F \cap U$ . This shows that  $F \cap U \subset E$ , which is only possible if  $E = F$ .

Since  $E = F$  implies  $w = \frac{1}{\alpha}v$ , the polytope  $P := {}^{n-1}\sqrt{\alpha}Q$  fulfills all assertions of the theorem.  $\square$

We now extend this result to arbitrary bodies  $K \in \mathcal{K}^n$ .

**Theorem 3.5.8.** *Let  $\mu$  be a finite Borel measure on  $S^{n-1}$  with centroid 0 and  $\dim \mu = n$ . Then, there exists a (up to a translation unique) body  $K \in \mathcal{K}^n$ , for which*

$$S_{n-1}(K, \cdot) = \mu.$$

*Proof.* Again, we only need to show the existence of  $K$ .

We make use of the fact that  $\mu$  can be approximated (in the weak convergence) by discrete measures (measures with finite support)  $\mu_j \rightarrow \mu$ , for  $j \rightarrow \infty$ , which also have centroid 0 and fulfill  $\dim \mu_j = n$ . The measure  $\mu_j$  can, for example, be constructed as follows. We divide  $S^{n-1}$  into finitely many pairwise disjoint Borel sets  $A_{ij}$ ,  $i = 0, 1, \dots, k(j)$ , such that  $\mu(A_{0j}) = 0$ , whereas  $\text{diam}(\text{cl conv } A_{ij}) < \frac{1}{j}$  and  $\mu(A_{ij}) > 0$ , for  $i = 1, \dots, k(j)$ . We then put

$$\mu_j := \sum_{i=1}^{k(j)} \mu(A_{ij}) \|x_{ij}\| \varepsilon_{u_{ij}},$$

where

$$x_{ij} := \frac{1}{\mu(A_{ij})} \int_{A_{ij}} u d\mu(u),$$

and  $u_{ij} := \frac{x_{ij}}{\|x_{ij}\|}$ . This definition makes sense since, for  $i \geq 1$ , it can be shown that  $0 \notin \text{cl conv } A_{ij}$  and therefore  $x_{ij} \neq 0$ . Moreover,  $\mu_j$  has centroid 0 and converges to  $\mu$  (see the exercises). Because of  $\dim \mu = n$ , we must have  $\dim \mu_j = n$ , for large enough  $j$ .

From Theorem 3.5.7, we obtain polytopes  $P_j$  with  $0 \in P_j$  and

$$\mu_j = S_{n-1}(P_j, \cdot), \quad j = 1, 2, \dots$$

We show that the sequence  $(P_j)_{j \in \mathbb{N}}$  is uniformly bounded. First,  $F(P_j) = \mu_j(S^{n-1}) \rightarrow \mu(S^{n-1})$  implies that

$$F(P_j) \leq C, \quad j = 1, 2, \dots,$$

for some  $C > 0$ . The isoperimetric inequality shows that then

$$V(P_j) \leq \tilde{C}, \quad j = 1, 2, \dots,$$

for another constant  $\tilde{C} > 0$ . Now let  $x \in S^{n-1}$  and  $\alpha_j \geq 0$  such that  $\alpha_j x \in P_j$ , hence  $[0, \alpha_j x] \in P_j$ . Since

$$h_{[0, \alpha_j x]} = \alpha_j \max(\langle x, \cdot \rangle, 0),$$

we get

$$\begin{aligned} V(P_j) &= \frac{1}{n} \sum_{i=1}^{k(j)} h_{P_j}(u_{ij}) V_{n-1}(P_j(u_{ij})) \\ &\geq \frac{1}{n} \sum_{i=1}^{k(j)} h_{[0, \alpha_j x]}(u_{ij}) V_{n-1}(P_j(u_{ij})) \\ &= \frac{\alpha_j}{n} \int_{S^{n-1}} \max(\langle x, u \rangle, 0) d\mu_j(u). \end{aligned}$$

The weak convergence implies

$$\frac{1}{n} \int_{S^{n-1}} \max(\langle x, u \rangle, 0) d\mu_j(u) \rightarrow \frac{1}{n} \int_{S^{n-1}} \max(\langle x, u \rangle, 0) d\mu(u).$$

Since  $\dim \mu = n$ , we have

$$f(x) := \frac{1}{n} \int_{S^{n-1}} \max(\langle x, u \rangle, 0) d\mu(u) > 0,$$

for all  $x \in S^{n-1}$ . The theorem of dominated convergence shows that  $f$  is continuous, hence

$$c := \min_{x \in S^{n-1}} f(x)$$

exists and we have  $c > 0$ . Therefore,  $\alpha_j \leq C'$  for all  $j \geq j_0$ , with suitable  $j_0$  and a certain constant  $C'$ . This shows that the sequence  $(P_j)_{j \in \mathbb{N}}$  is uniformly bounded.

By BLASCHKE's selection theorem, we can choose a converging subsequence  $P_{j_r} \rightarrow K$ ,  $r \rightarrow \infty$ ,  $K \in \mathcal{K}^n$ . Then

$$S_{n-1}(P_{j_r}, \cdot) \rightarrow S_{n-1}(K, \cdot),$$

but also

$$S_{n-1}(P_{j_r}, \cdot) \rightarrow \mu.$$

Therefore,  $S_{n-1}(K, \cdot) = \mu$ . □

### Exercises and problems

1. Let  $K, M, L \in \mathcal{K}^n$  such that  $K = M + L$ . Show that

$$S_j(M, \cdot) = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} S(\underbrace{K, \dots, K}_i, \underbrace{L, \dots, L}_{j-i}, \underbrace{B(1), \dots, B(1)}_{n-1-j}, \cdot),$$

for  $j = 0, \dots, n-1$ .

2. Let  $K \in \mathcal{K}^n$ ,  $\dim K = n$ ,  $O \in \mathcal{O}^n$  be an orthogonal matrix and

$$\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto Ox$$

the associated linear mapping. Show that  $\theta K$  is a translate of  $K$  if and only if the area measure of  $K$  is invariant under  $\theta$ , i.e.

$$S_{n-1}(K, A) = S_{n-1}(K, \theta^{-1}(A))$$

for all Borel sets  $A \subset S^{n-1}$ .

3. Let  $K \in \mathcal{K}^n$  and  $r(K)$  be the circumradius of  $K$ . Show that  $r(K) \leq 1$  if and only if  $V(K, M, \dots, M) \leq \frac{1}{n} F(M)$  for all  $M \in \mathcal{K}^n$ .
4. Let  $\alpha \in (0, 1)$  and  $M, L \in \mathcal{K}^n$  with  $\dim M = \dim L = n$ .

- (a) Show that there is a convex body  $K_\alpha \in \mathcal{K}^n$  with  $\dim K_\alpha = n$  and

$$S_{n-1}(K_\alpha, \cdot) = \alpha S_{n-1}(M, \cdot) + (1 - \alpha) S_{n-1}(L, \cdot).$$

(b) Show that

$$V(K_\alpha)^{\frac{n-1}{n}} \geq \alpha V(M)^{\frac{n-1}{n}} + (1-\alpha)V(L)^{\frac{n-1}{n}},$$

with equality if and only if  $M$  and  $L$  are homothetic.

5. Complete the proof of Theorem 3.5.8 by showing that the measures  $\mu_j$  are well-defined (i.e. that  $x_{ij} \neq 0$ ), have centroid 0, fulfill  $\dim \mu_j = n$ , for almost all  $j$ , and converge weakly to the given measure  $\mu$  (as  $j \rightarrow \infty$ ).

### 3.6 Projection functions

For a convex body  $K \in \mathcal{K}^n$  and a direction  $u \in S^{n-1}$ , we define

$$v(K, u) := V_{n-1}(K | u^\perp),$$

the content of the orthogonal projection of  $K$  onto the hyperplane  $u^\perp$ . The function  $v(K, \cdot)$  is called the *projection function* of  $K$ . We are interested in the information on the shape of  $K$  which can be deduced from the knowledge of its projection function  $v(K, \cdot)$ .

First, it is clear that translates  $K, K + x, x \in \mathbb{R}^n$ , have the same projection function. Second,  $K$  and  $-K$  have the same projection function, which shows that in general  $K$  is not determined by  $v(K, \cdot)$  (not even up to translations). The question occurs whether we get uniqueness up to translations and reflections. In order to give an answer, we need a representation of  $v(K, \cdot)$ .

