Point processes of cloud nets

So far we considered the cases $E = \mathbb{R}^n$ or $E = \mathbb{R}^n \times \mathbb{M}$. Now we concentrate on $E = F'(\mathbb{R}^n) = F$. Recall that Lemma 1.2 implies that $\Theta = \text{IE}_F$ is locally compact on $F'$. If $\Theta(F_c) < \infty$ for all $C \in \mathcal{C}$, we always assume that this condition is satisfied.

Theorem 2.27

1) Let $\Phi$ be a point process in $F'$ and $\Theta = \text{IE}_F$.

Then

$\Phi$ is stationary $\Rightarrow$ $\Theta$ is translation invariant,
$\Phi$ is isotropic $\Rightarrow$ $\Theta$ is rotation invariant.

2) Let $\Phi$ be a (simple) Poisson process in $F'$, $\Theta = \text{IE}_F$.

Then

$\Phi$ is stationary $(=)$ $\Theta$ is translation invariant,
$\Phi$ is isotropic $(=)$ $\Theta$ is rotation invariant.

3) Let $\Phi$ have Poisson distributed counting variables in $F'$, and assume that $\Phi(\{\mathbb{R}^n\}) = 0$ $\mathbb{P}$-a.s. and that $\Theta = \text{IE}_F$ is translation invariant. Then $\Phi$ is simple and therefore a stationary Poisson process.

Proof. a) is clear by definition.

$E[g(\Phi)] = g(\Theta) = \Theta$, $g \in G_u$. Hence Theorem 2.8.
implies the assertion.

c) Claim: $\Theta$ is free of atoms. Suppose $\{F\}, F \in F$, is an atom of $\Theta$. Then, by translation invariance, the same holds for $\{F\} + x = \{F + x\}$, where $x \in \mathbb{R}^n$. If $F \neq \mathbb{R}^n$, then there are infinitely many $x \in \mathbb{R}^n$ and a set $C \subseteq \mathbb{R}$ such that $(F + x) \cap C = \emptyset$ and such that the sets $F + x$ are mutually distinct. Therefore $\Theta (F_C) = \infty$, a contradiction. Therefore, necessarily $F = \mathbb{R}^n$ under the present assumptions. But

$$P(\bar{\Theta}(\{\mathbb{R}^n\}) > 0) = 1 - e^{-\Theta(\{\mathbb{R}^n\})} > 0,$$

which contradicts the hypotheses of c). Therefore $\Theta$ is simple (cf. Ex. 7.1). \(\blacksquare\)

Example. Let $\Phi$ be a point process in $F$ and let $A \subseteq F$ be fixed.

$$\Phi \cap A := \sum_{F \in \Phi} \delta_{F \cap A}$$

is a point process in $F$, the intersection process of $\Phi$ with $A$.

Theorem 2.28 Let $\Phi$ be a point process in $F$ and

$$Z_{\Phi} := \bigcup_{F \in \text{mpp} \Phi} F.$$

Then $Z_\Phi$ is a RACS in $\mathbb{R}^n$. Moreover, if $\Phi$ is stationary (isotropic) then $Z_\Phi$ is stationary (isotropic).
Proof. \( Z_\Phi \) is closed: Let \( \omega \in \Omega \) be fixed, \( (\omega_j)_j \in \Omega \) is compact.

Since \( F \) is compact in \( F^1 \) and \( \Phi(\Omega) \) is locally compact, it follows that \( \Phi(\Omega)(F) = \emptyset \). Hence there are only finitely many sets, say \( F_1, \ldots, F_k \in \text{supp} (\Phi) \) which have nonempty intersection with \( C \). This shows that \( \omega_j \in \bigcup_{i=1}^k F_i \), for \( j \in \mathbb{N} \), hence

\[
\omega \in \bigcup_{i=1}^k F_i \subset \bigcup_{i=1}^k Z_{\Phi}(F_i) \text{ is closed,}
\]

i.e. \( Z_{\Phi} : \Omega \rightarrow F \) is well-defined.

\( Z_{\Phi} \) is measurable: Let \( C \in \mathcal{C} \). Then \( Z_{\Phi}(\omega) \in F \) if \( \Phi(\omega)(F) = 0 \).

Thus \( \{ \omega \in \Omega : \Phi(\omega)(F) = 0 \} \) is measurable by assumption. \( \square \)

The following statements continue the analysis of infinitely divisible RACSs.