**Theorem 3.6.1.** *For  $K \in \mathcal{K}^n$  and  $u \in S^{n-1}$ , we have*

$$v(K, u) = \frac{1}{2} \int_{S^{n-1}} |\langle x, u \rangle| dS_{n-1}(K, x).$$

*Proof.* An application of FUBINI's theorem shows that

$$V(K + [-u, u]) = V(K) + 2v(K, u).$$

On the other hand, we have

$$V(K + [-u, u]) = \sum_{i=0}^n \binom{n}{i} V(\underbrace{K, \dots, K}_i, \underbrace{[-u, u], \dots, [-u, u]}_{n-i}).$$

From Exercise 3.3.1, we know that

$$V(\underbrace{K, \dots, K}_i, \underbrace{[-u, u], \dots, [-u, u]}_{n-i}) = 0,$$

for  $i = 0, \dots, n-2$ , hence

$$v(K, u) = \frac{n}{2} V(K, \dots, K, [-u, u]).$$

The assertion now follows from Theorem 3.5.2, since the segment  $[-u, u]$  has support function  $|\langle \cdot, u \rangle|$ .  $\square$

**Remarks.** A couple of properties of projection functions can be directly deduced from Theorem 3.6.1.

- (1) We have  $v(K, \cdot) = 0$ , if and only if  $\dim K \leq n - 2$ .  
 (2) If  $\dim K = n - 1$ ,  $K \subset x^\perp$ , then

$$v(K, \cdot) = V_{n-1}(K)|\langle x, \cdot \rangle|.$$

(3) If  $\dim K = n$  and  $K$  is not centrally symmetric (i.e.  $S_{n-1}(K, \cdot) \neq S_{n-1}(-K, \cdot)$ ), then there is an infinite family of bodies with the same projection function. Namely, for  $\alpha \in [0, 1]$ , there is a body  $K_\alpha \in \mathcal{K}^n$  with  $\dim K_\alpha = n$  and

$$S_{n-1}(K_\alpha, \cdot) = \alpha S_{n-1}(K, \cdot) + (1 - \alpha) S_{n-1}(-K, \cdot)$$

(see Exercise 3.5.4(a)). Then,

$$v(K_\alpha, \cdot) = \alpha v(K, \cdot) + (1 - \alpha) v(-K, \cdot) = v(K, \cdot).$$

This also shows that there is always a centrally symmetric body, namely  $K_{\frac{1}{2}}$ , with the same projection function as  $K$ .

(4) Since  $|\langle x, \cdot \rangle|$  is a support function, the function  $v(K, \cdot)$  is a positive combination of support functions, hence it is itself a support function of a convex body  $\Pi K$ ,

$$h_{\Pi K} := v(K, \cdot).$$

We call  $\Pi K$  the *projection body* of  $K$ . The projection body is always centrally symmetric to the origin and, if  $\dim K = n$ , then  $\dim \Pi K = n$ .

Before we continue to discuss projection functions, we want to describe projection bodies geometrically.

**Definition.** A finite sum of segments  $Z := s_1 + \cdots + s_k$  is called a *zonotope*. A *zonoid* is a convex body which is the limit (in the Hausdorff metric) of a sequence of zonotopes.

Zonotopes are polytopes and they are centrally symmetric. Namely, if  $s_i = [-y_i, y_i] + x_i$  is the representation of the segment  $s_i$  (with center  $x_i$  and endpoints  $-y_i + x_i$ ,  $y_i + x_i$ ), then

$$Z = \sum_{i=1}^k [-y_i, y_i] + \sum_{i=1}^k x_i.$$

Hence,  $x := \sum_{i=1}^k x_i$  is the center of  $Z$ . Zonoids, as limits of zonotopes, are also centrally symmetric. We assume w.l.o.g. that the center of zonotopes and zonoids is the origin and denote the correspondings set of zonoids by  $\mathcal{Z}^n$ .

The following results show that zonoids and projection bodies are closely related.



**Theorem 3.6.2.** *Let  $K \in \mathcal{K}^n$ . Then,  $K$  is a zonoid, if and only if there exists an even Borel measure  $\mu(K, \cdot)$  on  $S^{n-1}$  such that*

$$h_K(u) = \int_{S^{n-1}} |\langle x, u \rangle| d\mu(K, x).$$

For a zonoid  $K$ , the measure  $\mu(K, \cdot)$  is called the *generating measure* of  $K$ .

*Proof.* Suppose

$$h_K(u) = \int_{S^{n-1}} |\langle x, u \rangle| d\mu(K, x).$$

As in the proof of Theorem 3.5.8, we find a sequence of even, discrete measures  $\mu_j \rightarrow \mu(K, \cdot)$ ,

$$\mu_j := \frac{1}{2} \sum_{i=1}^{k(j)} \alpha_{ij} (\epsilon_{u_{ij}} + \epsilon_{-u_{ij}}), \quad u_{ij} \in S^{n-1}, \alpha_{ij} > 0.$$

Then,

$$Z_j := \sum_{i=1}^{k(j)} [-\alpha_{ij} u_{ij}, \alpha_{ij} u_{ij}]$$

is a zonotope and

$$\begin{aligned} h_{Z_j}(u) &= \int_{S^{n-1}} |\langle x, u \rangle| d\mu_j(x) \\ &\rightarrow \int_{S^{n-1}} |\langle x, u \rangle| d\mu(K, x) = h_K(u), \end{aligned}$$

for all  $u \in S^{n-1}$ . Therefore,  $Z_j \rightarrow K$  (as  $j \rightarrow \infty$ ), i.e.  $K$  is a zonoid.

Conversely, assume that  $K = \lim_{j \rightarrow \infty} Z_j$ ,  $Z_j$  zonotope. Then,

$$Z_j = \sum_{i=1}^{k(j)} [-y_{ij}, y_{ij}]$$

with suitable vectors  $y_{ij} \in \mathbb{R}^n$ . Consequently,

$$\begin{aligned} h_{Z_j}(u) &= \sum_{i=1}^{k(j)} |\langle y_{ij}, u \rangle| \\ &= \int_{S^{n-1}} |\langle x, u \rangle| d\mu_j(x), \end{aligned}$$

where

$$\mu_j := \frac{1}{2} \sum_{i=1}^{k(j)} \|y_{ij}\| (\epsilon_{u_{ij}} + \epsilon_{-u_{ij}})$$

and

$$u_{ij} := \frac{y_{ij}}{\|y_{ij}\|}.$$

We would like to show that the sequence  $(\mu_j)_{j \in \mathbb{N}}$  converges weakly.

We have

$$\int_{S^{n-1}} h_{Z_j}(u) du = \kappa_{n-1} V_1(Z_j) \rightarrow \kappa_{n-1} V_1(K).$$

Also, using FUBINI's theorem and Theorem 3.6.1 (for the unit ball), we get

$$\int_{S^{n-1}} h_{Z_j}(u) du = \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, u \rangle| du d\mu_j(x) = 2\kappa_{n-1} \mu_j(S^{n-1}).$$

Hence,  $\mu_j(S^{n-1})$  is bounded from above by a constant  $C$ , for all  $j$ . Now we use the fact that the set  $\mathcal{M}_C$  of all Borel measures  $\rho$  on  $S^{n-1}$  with  $\rho(S^{n-1}) \leq C$  is weakly compact (see, e.g., the books of Billingsley, *Convergence of probability measures*, Wiley 1968, p. 37; or Gänsler-Stute, *Wahrscheinlichkeitstheorie*, Springer 1977, p. 344). Therefore,  $(\mu_j)_{j \in \mathbb{N}}$  contains a convergent subsequence. W.l.o.g., we may assume that  $(\mu_j)_{j \in \mathbb{N}}$  converges to a limit measure which we denote by  $\mu(K, \cdot)$ . The weak convergence implies that

$$\begin{aligned} h_K(u) &= \lim_{j \rightarrow \infty} h_{Z_j}(u) \\ &= \lim_{j \rightarrow \infty} \int_{S^{n-1}} |\langle x, u \rangle| d\mu_j(x) \\ &= \int_{S^{n-1}} |\langle x, u \rangle| d\mu(K, x). \end{aligned}$$

□

**Remark.** As the above proof shows, we have  $\dim K = n$ , if and only if  $\dim \mu(K, \cdot) = n$ .

**Corollary 3.6.3.** *The projection body  $\Pi K$  of a convex body  $K$  is a zonoid. Reversely, if  $Z$  is a zonoid with  $\dim Z = n$ , then there is a convex body  $K$  with  $\dim K = n$  and which is centrally symmetric to the origin and fulfills*

$$Z = \Pi K.$$

*Proof.* The first result follows from Theorems 3.6.1 and 3.6.2. For the second, Theorem 3.6.2 shows that

$$h_Z(u) = \int_{S^{n-1}} |\langle x, u \rangle| d\mu(Z, x)$$

with an even measure  $\mu(Z, \cdot)$ ,  $\dim \mu(Z, \cdot) = n$ . By Theorem 3.5.8,

$$\mu(Z, \cdot) = S_{n-1}(K, \cdot),$$

for some convex body  $K \in \mathcal{K}^n$ ,  $\dim K = n$ , and hence  $Z = \Pi K$ . By Corollary 3.5.6,  $K$  is centrally symmetric.  $\square$

Finally, we want to show that the generating measure of a zonoid is uniquely determined. We first need two auxiliary lemmas. If  $A$  is the  $(n \times n)$ -matrix of an injective linear mapping in  $\mathbb{R}^n$ , we define

$$AZ := \{Ax : x \in Z\}$$

and denote by  $A\mu$ , for a measure  $\mu$  on  $S^{n-1}$ , the image measure of

$$\int_{(\cdot)} \|Ax\| d\mu(x)$$

under the mapping

$$x \mapsto \frac{Ax}{\|Ax\|}, \quad x \in S^{n-1}.$$

**Lemma 3.6.4.** *If  $Z \in \mathcal{K}^n$  is a zonoid and*

$$h_Z = \int_{S^{n-1}} |\langle x, \cdot \rangle| d\mu(Z, x),$$

*then  $AZ$  is a zonoid and*

$$h_{AZ} = \int_{S^{n-1}} |\langle x, \cdot \rangle| dA\mu(Z, x).$$

*Proof.* We have

$$\begin{aligned} h_{AZ} &= \sup_{x \in AZ} \langle u, x \rangle = \sup_{x \in Z} \langle u, Ax \rangle = \sup_{x \in Z} \langle A^T u, x \rangle = h_Z(A^T u) \\ &= \int_{S^{n-1}} |\langle x, A^T u \rangle| d\mu(Z, x) = \int_{S^{n-1}} |\langle Ax, u \rangle| d\mu(Z, x) \\ &= \int_{S^{n-1}} \left| \left\langle \frac{Ax}{\|Ax\|}, u \right\rangle \|Ax\| \right| d\mu(Z, x) = \int_{S^{n-1}} |\langle y, u \rangle| dA\mu(Z, y). \end{aligned}$$

$\square$

Let  $\mathcal{V}$  denote the vector space of functions

$$f = \int_{S^{n-1}} |\langle x, \cdot \rangle| d\mu(x) - \int_{S^{n-1}} |\langle x, \cdot \rangle| d\rho(x),$$

where  $\mu, \rho$  vary among all finite even Borel measures on  $S^{n-1}$ .  $\mathcal{V}$  is a subspace of the Banach space  $\mathbf{C}_e(S^{n-1})$  of even continuous functions on  $S^{n-1}$ .

**Lemma 3.6.5.** *The vector space  $\mathcal{V}$  is dense in  $\mathbf{C}_e(S^{n-1})$ .*

*Proof.* Choosing  $\mu = c\omega_{n-1}$ , for  $c \geq 0$ , and  $\rho = 0$  (or vice versa), we see that  $\mathcal{V}$  contains all constant functions.

By Lemma 3.6.4, the support functions  $h_{AB(1)}$  lie in  $\mathcal{V}$ , for all regular  $(n \times n)$ -matrices  $A$  (the body  $AB(1)$  is an ellipsoid). Since

$$h_{AB(1)}(u) = \|A^T u\|, \quad u \in S^{n-1},$$

we obtain all functions

$$\begin{aligned} f(B, u) &:= \sqrt{\langle Au, Au \rangle} = \sqrt{\langle A^T A u, u \rangle} \\ &= \sqrt{\langle Bu, u \rangle} = \left( \sum_{i,j=1}^n b_{ij} u_i u_j \right)^{\frac{1}{2}}, \end{aligned}$$

where  $B = A^T A = ((b_{ij}))$  varies among the positive definite  $(n \times n)$ -matrices  $B$ . For  $\epsilon > 0$ , let  $\tilde{B} = ((\tilde{b}_{ij}))$  with

$$\tilde{b}_{ij} := \begin{cases} b_{ij} + \epsilon & \text{if } (i, j) \in \{(i_0, j_0), (j_0, i_0)\}, \\ b_{ij} & \text{if } (i, j) \notin \{(i_0, j_0), (j_0, i_0)\}. \end{cases}$$

Then,  $\tilde{B}$  is symmetric and positive definite, for small enough  $\epsilon$ . Consequently,

$$\frac{f(\tilde{B}, \cdot) - f(B, \cdot)}{\epsilon} \in \mathcal{V}$$

and

$$\lim_{\epsilon \rightarrow 0} \frac{f(\tilde{B}, \cdot) - f(B, \cdot)}{\epsilon} = \frac{\partial f}{\partial b_{i_0, j_0}}(B, \cdot) \in \text{cl } \mathcal{V}.$$

A direct computation yields

$$\frac{\partial f}{\partial b_{i_0, j_0}}(B, u) = \frac{u_{i_0} u_{j_0}}{f(B, u)}.$$

Repeating this argument with  $b_{i_1, j_1}$  etc., we obtain that all partial derivatives of  $f$  w.r.t. variables in  $B$  are in  $\text{cl } \mathcal{V}$ , hence all functions

$$u \mapsto \frac{u_1^{i_1} \cdots u_n^{i_n}}{f(B, u)^k}, \quad i_1 + \cdots + i_n = 2k, k = 1, 2, \dots$$

Now we choose  $B$  to be the unit matrix. Then  $f(B, \cdot) = 1$ , hence all even polynomials are in  $\text{cl } \mathcal{V}$ . The theorem of STONE-WEIERSTRASS now shows that  $\text{cl } \mathcal{V} = \mathbf{C}_e(S^{n-1})$ .  $\square$

**Theorem 3.6.6.** *For a zonoid  $Z \in \mathcal{K}^n$ , the generating measure is uniquely determined.*

*Proof.* Assume we have two measures  $\mu := \mu(Z, \cdot)$  and  $\rho$  on  $S^{d-1}$  with

$$\int_{S^{n-1}} |\langle x, \cdot \rangle| d\mu(x) = \int_{S^{n-1}} |\langle x, \cdot \rangle| d\rho(x).$$

Then,

$$\int_{S^{n-1}} \int_{S^{n-1}} |\langle x, u \rangle| d\mu(x) d\tilde{\mu}(u) = \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, u \rangle| d\rho(x) d\tilde{\mu}(u),$$

for all measures  $\tilde{\mu}$  on  $S^{n-1}$ . Replacing  $\tilde{\mu}$  by a difference of measures and applying FUBINI'S theorem, we obtain

$$\int_{S^{n-1}} f(x) d\mu(x) = \int_{S^{n-1}} f(x) d\rho(x),$$

for all functions  $f \in \mathcal{V}$ . Lemma 3.6.5 shows that this implies  $\mu = \rho$ .  $\square$

Combining Theorem 3.6.6 with Theorems 3.6.1, 3.6.2 and 3.5.8, we get directly our final result in this chapter.

**Corollary 3.6.7.** *A centrally symmetric convex body  $K \in \mathcal{K}^n$  with  $\dim K = n$  is uniquely determined (up to translations) by its projection function  $v(K, \cdot)$ .*

### Exercises and problems

1. Let  $n \geq 3$  and  $P \in \mathcal{K}^n$  be a polytope. Show that  $P$  is a zonotope, if and only if all 2-faces of  $P$  are centrally symmetric.
2. Let  $Z \in \mathcal{K}^n$  be a zonoid and  $u_1, \dots, u_k \in S^{n-1}$ .
  - (a) Show that there exists a zonotope  $P$  such that

$$h_Z(u_i) = h_P(u_i), \quad i = 1, \dots, k.$$

*Hint:* Use CARATHÉODORY'S theorem for a suitable subset  $A$  of  $\mathbb{R}^k$ .

- (b) Show in addition that  $P$  can be chosen to be the sum of at most  $k$  segments.

*Hint:* Replace CARATHÉODORY'S theorem by the theorem of BUNDT.

3. Let  $P, Q \in \mathcal{P}^n$  be zonotopes and  $K \in \mathcal{K}^n$  a convex body such that

$$P = K + Q.$$

Show that  $K$  is also a zonotope.

- \* 4. Let  $P \in \mathcal{P}^n$  be a polytope. Show that  $P$  is a zonotope, if and only if  $h_P$  fulfills the HLAWKA inequality:

$$(*) \quad h_P(x) + h_P(y) + h_P(z) + h_P(x + y + z) \geq h_P(x + y) + h_P(x + z) + h_P(y + z),$$

for all  $x, y, z \in \mathbb{R}^n$ .

*Hint:* For one direction, show first that  $(*)$  implies the central symmetry of  $P$  and then that  $(*)$  implies the HLAWKA inequality for each face  $P(u)$ ,  $u \in S^{n-1}$ . Then use Exercise 1 above.

5. Let  $Z \in \mathcal{K}^n$  be a zonoid.

- (a) For  $u \in S^{n-1}$ , show that  $Z(u)$  is a zonoid and that

$$h_{Z(u)} = \int_{S^{n-1} \cap u^\perp} |\langle x, \cdot \rangle| \mu(Z, dx) + \langle x_u, \cdot \rangle,$$

where

$$x_u := 2 \int_{\{x \in S^{n-1}; \langle x, u \rangle > 0\}} x d\mu(Z, x).$$

- (b) Use (a) to show that a zonoid which is a polytope must be a zonotope.

# Chapter 4

## Integral geometric formulas

In this final chapter, we discuss integral formulas for intrinsic volumes  $V_j(K)$ , which are based on sections and projections of convex bodies  $K$ . We shall also discuss some applications of stereological nature.