First we state both theorems and then we provide the proofs.
Theorem 2.29 For a RACS \( Z \in \mathbb{R}^n \) the following conditions are equivalent:

a) There is a point process \( \Phi \) with Poisson distributed counting variables such that \( Z \sim Z \Phi \).

b) There is a Poisson process \( \Phi \) such that \( Z \sim Z \Phi \).

c) \( Z \) has no fixed points and is infinitely divisible.

If these conditions are satisfied, then \( \Theta \) is the intensity measure of \( \Phi \), i.e., \( \Theta = \text{int} \Phi \).

Theorem 2.30 For a stationary RACS \( Z \in \mathbb{R}^n \) with \( Z \not\equiv 0 \), the following conditions are equivalent:

a) There is a simple Poisson process \( \Phi \) in \( \mathbb{R}^1 \) so that \( Z \sim Z \Phi \).

b) There is a locally finite measure \( \Theta \) on \( \mathbb{R}(\mathbb{R}) \) which is free of atoms and such that \( T_2(c) = 1 - e^{-\Theta(F_c)} \), \( c \in \mathbb{C} \).

c) \( Z \) has no atoms and is infinitely divisible.

If these conditions are satisfied, then \( \Theta \) is translation invariant and \( \Phi \) is stationary.
Proof of Theorem 2.23. 

(a), (a') $\Rightarrow$ (b).

Let $C \in \mathcal{E}$, then

$$T_2(C) = \mathbb{P}(Z \cap C \neq \emptyset) = \mathbb{P}(Z_{\bar{\Phi}} \cap C \neq \emptyset) = \mathbb{P}(\bar{\Phi}(F_C) > 0) = 1 - \mathbb{P}(\bar{\Phi}(F_C) = 0) = 1 - e^{-\Theta(F_C)},$$

where $\Theta = \mathbb{E}\bar{\Phi}$.

$\Theta(F_C) < \infty$, i.e., this is an overall assumption on a point process $\bar{\Phi}$.

(b) $\Rightarrow$ (a), (a'). There is a Poisson process $\bar{\Phi}$ in $\mathcal{F}$ such that $\Theta = \mathbb{E}\bar{\Phi}$. Hence $T_{Z_{\bar{\Phi}}}(C) = 1 - e^{-\Theta(F_C)} = T_2(C)$ for $C \in \mathcal{E}$, i.e., $Z_{\bar{\Phi}} \sim Z$.

(b) $\Rightarrow$ (c). Suppose (b) holds. For $m \in \mathbb{N}$, we define $-\Theta^{(m)}_{\bar{\Phi}} := 1 - (1 - T_2)^{-1}$

and therefore

$$T_{Z_{\bar{\Phi}^{(m)}}}(C) = 1 - e^{\Theta(F_C)/m} = 1 - e^{-\Theta_m(F_C)},$$

where $\Theta_m := \frac{1}{m} \Theta$. There is a Poisson process $\bar{\Phi}_{\bar{\Phi}^{(m)}}$ such that $\mathbb{E}\bar{\Phi}_{\bar{\Phi}^{(m)}} = \Theta_m$. Hence, $Z_{\bar{\Phi}^{(m)}}$ has the capacity functional $T^{(m)}$ by (a1) $\Rightarrow$ (b1). Lemma 1.24 yields that $Z$ is infinitely divisible. From

$$\mathbb{P}(x \in Z) = T_2(\{x\}) = 1 - e^{-\Theta(F_{\{x\}})} < 1,$$

as $\Theta$ is locally finite ($x \in \bar{\Omega}$), it follows that $Z$ has no fixed points.

(a1) $\Rightarrow$ (b1). This is implied by Theorem 1.25. \qed
**Theorem 2.30:**

**Proof.** (a) $\Rightarrow$ (b) follows from Thm. 2.29 (c) $\Rightarrow$ (b)

and the fact that $\Theta = E \Phi$, $\Phi$ is a Poisson process and

**Remark:** A variant of Thm. 2.30 can be stated without the assumption that $\mathbf{Z} \neq \mathbf{R}^n$ $\mathbb{P}$-a.s. Then, of course, $\Phi$ need not be simple and $\Theta$ may have atoms by Sec. 7.1.
3 Geometric Models

We consider geometric point processes, i.e. point processes in \( \mathbb{F} = F(\mathbb{R}^n) \). Atoms of (random) counting measures are therefore closed sets in \( \mathbb{R}^n \). In particular, we consider processes of flats and particles under the assumption of stationarity.