### 4.1 Invariant measures

As a motivation, we start with the formula for the projection function  $v(K, \cdot)$  from Theorem 3.6.1. Integrating  $v(K, u)$  over all  $u \in S^{n-1}$  (with respect to the spherical LEBESGUE measure  $\omega_{n-1}$ ), we obtain

$$\int_{S^{n-1}} v(K, u) du = \kappa_{n-1} \int_{S^{n-1}} dS_{n-1}(K, x) = 2\kappa_{n-1} V_{n-1}(K).$$

Since  $v(K, u) = V_{n-1}(K|u^\perp)$ , we may replace the integration over  $S^{n-1}$  by one over the space  $\mathcal{L}_{n-1}^n$  of hyperplanes (through 0) in  $\mathbb{R}^n$ , namely by considering the normalized image measure  $\nu_{n-1}$  of  $\omega_{n-1}$  under the mapping  $u \mapsto u^\perp$ . Denoting the integration by  $\nu_{n-1}$  shortly as  $dL_{n-1}$ , we then get

$$\int_{\mathcal{L}_{n-1}^n} V_{n-1}(K|L_{n-1}) dL_{n-1} = \frac{2\kappa_{n-1}}{n\kappa_n} V_{n-1}(K).$$

This is known as CAUCHY's surface area formula for convex bodies. Our first goal is to generalize this projection formula to other intrinsic volumes  $V_j$  and to projection flats  $L_q$  of lower dimensions. This requires a natural measure  $\nu_q$  on the space of  $q$ -dimensional subspaces first. Later we will also consider integrals over sections of  $K$  with affine flats and integrate those with a natural measure  $\mu_q$  on (affine)  $q$ -flats. This first section discusses how the measures  $\nu_q$  and  $\mu_q$  can be introduced in an elementary way.

We start with the set  $\mathcal{L}_q^n$  of  $q$ -dimensional (linear) subspaces of  $\mathbb{R}^n$ ,  $q \in \{0, \dots, n-1\}$ .  $\mathcal{L}_q^n$  becomes a compact metric space, if we define the distance  $d(L, L')$ , for

$L, L' \in \mathcal{L}_q^n$ , as the HAUSDORFF distance of  $L \cap B(1)$  and  $L' \cap B(1)$ . We want to introduce an *invariant probability measure*  $\nu_q$  on  $\mathcal{L}_q^n$ . Here, probability measure refers to the Borel  $\sigma$ -algebra generated by the metric structure and invariance refers to the rotation group  $SO_n$  and means that

$$\nu_q(\vartheta\mathcal{A}) = \nu_q(\mathcal{A}),$$

for all  $\vartheta \in SO_n$  and all Borel sets  $\mathcal{A} \subset \mathcal{L}_q^n$  (with  $\vartheta\mathcal{A} := \{\vartheta L : L \in \mathcal{A}\}$ ). We will obtain  $\nu_q$  as the image measure of an invariant measure  $\nu$  on  $SO_n$ .

The rotation group  $SO_n$  can be viewed as a subset of  $(S^{n-1})^n \subset \mathbb{R}^{n^2}$ , if we identify rotations  $\vartheta$  with orthogonal matrices  $A$  (with  $\det A = 1$ ) and then replace  $A$  by the  $n$ -tuple  $(a_1, \dots, a_n) \in (S^{n-1})^n$  of column vectors. The euclidean metric on  $\mathbb{R}^{n^2}$  therefore induces a metric on  $SO_n$  and  $SO_n$  becomes a compact metric space in this way. It is easy to see that the operations of multiplication and inversion in  $SO_n$  (i.e. the mappings  $(\vartheta, \eta) \mapsto \vartheta\eta$  and  $\vartheta \mapsto \vartheta^{-1}$ ) are continuous. This shows that  $SO_n$  (with the given metric) is a compact *topological group*. A general theorem in the theory of topological groups implies the existence and uniqueness of an invariant probability measure  $\nu$  on  $SO_n$  (called the HAAR measure). Since  $SO_n$  is not commutative, invariance means here

$$\nu(\vartheta\mathcal{A}) = \nu(\mathcal{A}), \quad \nu(\mathcal{A}\vartheta) = \nu(\mathcal{A}), \quad \nu(\mathcal{A}^{-1}) = \nu(\mathcal{A}),$$

for all  $\vartheta \in SO_n$  and all Borel sets  $\mathcal{A} \subset SO_n$ , where

$$\vartheta\mathcal{A} := \{\vartheta\eta : \eta \in \mathcal{A}\}, \quad \mathcal{A}\vartheta := \{\eta\vartheta : \eta \in \mathcal{A}\}, \quad \mathcal{A}^{-1} := \{\eta^{-1} : \eta \in \mathcal{A}\}.$$

However, we can show the existence of  $\nu$  also by a direct construction.

**Lemma 4.1.1.** *There is an invariant probability measure  $\nu$  on  $SO_n$ .*

*Proof.* We consider the set  $LU_n \subset (S^{n-1})^n$  of linearly independent  $n$ -tuples.  $LU_n$  is open and the complement has measure zero with respect to  $\omega_{n-1} \otimes \dots \otimes \omega_{n-1}$ . On  $LU_n$  we define the mapping  $T$  onto  $SO_n$  by

$$(*) \quad T(x_1, \dots, x_n) := \left( \frac{y_1}{\|y_1\|} : \dots : \frac{y_n}{\|y_n\|} \right),$$

where  $(y_1, \dots, y_n)$  is the  $n$ -tuple obtained from  $(x_1, \dots, x_n)$  by the GRAM-SCHMIDT orthogonalization procedure (and where, in addition, the sign of  $y_n$  is chosen such that the matrix on the right side of  $(*)$  has determinant 1). Up to the sign of  $y_n$ , we thus have

$$y_k := x_k - \sum_{i=1}^{k-1} \langle x_k, y_i \rangle \frac{y_i}{\|y_i\|^2}, \quad k = 2, \dots, n,$$



and  $y_1 := x_1$ .  $T$  is almost everywhere defined (with respect to  $\omega_{n-1} \otimes \cdots \otimes \omega_{n-1}$ ) and continuous. Let  $\bar{\nu}$  be the image measure of  $\omega_{n-1} \otimes \cdots \otimes \omega_{n-1}$  under  $T$ . For each continuous function  $f$  on  $SO_n$  and  $\vartheta \in SO_n$ , we then get

$$\begin{aligned} \int_{SO_n} f(\vartheta\eta) d\bar{\nu}(\eta) &= \int_{S^{n-1}} \cdots \int_{S^{n-1}} f(\vartheta T(x_1, \dots, x_n)) dx_1 \cdots dx_n \\ &= \int_{S^{n-1}} \cdots \int_{S^{n-1}} f\left(\left(\frac{\vartheta y_1}{\|y_1\|} : \cdots : \frac{\vartheta y_n}{\|y_n\|}\right)\right) dx_1 \cdots dx_n. \end{aligned}$$

Obviously,

$$\left(\frac{\vartheta y_1}{\|y_1\|} : \cdots : \frac{\vartheta y_n}{\|y_n\|}\right) = T(\vartheta x_1, \dots, \vartheta x_n),$$

and we obtain

$$\int_{SO_n} f(\vartheta\eta) d\bar{\nu}(\eta) = \int_{SO_n} f(\eta) d\bar{\nu}(\eta).$$

This shows that  $\bar{\nu}$  is invariant from the left. Invariance from the right and inversion invariance can be shown in a similar manner (see the Exercises).

The normalized measure  $\nu := (1/n\kappa_n)^n \bar{\nu}$  fulfills now all assertions of the lemma.  $\square$

For the rest of this chapter we choose a fixed subspace  $L_q^0 \in \mathcal{L}_q^n$  as a reference space and define

$$\nu_q := \Phi \circ \nu,$$

where

$$\Phi : SO_n \rightarrow \mathcal{L}_q^n, \quad \vartheta \mapsto \vartheta L_q^0.$$

It is easy to see that  $\Phi$  is continuous and therefore measurable. The definition of  $\nu_q$  is based on the fact that the rotation group  $SO_n$  operates *transitively* on  $\mathcal{L}_q^n$ , which means that for given  $L, L' \in \mathcal{L}_q^n$  there is always a rotation  $\vartheta$  with  $L' = \vartheta L$ . This implies that the images  $\vartheta L_q^0, \vartheta \in SO_n$ , run through all elements of  $\mathcal{L}_q^n$ . We abbreviate the integration with respect to  $\nu_q$  by  $dL_q$ .

**Theorem 4.1.2.** *For  $q \in \{1, \dots, n-1\}$ , the measure  $\nu_q$  is an invariant probability measure. It is the only invariant probability measure on  $\mathcal{L}_q^n$ .*

*Moreover, for a continuous function  $f$  on  $\mathcal{L}_q^n$ , we have*

$$\int_{\mathcal{L}_q^n} f(L_q) dL_q = \int_{\mathcal{L}_{n-q}^n} f(L_{n-q}^\perp) dL_{n-q},$$

for  $1 \leq q \leq n-1$ , and

$$\int_{\mathcal{L}_q^n} f(L_q) dL_q = \int_{\mathcal{L}_m^n} \left( \int_{\mathcal{L}_q^n(L_m)} f(L_q) dL_q \right) dL_m,$$

for  $0 \leq q < m \leq n-1$ . (Here  $\mathcal{L}_q^n(L_m) := \{L_q \in \mathcal{L}_q^n : L_q \subset L_m\}$  and we identify this set with  $\mathcal{L}_q^m$ .)