Plan:  
3.1 Processes of flats, sections with a fixed plane
- Section process, intensities and direction distribution of the resulting processes.

3.2 Particle processes, intensity and shape distribution, functional densities

3.3 Connection between particle processes and marked point processes, germ-grain models, Boolean models

For simplicity, we consider multiple point processes and assume that all intensity measures are locally finite.

3.1 Processes of flats

Let \( k \in \{1, \ldots, n-1\} \). A process of flats \( \mathcal{F} \) in \( \mathbb{R}^n \) is a point process in \( \mathbb{F}' = F(\mathbb{R}^n) \) with \( \Theta(F' \setminus E_k) = 0 \), where

\[ \Theta = \sum_{\mathcal{F}} \quad \text{and} \quad E_k \text{ is the set of } k \text{-flats in } \mathbb{R}^n. \]
We need the following spaces:
\[ L^k = \{ L \in \mathbb{R}^n : L \text{ is a } k\text{-dim. linear subspace} \} , \]
(Gassmann's of \( k\)-dim. linear subspace)
\[ E^k = \{ L + x \in \mathbb{R}^n : L \in L^k, x \in \mathbb{R}^n \} . \]
There is an endowed with the subspace topology from \( F' \).
Moreover, the topology on \( E^k \) is the largest topology so that
\[ \Phi_k : G_n \to E^k, \quad \gamma \mapsto \gamma L_0, \]
is continuous; here \( G_n \) is the motion group and \( L_0 \in \mathcal{E} \) is fixed (and arbitrary). Similarly, the topology on \( L^k \) is
the largest topology so that
\[ \Phi_k : O(n) \to L^k, \quad S \mapsto S L_0, \]
is continuous, where \( O(n) \) is the rotation group and \( L_0 \in L^k \)
is fixed, but arbitrary.
Then \( E^k \cup \{ 0 \} \) is a compact subset of \( F \), \( E^k \) is locally
compact, \( L^k \subseteq E^k \) and \( L^k \) is a compact space.
We define
\[ \tau : \bigcup_{k=0}^n E^k \to \bigcup_{k=0}^n L^k, \quad E \mapsto E + t \]
where \( t \in \mathbb{R}^n \) is such that \( 0 \in \mathbb{R}^+ t \).

The following theorem exhibits the structure of translation invariant measures on \( \mathbb{E}^n \).

**Theorem 3.1** Let \( \Theta \) be a locally finite, translation invariant measure on \( \mathbb{E}^n \). Then there is a unique finite measure \( \Theta_0 \) on \( \mathbb{E}^n \) such that

\[
\Theta(A) = \int_{\mathbb{E}^n} \int_{\mathbb{R}^n} \mathbf{1}_{\{L + x \in A\}} \lambda_n(dx) \, \Theta(dL),
\]

for all measurable \( A \subset \mathbb{E}^n \).

**Proof (sketch)** For \( A \subset \mathcal{E}^n \), let

\[ L_u := \{ L \in \mathbb{E}^n : L \cap U = \{0\} \}, \]

\[ \mathcal{E}_u := \{ L + x : L \in \mathcal{E}_u, x \in U \} \text{ translation invariant set}. \]

The map \( \varphi : L_u \times U \to \mathcal{E}_u, \ (L,x) \mapsto L + x \), is a homeomorphism. Consider, for \( A \subset \mathcal{E}_u \) measurable,

\[
\Theta(A) = \Theta(\varphi(A \times U)), \quad A \subset \mathcal{E}_u \text{ measurable}.
\]

Then \( \Theta_0 \) is a locally finite, translation invariant measure on \( \mathcal{B}(U) \). Therefore, a well-known characteristic theorem of Lebesgue measure shows that
\[ \Theta \left( \psi \left( A \times B \right) \right) = \gamma _{\text{A}} \left( B \right) = \nu _{\text{A}} \left( A \right) \cdot \lambda _{\text{A}} \left( B \right), \]

then \( \lambda _{\text{A}} \) is Lebesgue measure on \( \text{A} \) and \( \nu _{\text{A}} \left( A \right) \) is a constant depending on \( \text{A} \). Since \( \psi \) is a homeomorphism and \( \nu _{\text{A}} \left( \cdot \right) \) is a measure on \( \mathcal{A}_\lambda \) and finite, it follows that

\[ \int _{\mathcal{A}_\lambda } f \, d\Theta = \int _{\mathcal{A}_\lambda } \int _{\mathcal{A}_\lambda } f \left( L \chi \right) \lambda _{\text{A}} \left( \, dx \right) \nu _{\text{A}} \left( d\lambda \right). \]