*Proof.* Obviously,  $\nu_q$  is a probability measure. To show its invariance, let  $f$  be a continuous function on  $\mathcal{L}_q^n$  and  $\eta \in SO_n$ . Then

$$\begin{aligned} \int_{\mathcal{L}_q^n} f(\eta L_q) dL_q &= \int_{SO_n} f(\eta \vartheta L_q^0) d\nu(\vartheta) \\ &= \int_{SO_n} f(\rho L_q^0) d\nu(\rho) \\ &= \int_{\mathcal{L}_q^n} f(L_q) dL_q. \end{aligned}$$

Next, we show the uniqueness. Assume that  $\nu'_q$  is also an invariant probability measure on  $\mathcal{L}_q^n$ . Then

$$\begin{aligned} \int_{\mathcal{L}_q^n} f(L_q) d\nu'_q(L_q) &= \int_{SO_n} \int_{\mathcal{L}_q^n} f(\vartheta L_q) d\nu'_q(L_q) d\nu(\vartheta) \\ &= \int_{\mathcal{L}_q^n} \int_{SO_n} f(\vartheta L_q) d\nu(\vartheta) d\nu'_q(L_q) \end{aligned}$$

For  $L_q, L'_q \in \mathcal{L}_q^n$  there exists an  $\eta \in SO_n$  with  $L_q = \eta L'_q$ , hence

$$\begin{aligned} \int_{SO_n} f(\vartheta L_q) d\nu(\vartheta) &= \int_{SO_n} f(\vartheta \eta L'_q) d\nu(\vartheta) \\ &= \int_{SO_n} f(\vartheta L'_q) d\nu(\vartheta). \end{aligned}$$

This shows that the function

$$L_q \mapsto \int_{SO_n} f(\vartheta L_q) d\nu(\vartheta)$$

is a constant  $c(f)$ , which implies that

$$\int_{\mathcal{L}_q^n} f(L_q) d\nu'_q(L_q) = c(f) \int_{\mathcal{L}_q^n} d\nu'_q(L_q) = c(f).$$

In the same way, we get

$$\int_{\mathcal{L}_q^n} f(L_q) dL_q = c(f),$$

hence

$$\int_{\mathcal{L}_q^n} f(L_q) d\nu'_q(L_q) = \int_{\mathcal{L}_q^n} f(L_q) dL_q,$$

for all continuous functions  $f$  on  $\mathcal{L}_q^n$ . Therefore,  $\nu'_q = \nu_q$ .

The two integral formulas now follow from the uniqueness of  $\nu_q$ . Namely,

$$f \mapsto \int_{\mathcal{L}_{n-q}^n} f(L_{n-q}^\perp) dL_{n-q}$$

defines a probability measure  $\nu'_q$  on  $\mathcal{L}_q^n$  (by the RIESZ representation theorem). The invariance of  $\nu_{n-q}$  shows that  $\nu'_q$  is invariant, hence  $\nu'_q = \nu_q$ . In the same manner, we obtain the second, iterated integral formula. Here, the uniqueness result is already used to show that the invariant measure  $\nu_q^{\vartheta L_m}$  on  $\mathcal{L}_q^n(\vartheta L_m)$  is the image under  $L \mapsto \vartheta L$  of the invariant measure  $\nu_q^{L_m}$  on  $\mathcal{L}_q^n(L_m)$ .  $\square$

Now we consider the set  $\mathcal{E}_q^n$  of affine  $q$ -dimensional subspaces ( $q$ -flats, for short) in  $\mathbb{R}^n$ . Each  $E_q \in \mathcal{E}_q^n$  has a unique representation  $E_q = L_q + x$ ,  $L_q \in \mathcal{L}_q^n$ ,  $x \in L_q^\perp$ . This allows us to define a metric on  $\mathcal{E}_q^n$ , namely as

$$d(E_q, E'_q) := d(L_q, L'_q) + d(x, x').$$

The metric space  $\mathcal{E}_q^n$  is locally compact but not compact. We define the measure  $\mu_q$  as the image

$$\mu_q := \Psi \circ (\nu \otimes \lambda_{n-q}),$$

where

$$\Psi : SO_n \times (L_q^0)^\perp \rightarrow \mathcal{E}_q^n, \quad (\vartheta, x) \mapsto \vartheta(L_q^0 + x)$$

and  $\lambda_{n-q}$  is the LEBESGUE measure on  $(L_q^0)^\perp$ . Apparently,  $\mu_q(\mathcal{E}_q^n) = \infty$ , but the set  $\mathcal{E}_q^n(B(1))$  of  $q$ -flats intersecting the unit ball has finite measure,

$$\mu_q(\mathcal{E}_q^n(B(1))) = \kappa_{n-q}.$$

For the measure  $\mu_q$ , invariance refers to the group  $G_n$  of rigid motions, that is

$$\mu_q(g\mathcal{A}) = \mu_q(\mathcal{A}),$$

for all  $g \in G_n$  and all Borel sets  $\mathcal{A} \subset \mathcal{E}_q^n$  (again  $g\mathcal{A} := \{gL : L \in \mathcal{A}\}$ ). As in the case of  $\nu_q$ , we will denote integration by  $\mu_q$  simply as  $dE_q$ . For a flat  $E_m \in \mathcal{E}_m^n$ ,  $q < m \leq n-1$ , we put  $\mathcal{E}_q^n(E_m) := \{E_q \in \mathcal{E}_q^n : E_q \subset E_m\}$ . Because of the unique decomposition  $E_m = L_m + x$ ,  $L_m \in \mathcal{L}_m^n$ ,  $x \in L_m^\perp$ , we may identify  $\mathcal{E}_q^n(E_m)$  with  $\mathcal{E}_q^m$  (by mapping  $x$  to the origin). We denote by  $dL_m$  the integration with respect to the corresponding measure on  $\mathcal{E}_q^m(E_m)$ .

**Theorem 4.1.3.** *For  $q \in \{0, \dots, n-1\}$ ,  $\mu_q$  is an invariant measure.*

*For a continuous function  $f$  on  $\mathcal{E}_q^n$  with compact support, we have*

$$\int_{\mathcal{E}_q^n} f(E_q) dE_q = \int_{\mathcal{L}_q^n} \int_{L_q^\perp} f(L_q + x) dx dL_q.$$

Furthermore,

$$\int_{\mathcal{E}_q^n} f(E_q) dE_q = \int_{\mathcal{E}_m^n} \left( \int_{\mathcal{E}_q^n(E_m)} f(E_q) dE_q \right) dE_m,$$

for  $0 \leq q < m \leq n - 1$ .

*Proof.* For the invariance of  $\mu_q$ , we consider  $g \in G_n$  and a continuous function  $f$  on  $\mathcal{E}_q^n$  with compact support. By definition of  $\mu_q$ ,

$$\int_{\mathcal{E}_q^n} f(gE_q) dE_q = \int_{SO_n} \int_{(L_q^0)^\perp} f(g\vartheta(L_q^0 + x)) dx d\vartheta.$$

We decompose  $g$  into rotation and translation,

$$g : z \mapsto \eta(z + y), \quad \eta \in SO_n, y \in \mathbb{R}^n,$$

and put  $x' := \vartheta^{-1}y|(L_q^0)^\perp$ . Then,

$$g\vartheta(L_q^0 + x) = \eta\vartheta(L_q^0 + x + x'),$$

hence

$$\begin{aligned} \int_{\mathcal{E}_q^n} f(gE_q) dE_q &= \int_{SO_n} \int_{(L_q^0)^\perp} f(\rho(L_q^0 + z)) dz d\rho \\ &= \int_{\mathcal{E}_q^n} f(E_q) dE_q. \end{aligned}$$

The first integral formula follows from

$$\begin{aligned} \int_{\mathcal{E}_q^n} f(E_q) dE_q &= \int_{SO_n} \int_{(L_q^0)^\perp} f(\vartheta(L_q^0 + x)) dx d\nu(\vartheta) \\ &= \int_{SO_n} \int_{(\vartheta L_q^0)^\perp} f(\vartheta L_q^0 + x) dx d\nu(\vartheta) \\ &= \int_{\mathcal{L}_q^n} \int_{L_q^\perp} f(L_q + x) dx dL_q. \end{aligned}$$

For the second integral formula, we consider  $E_q \in \mathcal{E}_q^n(E_m)$ . Because of  $E_m = L_m + x$ ,  $L_m \in \mathcal{L}_m^n$ ,  $x \in L_m^\perp$ , we get

$$E_q = L_q + x + y$$

with  $L_q \in \mathcal{L}_q^n(L_m)$  and  $y \in L_q^\perp \cap L_m$ . Therefore,

$$\begin{aligned}
& \int_{\mathcal{E}_m^n} \left( \int_{\mathcal{E}_q^n(E_m)} f(E_q) dE_q \right) dE_m \\
&= \int_{\mathcal{L}_m^n} \int_{L_m^\perp} \left( \int_{\mathcal{L}_q^n(L_m)} \int_{L_q^\perp \cap L_m} f(L_q + x + y) dy dL_q \right) dx dL_m \\
&= \int_{\mathcal{L}_m^n} \int_{\mathcal{L}_q^n(L_m)} \left( \int_{L_m^\perp} \int_{L_q^\perp \cap L_m} f(L_q + x + y) dy dx \right) dL_q dL_m \\
&= \int_{\mathcal{L}_m^n} \int_{\mathcal{L}_q^n(L_m)} \left( \int_{L_q^\perp} f(L_q + z) dz \right) dL_q dL_m \\
&= \int_{\mathcal{L}_q^n} \int_{L_q^\perp} f(L_q + z) dz dL_q \\
&= \int_{\mathcal{E}_q^n} f(E_q) dE_q,
\end{aligned}$$

where we have used the first integral formula and also Theorem 4.1.2.  $\square$

We remark that it is also possible to prove a uniqueness result for the measure  $\mu_q$ , but the proof is a bit more involved. We also remark, that both measures  $\nu_q$  and  $\mu_q$  are actually independent of the choice of the reference space  $L_q^0$ .