Let \( \pi _{\lambda} : \mathcal{A} \to L^1 \) denote the (linear) orthogonal projection from \( \mathcal{A} \) onto \( L^1 \), hence \( \mathcal{A} \cap L = \{ 0 \} \), this map is a bijection. The transformation formula shows that

\[ \int _{\mathcal{A}} g \circ \pi _{\lambda} \left( u \right) f \circ \pi _{\lambda} \left( u \right) \lambda _{\text{A}} \left( du \right) = \int _{L} g \left( \chi \right) \lambda _{L} \left( d\chi \right), \]

where

\[ \int _{L} f \left( \chi \right) \lambda _{L} \left( d\chi \right) = \left| \left< L^1, \text{A} \right> \right| = \left| \det \left( \left< a_i, e_j \right> : j=1 \right) \right|, \]

\[ a_1, \ldots, a_m \quad \text{is an orthonormal basis of } L^1, \]

\[ e_1, \ldots, e_n \quad \text{is an orthonormal basis of } \mathcal{A}. \]

Observe that \( \pi _{\lambda} \left( u \right) = u + l \left( u \right), \quad l \left( u \right) \in L \). Hence

\[ \int _{\mathcal{A}} f \left( L \chi \right) \lambda _{\text{A}} \left( du \right) = \int _{L} f \left( L \chi \right) \lambda _{L} \left( d\chi \right) \cdot \left| \left< L^1, \text{A} \right> \right|^{-1}. \]
This finally yields that

\[ \int_{E_k} \int d\theta = \int \int \int (L \times \theta) \Lambda_{L^u} (d\omega) \Theta_u (dL), \]

where \( \Theta_u = \langle L^u, u \rangle^{-1} \cdot v_u. \)

For the general case, one can direct \( E_k \) by using finitely many subspaces \( U_1, \ldots, U_n \in L^u_k. \)

For \( A \in \mathcal{B}(E_u^k) \), we put

\[ \Theta (F_u^k \cap \omega^{-1} (A)) = \kappa_n \cdot \Theta_u (A) \]

which shows that \( \Theta_u \) is uniquely determined. \( \square \)

We now discuss various consequences for stationary processes of \( k \)-flats. Let \( \Phi \) be a stationary process of \( k \)-flats in \( \mathbb{R} \) with intensity measure \( \Theta \neq 0 \). Then there exist \( r \in (0, \infty) \) and a probability measure \( P_0 \) on \( L^u_k \) such that

\[ \Theta (A) = r \int \int \int \{ L \times \in \mathcal{A} \} \Lambda_{L^u} (d\omega) P_0 (dL), \]

\( A \in \mathcal{B}(E_u^k). \) We call \( r \) the intensity and \( P_0 \) the direction distribution of \( \Phi. \)
Remarks:

1. If \( X \) is isotropic, then \( \Theta \) is rotation invariant and therefore also \( R_0 \). But then \( R_0 = v_0 \), the unique rotation invariant probability measure on \( \mathbb{R}_+^n \). \( \rightarrow \) Exercises

If \( \Theta = 0 \), then \( \gamma = 0 \)

2. In general, we have

\[
\gamma R_0 = \frac{1}{k_{n-1}} \mathbb{E} \left[ \Phi \left( F_{B^n} \cap \Pi_0^{-1}(\cdot) \right) \right],
\]

\[
\gamma = \frac{1}{k_{n-1}} \mathbb{E} \left[ \Phi \left( F_{B^n} \right) \right],
\]

\[
R_0 = \frac{\mathbb{E} \left[ \Phi \left( F_{B^n} \cap \Pi_0^{-1}(\cdot) \right) \right]}{\mathbb{E} \left[ \Phi \left( F_{B^n} \right) \right]}.
\]

The preceding remarks provide some intuitive interpretation of the intensity and the direction distribution of processes of flats. Another interpretation follows from

Theorem 3.2 Let \( \Phi \) be a stationary process of \( k \)-flats in \( \mathbb{R}^n \). Then

\[
\mathbb{E} \left[ \sum_{E \in \Phi} \delta_E(\cdot) \right] = \gamma \cdot \delta^n.
\]
Proof. Let $(\Omega, \mathcal{A}, \mathbb{P})$ denote the underlying probability space. If $A \in \mathcal{A}^n$ is compact, then

$$E \rightarrow \int_{E} \mathbb{P}(A) \ dE \in \mathbb{R}^n$$

is measurable, since it is upper semi-continuous.

For arbitrary $A \in \mathcal{B}(\mathbb{R}^n)$ measurability then follows from a Dynkin argument. (Exercise)

Now Campbell's theorem yields that

$$E : [0, \infty) \rightarrow \sum_{E \in \Phi} \mathbb{P}(A)$$

is measurable. Therefore we obtain that

$$\mathbb{E} \left[ \sum_{E \in \Phi} \mathbb{P}(A) \right] = \int_{\mathbb{R}^n} \mathbb{P}(A) \Theta \left( dE \right)$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^+} \mathbb{P}(A) \lambda_{\mathbb{R}^n} \left( dx \right) \mathbb{P} \left( dE \right)$$

$$= \int_{\mathbb{R}^n} \lambda_{\mathbb{R}^n} \left( A \right) \mathbb{P} \left( dE \right)$$

$$= \mathbb{E} \left[ \lambda_{\mathbb{R}^n} \left( A \right) \right]$$

by Fubini's theorem.

Corollary 3.3. Let $\gamma \in (0, \infty)$ and $\mathbb{P}_0$ be a probability measure on $\mathbb{R}^n$. Then there is a unique (up to equivalence) stationary Poisson process of $k$-flats $\mathbb{F}$ in $\mathbb{R}^n$ with intensity $\gamma$ and direction distribution $\mathbb{P}_0$. It is isotropic if $\mathbb{P}_0 = \nu_\gamma$. 
Definition. Two $k$-flats $E_1, E_2$ are said to be in general position if their direction spaces $L_1, L_2$ with $L_i = \overline{v_0(E_i)}$ satisfy:

$$\dim \{L_1 \cup L_2\} = \mathbb{R}^n \quad \text{or} \quad \dim \{L_1 \cap L_2\} = 0.$$ 

The following result is very useful. 
\textbf{Theorem 3.4.} Let $\Phi$ be a stationary Poisson process of $k$-flats in $\mathbb{R}^n$. 

(a) If $k < \frac{n}{2}$, then any two $k$-flats in $\Phi$ are disjoint a.s.

(b) $\Phi$ has no atoms $\Rightarrow$ Any two $k$-flats in $\Phi$ are not translates of each other a.s.

(c) $\Phi$ is absolutely continuous with respect to $\nu_k \Rightarrow$ Any two subspaces / $k$-flats in $\Phi$ are in general relative position a.s.

Proof. Let $A \in \mathcal{B}(E_k \times E_{k})$. Then we obtain

$$\mathbb{P}^2 = \Theta^2$$

for the Poisson process $\Phi$, and hence

$$E \left[ \sum_{(E_1, E_2) \in \Phi^2} 1 \{ (E_1, E_2) \in A \} \right]$$

is mutually different pairs of the triple point process $\Phi$. 

\[
\begin{align*}
&= \int_{\mathcal{E}_n^+} \int_{\mathcal{E}_n^-} [ \{ \mathbf{E}_1, \mathbf{E}_2 \} \in \mathcal{A} ] \, \Theta(d\mathbf{E}_1) \, \Theta(d\mathbf{E}_2) \\
&= \mathfrak{F}^2 \int_{\mathcal{L}_n^+} \int_{\mathcal{L}_n^-} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} [ \{ \mathbf{L}_1 + \mathbf{x}_1, \mathbf{L}_2 + \mathbf{x}_2 \} \in \mathcal{A} ] \\
&\quad \lambda_{\mathcal{L}_1} (d\mathbf{x}_1) \, \lambda_{\mathcal{L}_2} (d\mathbf{x}_2) \, \mathcal{P}_0 (d\mathbf{L}_1) \, \mathcal{P}_0 (d\mathbf{L}_2).
\end{align*}
\]

a) \[ \mathcal{A} := \{ \{ \mathbf{E}_1, \mathbf{E}_2 \} \in (\mathcal{E}_n^+)^2 : \mathbf{E}_1 \cap \mathbf{E}_2 \neq \emptyset \}. \]