### Exercises and problems

1. Fill in the arguments omitted in the proof of Lemma 4.1.1 (invariance from the right and inversion invariance) by showing that

$$\nu(\mathcal{A}\vartheta) = \nu(\mathcal{A}), \quad \text{and} \quad \nu(\mathcal{A}^{-1}) = \nu(\mathcal{A}),$$

for all  $\vartheta \in SO_n$  and all Borel sets  $\mathcal{A} \subset SO_n$ .

2. Show that

$$\int_{SO_n} \int_{(L_q^1)^\perp} f(\vartheta(L_q^1 + x)) dx d\nu(\vartheta) = \int_{SO_n} \int_{(L_q^0)^\perp} f(\vartheta(L_q^0 + x)) dx d\nu(\vartheta),$$

for  $L_q^0, L_q^1 \in \mathcal{L}_q^n$  and a continuous function  $f$  on  $\mathcal{E}_q^n$  with compact support (independence of the reference space).

- \* 3. Show that  $\mu_q$  is the only invariant measure on  $\mathcal{E}_q^n$  with

$$\mu_q(\mathcal{E}_q^n(B(1))) = \kappa_{n-q}.$$

## 4.2 Projection formulas

**Theorem 4.2.1 (CAUCHY-KUBOTA).** *Let  $K \in \mathcal{K}^n$ ,  $q \in \{0, \dots, n-1\}$  and  $j \in \{0, \dots, q\}$ . Then, we have*

$$\int_{\mathcal{L}_q^n} V_j(K|L_q) dL_q = \beta_{njq} V_j(K)$$

with

$$\beta_{njq} = \frac{\binom{q}{j} \kappa_q \kappa_{n-j}}{\binom{n}{j} \kappa_n \kappa_{q-j}}.$$

*Proof.* The mapping  $L_q \mapsto K|L_q$  is continuous, therefore

$$L_q \mapsto V_j(K|L_q)$$

is continuous.

We first consider the case  $q = n-1$ . For  $j = q$ , we get CAUCHY's surface formula which has been proved already at the beginning of section 4.1. For  $j < q$ , we combine this with the STEINER formula (in dimension  $n-1$ ). We obtain

$$\begin{aligned} V_{n-1}(K + B(\alpha)) &= \frac{n\kappa_n}{2\kappa_{n-1}} \int_{\mathcal{L}_{n-1}^n} V_{n-1}(K|L_{n-1} + [B(\alpha) \cap L_{n-1}]) dL_{n-1} \\ &= \frac{n\kappa_n}{2\kappa_{n-1}} \sum_{j=0}^{n-1} \alpha^{n-1-j} \kappa_{n-1-j} \int_{\mathcal{L}_{n-1}^n} V_j(K|L_{n-1}) dL_{n-1}. \end{aligned}$$

(Here, we make use of the fact that  $V_{n-1}$  is dimension invariant, hence  $V_{n-1}(K|L_{n-1})$  yields the same value in  $L_{n-1}$  as in  $\mathbb{R}^n$ .) On the other hand, Corollary 3.5.4 (or Exercise 3.3.7) show that

$$\begin{aligned} V_{n-1}(K + B(\alpha)) &\left( = \frac{1}{2} F(K + B(\alpha)) \right) \\ &= \sum_{j=0}^{n-1} \alpha^{n-1-j} \frac{(n-j)\kappa_{n-j}}{2} V_j(K). \end{aligned}$$

The formula for  $j < q = n-1$  follows now by comparing the coefficients in these two polynomial expansions.

Finally, the case  $q < n-1$  is obtained by a recursion. Namely, assume that the formula holds for  $q+1$ . Then, using Theorem 4.1.2 we obtain

$$\int_{\mathcal{L}_q^n} V_j(K|L_q) dL_q = \int_{\mathcal{L}_{q+1}^n} \left( \int_{\mathcal{L}_q^n(L_{q+1})} V_j(K|L_q) dL_q \right) dL_{q+1}.$$

The inner integral refers to the hyperplane case (in dimension  $q + 1$ ) which we have proved already. Therefore,

$$\begin{aligned} \int_{\mathcal{L}_q^n} V_j(K|L_q) dL_q &= \beta_{(q+1)jq} \int_{\mathcal{L}_{q+1}^n} V_j(K|L_{q+1}) dL_{q+1} \\ &= \beta_{(q+1)jq} \beta_{nj(q+1)} V_j(K) \\ &= \beta_{njq} V_j(K), \end{aligned}$$

where we have used the assertion for  $q+1$  and the fact that  $K|L_q = (K|L_{q+1})|L_q$ .  $\square$

**Remarks.** (1) For  $j = q$ , the CAUCHY-KUBOTA formulas yield

$$V_j(K) = \frac{1}{\beta_{njj}} \int_{\mathcal{L}_j^n} V_j(K|L_j) dL_j,$$

hence  $V_j(K)$  is proportional to the mean content of the projections of  $K$  onto  $j$ -dimensional subspaces. Since  $V_j(K|L_j)$  is also the content of the base of the cylinder circumscribed to  $K$  (with direction space  $L^\perp$ ),  $V_j(K|L_j)$  was called the ‘quermass’ of  $K$  in direction  $L^\perp$ . This explains the name ‘quermassintegral’ for the functionals  $W_{n-j}(K) = c_{nj} V_j(K)$ .

(2) For  $j = q = 1$ , we obtain

$$V_1(K) = \frac{1}{\beta_{n11}} \int_{\mathcal{L}_1^n} V_1(K|L_1) dL_1.$$

This gives now a rigorous proof for the fact that  $V_1(K)$  is proportional to the mean width of  $K$ .

### Exercises and problems

1. Prove the following generalizations of the CAUCHY-KUBOTA formulas:

$$(a) \quad \int_{\mathcal{L}_q^n} V^{(q)}(K_1|L_q, \dots, K_q|L_q) dL_q = \gamma_{nq} V(K_1, \dots, K_q, \underbrace{B(1), \dots, B(1)}_{n-q}),$$

for  $K_1, \dots, K_q \in \mathcal{K}^n$ ,  $0 \leq q \leq n - 1$ , and a certain constant  $\gamma_{nq}$ ,

$$(b) \quad \int_{\mathcal{L}_q^n} S_j^{(q)}(K|L_q, A \cap L_q) dL_q = \delta_{njq} S_j(K, A),$$

for  $K \in \mathcal{K}^n$ , a Borel set  $A \subset S^{n-1}$ ,  $0 \leq j < q \leq n - 1$ , and a certain constant  $\delta_{njq}$ .

### 4.3 Section formulas

**Theorem 4.3.1 (CROFTON).** *Let  $K \in \mathcal{K}^n$ ,  $q \in \{0, \dots, n-1\}$  and  $j \in \{0, \dots, q\}$ . Then, we have*

$$\int_{\mathcal{E}_q^n} V_j(K \cap E_q) dE_q = \alpha_{njq} V_{n+j-q}(K)$$

with

$$\alpha_{njq} = \frac{\binom{q}{j} \kappa_q \kappa_{n+j-q}}{\binom{n}{q-j} \kappa_n \kappa_j}.$$

*Proof.* Here, we start with the case  $j = 0$ . From Theorem 4.1.3, we get

$$\int_{\mathcal{E}_q^n} V_0(K \cap E_q) dE_q = \int_{\mathcal{L}_q^n} \int_{L_q^\perp} V_0(K \cap (L_q + x)) dx dL_q.$$

On the right-hand side, the integrand fulfills

$$V_0(K \cap (L_q + x)) = \begin{cases} 1 & \text{if } x \in K|L_q^\perp, \\ 0 & \text{if } x \notin K|L_q^\perp. \end{cases}$$

Hence, using Theorems 4.1.2 and 4.2.1, we obtain

$$\begin{aligned} \int_{\mathcal{E}_q^n} V_0(K \cap E_q) dE_q &= \int_{\mathcal{L}_q^n} V_{n-q}(K|L_q^\perp) dL_q \\ &= \int_{\mathcal{L}_{n-q}^n} V_{n-q}(K|L_{n-q}) dL_{n-q} \\ &= \beta_{n(n-q)(n-q)} V_{n-q}(K) \\ &= \alpha_{n0q} V_{n-q}(K). \end{aligned}$$

This proves the result for  $j = 0$ .

Now, let  $j > 0$ . We use the result just proven for  $K \cap E_q$  (in  $E_q$ ) and obtain

$$V_j(K \cap E_q) = \frac{1}{\beta_{qjj}} \int_{\mathcal{E}_{q-j}^n(E_q)} V_0(K \cap E_{q-j}) dE_{q-j}.$$

Hence,

$$\begin{aligned} \int_{\mathcal{E}_q^n} V_j(K \cap E_q) dE_q &= \frac{1}{\beta_{qjj}} \int_{\mathcal{E}_q^n} \int_{\mathcal{E}_{q-j}^n(E_q)} V_0(K \cap E_{q-j}) dE_{q-j} dE_q \\ &= \frac{1}{\beta_{qjj}} \int_{\mathcal{E}_{q-j}^n} V_0(K \cap E_{q-j}) dE_{q-j} \\ &= \frac{\beta_{n(n+j-q)(n+j-q)}}{\beta_{qjj}} V_{n+j-q}(K) \\ &= \alpha_{njq} V_{n+j-q}(K), \end{aligned}$$

where we have first used Theorem 4.1.3 and then again the result above.  $\square$



**Remarks.** (1) Replacing the pair  $(j, q)$  by  $(0, n - j)$ , we obtain

$$\begin{aligned} V_j(K) &= \frac{1}{\alpha_{n0(n-j)}} \int_{\mathcal{E}_{n-j}^n} V_0(K \cap E_{n-j}) dE_{n-j} \\ &= \frac{1}{\alpha_{n0(n-j)}} \int_{K \cap E_{n-j} \neq \emptyset} dE_{n-j} \\ &= \frac{1}{\alpha_{n0(n-j)}} \mu_{n-j}(\{E_{n-j} \in \mathcal{E}_{n-j}^n : K \cap E_{n-j} \neq \emptyset\}). \end{aligned}$$

Hence,  $V_j(K)$  is (up to a constant) the measure of all  $(n - j)$ -flats which meet  $K$ .

(2) We can give another interpretation of  $V_j(K)$  in terms of flats touching  $K$ . Namely, consider the set

$$A(\alpha) := (\{E_{n-j-1} \in \mathcal{E}_{n-j-1}^n : K \cap E_{n-j-1} = \emptyset, K + B(\alpha) \cap E_{n-j-1} \neq \emptyset\}).$$

These are the  $(n - j - 1)$ -flats meeting the parallel body  $K + B(\alpha)$  but not  $K$ . If the limit

$$\lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \mu_{n-j-1}(A(\alpha))$$

exists, we can interpret it as the measure of all  $(n - j - 1)$ -flats touching  $K$ . Now (1) and Exercise 3.3.7 show that

$$\begin{aligned} \frac{1}{\alpha} \mu_{n-j-1}(A(\alpha)) &= \frac{\alpha_{n0(n-j-1)}}{\alpha} [V_{j+1}(K + B(\alpha)) - V_{j+1}(K)] \\ &= \frac{\alpha_{n0(n-j-1)}}{\alpha} \sum_{i=0}^j \alpha^{j+1-i} \binom{n-i}{n-j-1} \frac{\kappa_{n-i}}{\kappa_{n-j-1}} V_i(K) \\ &\rightarrow \alpha_{n0(n-j-1)} (n-j) \frac{\kappa_{n-j}}{\kappa_{n-j-1}} V_j(K), \end{aligned}$$

as  $\alpha \rightarrow 0$ .

(3) We can use (1) to solve some problems of *Geometrical Probability*. Namely, if  $K, K_0 \in \mathcal{K}^n$  are such that  $K \subset K_0$  and  $V(K_0) > 0$ , we can restrict  $\mu_q$  to  $\{E_q \in \mathcal{E}_q^n : K_0 \cap E_q \neq \emptyset\}$  and normalize it to get a probability measure. A random  $q$ -flat  $X_q$  with this distribution is called a *random  $q$ -flat in  $K_0$* . We then get

$$\text{Prob}(X_q \cap K \neq \emptyset) = \frac{V_{n-q}(K)}{V_{n-q}(K_0)}.$$

As an example, we mention the BUFFON needle problem. Originally the problem was formulated in the following way: Given an array of parallel lines in the plane  $\mathbb{R}^2$  with distance 1, what is the probability that a randomly thrown needle of length  $L < 1$  intersects one of the lines? If we consider the disc of radius  $\frac{1}{2}$  around the center of the needle, there will be almost surely exactly one line of the array intersecting

this disc. Hence, the problem can be formulated in an equivalent way: Assume the needle  $N$  is fixed with center at 0. What is the probability that a random line  $X_1$  in  $B(\frac{1}{2})$  intersects  $N$ ? The answer is

$$\begin{aligned} \text{Prob}(X_1 \cap N \neq \emptyset) &= \frac{V_1(N)}{V_1(B(\frac{1}{2}))} \\ &= \frac{L}{\pi/2} \\ &= \frac{2L}{\pi}. \end{aligned}$$

(4) In continuation of (3), we can consider, for  $K, K_0 \in \mathcal{K}^n$  with  $K \subset K_0$  and  $V(K_0) > 0$  and for a random  $q$ -flat  $X_q$  in  $K_0$ , the expected  $j$ -th intrinsic volume of  $K \cap X_q$ ,  $j \in \{0, \dots, q\}$ . We get

$$\begin{aligned} \mathbb{E}V_j(K \cap X_q) &= \frac{\int V_j(K \cap E_q) dE_q}{\int V_0(K_0 \cap E_q) dE_q} \\ &= \frac{\alpha_{njq} V_{n+j-q}(K)}{\alpha_{n0q} V_{n-q}(K_0)}. \end{aligned}$$

If  $K_0$  is supposed to be known (and  $K$  is unknown) and if  $V_j(K \cap X_q)$  is observable, then

$$\frac{\alpha_{n0q} V_{n-q}(K_0)}{\alpha_{njq}} V_j(K \cap X_q)$$

is an unbiased estimator of  $V_{n+j-q}(K)$ . Varying  $q$ , we get in this way three estimators for the volume  $V(K)$ , two for the surface area  $F(K)$  and one for the mean width  $\bar{B}(K)$ .

The estimation formulas in Remark (4) would be of practical interest, if the set  $K$  under consideration was not supposed to be convex. In this final part, we therefore want to generalize the CROFTON formulas to certain non-convex sets. The set class which we consider is the *convex ring*  $\mathcal{R}^n$ , which consists of finite unions of convex bodies,

$$\mathcal{R}^n := \left\{ \bigcup_{i=1}^k K_i : k \in \mathbb{N}, K_i \in \mathcal{K}^n \right\}.$$

We assume  $\emptyset \in \mathcal{K}^n$ , hence  $\mathcal{R}^n$  is closed against finite unions and intersections, and it is the smallest set class with this property and containing  $\mathcal{K}^n$ . It is easy to see that  $\mathcal{R}^n$  is dense in the class  $\mathcal{C}^n$  of compact subsets of  $\mathbb{R}^n$  (in the Hausdorff metric), hence any compact set can be well approximated by elements of  $\mathcal{R}^n$ .

Our first goal is to extend the intrinsic volumes  $V_j$  to sets in  $\mathcal{R}^n$ . Since  $V_j$  is additive on  $\mathcal{K}^n$  (see the exercises), we seek an additive extension. Here a functional  $\varphi$  on  $\mathcal{R}^n$  (or  $\mathcal{K}^n$ ) is called *additive*, if

$$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M).$$

On  $\mathcal{R}^n$ , we require that this relation holds for all  $K, M \in \mathcal{R}^n$ , whereas on  $\mathcal{K}^n$  we can only require it for  $K, M \in \mathcal{K}^n$  with  $K \cup M \in \mathcal{K}^n$ . In addition, we assume that an additive functional  $\varphi$  fulfills  $\varphi(\emptyset) = 0$ . If  $\varphi : \mathcal{R}^n \rightarrow \mathbb{R}$  is additive, the inclusion-exclusion principle (which follows by induction) shows that, for  $A \in \mathcal{R}^n$ ,  $A = \bigcup_{i=1}^k K_i$ ,  $K_i \in \mathcal{K}^n$ , we have

$$(*) \quad \varphi(A) = \sum_{v \in S(k)} (-1)^{|v|-1} \varphi(K_v).$$

Here, we have used the following notation:  $S(k)$  is the set of all non-empty finite subsets of  $\{1, \dots, k\}$ ,  $|v|$  is the cardinality of  $v$ , and  $K_v$ , for  $v = \{i_1, \dots, i_m\}$ , is the intersection  $K_{i_1} \cap \dots \cap K_{i_m}$ . (\*) shows that the values of  $\varphi$  on  $\mathcal{R}^n$  depend only on the behavior of  $\varphi$  on  $\mathcal{K}^n$ . In particular, if an additive functional  $\varphi : \mathcal{K}^n \rightarrow \mathbb{R}$  has an additive extension to  $\mathcal{R}^n$ , then this extension is unique. On the other hand, (\*) cannot be used to show the existence of such an additive extension, since the right-hand side may depend on the special representation  $A = \bigcup_{i=1}^k K_i$  (and, in general, a set  $A \in \mathcal{R}^n$  can have many different representations as a finite union of convex bodies). There is a general theorem of GROEMER which guarantees the existence of an additive extension for all functionals  $\varphi$  which are additive and continuous on  $\mathcal{K}^n$ . For the intrinsic volumes  $V_j$ , however, we can use a direct approach due to HADWIGER.

**Theorem 4.3.2.** *For  $j = 0, \dots, n$ , there is a unique additive extension of  $V_j$  onto  $\mathcal{R}^n$ .*

*Proof.* It remains to show the existence.

We begin with the Euler characteristic  $V_0$  and prove the existence by induction on  $n$ ,  $n \geq 0$ .

It is convenient to start with the dimension  $n = 0$  since  $\mathcal{R}^0 = \{\emptyset, \{0\}\} (= \mathcal{K}^0)$ . Because of  $V_0(\emptyset) = 0$  and  $V_0(\{0\}) = 1$ ,  $V_0$  is additive on  $\mathcal{R}^0$ .

For the step from dimension  $n - 1$  to dimension  $n$ ,  $n \geq 1$ , we choose a fixed direction  $u_0 \in S^{n-1}$  and consider the family of hyperplanes  $E_\alpha := \{\langle \cdot, u_0 \rangle = \alpha\}$ ,  $\alpha \in \mathbb{R}$ . Then, for  $A \in \mathcal{R}^n$ ,  $A = \bigcup_{i=1}^k K_i$ ,  $K_i \in \mathcal{K}^n$ , we have

$$A \cap E_\alpha = \bigcup_{i=1}^k (K_i \cap E_\alpha)$$

and by induction hypothesis the additive extension  $V_0(A \cap E_\alpha)$  exists. From (\*) we obtain that the function  $f_A : \alpha \mapsto V_0(A \cap E_\alpha)$  is integer-valued and bounded from below and above. Therefore,  $f_A$  is piecewise constant and (\*) shows that the value of  $f_A(\alpha)$  can only change if the hyperplane  $E_\alpha$  supports one of the convex bodies  $K_v$ ,  $v \in S(k)$ . We define the ‘jump function’

$$g_A(\alpha) := f_A(\alpha) - \lim_{\beta \searrow \alpha} f_A(\beta), \quad \alpha \in \mathbb{R},$$

and put

$$V_0(A) := \sum_{\alpha \in \mathbb{R}} g_A(\alpha).$$

This definition makes sense since  $g_A(\alpha) \neq 0$  only for finitely many values of  $\alpha$ . Moreover, for  $k = 1$ , that is  $A = K \in \mathcal{K}^n$ ,  $K \neq \emptyset$ , we have  $V_0(K) = 0 + 1 = 1$ , hence  $V_0$  is an extension of the Euler characteristic. By induction hypothesis,  $A \mapsto f_A(\alpha)$  is additive on  $\mathcal{R}^n$  for each  $\alpha$ . Therefore, as a limit,  $A \mapsto g_A(\alpha)$  is additive and so  $V_0$  is additive. The uniqueness, which we have already obtained from (\*), shows that this construction does not depend on the choice of the direction  $u_0$ .

Now we consider the case  $j > 0$ . For  $A \in \mathcal{R}^n$ ,  $A = \bigcup_{i=1}^k K_i$ ,  $K_i \in \mathcal{K}^n$ , and  $\alpha > 0$ ,  $x \in \mathbb{R}^n$ , we have

$$A \cap (B(\alpha) + x) = \bigcup_{i=1}^k (K_i \cap (B(\alpha) + x)).$$

Therefore, (\*) implies

$$V_0(A \cap (B(\alpha) + x)) = \sum_{v \in S(k)} (-1)^{|v|-1} V_0(K_v \cap (B(\alpha) + x)).$$

Since  $V_0(K_v \cap (B(\alpha) + x)) = 1$ , if and only if  $x \in K_v + B(\alpha)$ , we then get from the STEINER formula

$$\begin{aligned} \int_{\mathbb{R}^n} V_0(A \cap (B(\alpha) + x)) dx &= \sum_{v \in S(k)} (-1)^{|v|-1} \int_{\mathbb{R}^n} V_0(K_v \cap (B(\alpha) + x)) dx \\ &= \sum_{v \in S(k)} (-1)^{|v|-1} V_n(K_v + B(\alpha)) \\ &= \sum_{v \in S(k)} (-1)^{|v|-1} \left( \sum_{j=0}^n \alpha^{n-j} \kappa_{n-j} V_j(K_v) \right) \\ &= \sum_{j=0}^n \alpha^{n-j} \kappa_{n-j} \left( \sum_{v \in S(k)} (-1)^{|v|-1} V_j(K_v) \right). \end{aligned}$$

If we define

$$V_j(A) := \sum_{v \in S(k)} (-1)^{|v|-1} V_j(K_v),$$

then

$$\int_{\mathbb{R}^n} V_0(A \cap (B(\alpha) + x)) dx = \sum_{j=0}^n \alpha^{n-j} \kappa_{n-j} V_j(A).$$

Since this equation holds for all  $\alpha > 0$ , the values  $V_j(A)$ ,  $j = 0, \dots, n$ , depend only on  $A$  and not on the special representation, and moreover  $V_j$  is additive.  $\square$

**Remarks.** (1) The formula

$$\int_{\mathbb{R}^n} V_0(A \cap (B(\alpha) + x)) dx = \sum_{j=0}^n \alpha^{n-j} \kappa_{n-j} V_j(A),$$

which we used in the above proof, is a *generalized STEINER formula*; it reduces to the classical STEINER formula if  $A \in \mathcal{K}^n$ .

(2) The extended EULER characteristic  $V_0$  (also called the EULER-POINCARÉ characteristic) plays also an important role in topology. In  $\mathbb{R}^2$  and for  $A \in \mathcal{R}^2$ ,  $V_0(A)$  equals the number of connected components minus the number of ‘holes’ in  $A$ .

(3) On  $\mathcal{R}^n$ ,  $V_n$  is still the volume (Lebesgue measure) and  $F = 2V_{n-1}$  can still be interpreted as the surface area. The other (extended) intrinsic volumes  $V_j$  do not have a direct geometric interpretation.

Since union and intersection can be interchanged (as we have used already in the above arguments), the additivity of  $V_j$  allows us directly to extend the Crofton formulas to the convex ring.

**Theorem 4.3.3 (CROFTON).** *Let  $A \in \mathcal{R}^n$ ,  $q \in \{0, \dots, n-1\}$  and  $j \in \{0, \dots, q\}$ . Then, we have*

$$\int_{\mathcal{E}_q^n} V_j(A \cap E_q) dE_q = \alpha_{njq} V_{n+j-q}(A).$$

As we have explained in a previous remark, these formulas can be used to give unbiased estimators for  $V_{n+j-q}(A)$  based on intersections  $A \cap X_q$  with random  $q$ -flats in the reference body  $K_0$ . This can be used in practical situations to estimate the surface area of a complicated tissue  $A$  in, say, a cubical specimen  $K_0$  by measuring the boundary length  $L(A \cap X_2)$  of a planar section  $A \cap X_2$ . Since the latter quantity is still complicated to obtain, one uses the CROFTON formulas again and estimates  $L(A \cap X_2)$  from counting intersections with random lines  $X_1$  in  $K_0 \cap X_2$ . Such *stereological formulas* are used and have been developed further in many applied sciences including medicine, biology, geology, metallurgy and material science.

### Exercises and problems

1. Calculate the probability that a random secant of  $B(1)$  is longer than  $\sqrt{3}$ . (According to the interpretation of a ‘random secant’, one might get here the values  $\frac{1}{2}$ ,  $\frac{1}{3}$  or  $\frac{1}{4}$ . Explain why  $\frac{1}{2}$  is the right, ‘rigid motion invariant’ answer.)
2. Let  $K, K' \in \mathcal{K}^n$  and  $K \cup K' \in \mathcal{K}^n$ . Show that:

$$(a) \quad (K \cap K') + (K \cup K') = K + K',$$

(b)  $(K \cap K') + M = (K + M) \cap (K' + M)$ , for all  $M \in \mathcal{K}^n$ .

(c)  $(K \cup K') + M = (K + M) \cup (K' + M)$ , for all  $M \in \mathcal{K}^n$ .

3. Let  $\varphi(K) := V(\underbrace{K, \dots, K}_{j\text{-mal}}, M_{j+1}, \dots, M_n)$ , where  $K, M_{j+1}, \dots, M_n \in \mathcal{K}^n$ . Show that  $\varphi$  is additive, that is

$$\varphi(K \cap K') + \varphi(K \cup K') = \varphi(K) + \varphi(K')$$

for all  $K, K' \in \mathcal{K}^n$  with  $K \cup K' \in \mathcal{K}^n$ .

4. Show that the mappings  $K \mapsto S_j(K, A)$  are additive on  $\mathcal{K}^n$ , for all  $j \in \{0, \dots, n\}$  and all Borel sets  $A \subset S^{n-1}$ .
5. Show that the convex ring  $\mathcal{R}^n$  is dense in  $\mathcal{C}^n$  in the HAUSDORFF metric.
6. Let  $\varphi : \mathcal{R}^n \rightarrow \mathbb{R}$  be additive and  $A \in \mathcal{R}^n$ ,  $A = \bigcup_{i=1}^k K_i$ ,  $K_i \in \mathcal{K}^n$ . Give a proof for the inclusion-exclusion formula

$$(*) \quad \varphi(A) = \sum_{v \in S(k)} (-1)^{|v|-1} \varphi(K_v).$$