Thus,
\[
\int_{\mathcal{L}_1} [ \{ \mathbf{L}_1 + \mathbf{x}_1, \mathbf{E}_2 \} \in \mathcal{A} ] \, \lambda_{\mathcal{L}_1} (d\mathbf{x}_1) = 0
\]

since \( \{ \mathbf{L}_1 + \mathbf{x}_1 \} \cap \mathbf{E}_2 \neq \emptyset \Rightarrow x_1 \in \mathbf{E}_2 \cap \mathcal{L}_1 \)

and \( \mathfrak{F} = \min(\mathcal{L}_1, \mathcal{L}_2) \leq \mathfrak{a} \leq \mathcal{L}_1 - \mathfrak{a} \). Then the estimate follows.

b) \[ \mathcal{A} := \{ \{ \mathbf{E}_1, \mathbf{E}_2 \} \in (\mathcal{E}_n^+)^2 : \mathbf{E}_1 \text{ is a translate of } \mathbf{E}_2 \}. \]

\[ \Rightarrow \mathcal{P}_0 \left( \{ \mathbf{L}_1 \in \mathcal{L}_n^- : \mathbf{L}_1 = \mathbf{L}_2 \} \right) = \mathcal{P}_0 (\mathcal{L}_1^+) = 0. \]

c) \[ \mathcal{A} := \{ \{ \mathbf{E}_1, \mathbf{E}_2 \} \in (\mathcal{E}_n^+)^2 : \mathbf{E}_1, \mathbf{E}_2 \text{ are not in general } \]

\[ \text{relative position} \} \]

\[ \mathcal{A}(\mathbf{L}_2) := \{ \mathbf{L}_1 \in \mathcal{L}_n^+ : \{ \mathbf{L}_1, \mathbf{L}_2 \} \in \mathcal{A} \} \Rightarrow \mathfrak{f}_a (\mathcal{A}(\mathbf{L}_2)) = 0. \]

For the last implication we use \( \mathfrak{f} \) 13.2.1 in Schneider & \[ \square \]
Semi-process with a fixed flat:

Let \( \Phi \) be a stationary process of \( k \)-flats in \( \mathbb{R}^n \) with \( k \in \{1, \ldots, n-1\} \) and \( \mathcal{E} \in \mathbb{L}_{n-k+1} \), \( j \in \{0, \ldots, k-1\} \). Intersecting the flats of \( \Phi \) with \( S \) should generically yield a process of \( j \)-flats in \( S \).

**Definition:**

\[
\Phi \cap S = \bigcup_{E \in \Phi, E \cap S \neq \emptyset} S_{E \cap S}
\]

If \( \Phi = \bigcup_i S_{E_i} \), then \( \Phi \cap S = \bigcup_{E \cap S \neq \emptyset} S_{E \cap S} \)

If \( E \in \Phi \) and \( S \) are in general relative position, then \( \dim(E \cap S) = j \). It can be shown that if \( \Phi \) is a set of \( j \)-flats in \( \mathbb{R}^n \) and \( S \) are in general relative position, then \( \Phi \) also \( \Phi \cap S \) is simple.

Therefore consider \( \Phi \cap S \) as a stationary process of \( j \)-flats in \( S \).

Recall: \( \mathbf{L}^n_+ \subseteq \mathbb{L}^n_+ \):

\[
|<S, L^+>| = |<L, S^+>| = \det(S_L) = [S_L] = [L_S].
\]
Theorem 3.5  Let \( k \in \{1, \ldots, n-1\} \) and let \( \Phi \) be a stationary process of \( k \)-flats in \( \mathbb{R}^n \) with intensity \( \varrho \) and direction distribution \( P_0 \). Let \( S \in 2^{\mathbb{R}^n} \) and let \( \Phi \cap S \) denote the intensity of the point process \( \Phi \cap S \). Then

\[
\gamma \Phi \cap S = \gamma \int_{\mathbb{R}^n} \mathbf{1}_{\langle S, L \rangle > 1} P_0 (dL).
\]

Proof. Put \( \mathcal{B}^{n-k} := S \cap \mathcal{B}^n \). Then we get for the point process \( \Phi \cap S \) in \( S \):

\[
\kappa_{n-k} \gamma \Phi \cap S = \mathbb{E} \left[ (\Phi \cap S)(\mathcal{B}^{n-k}) \right] = \mathbb{E} \left[ (\Phi \cap S)(\mathcal{F}_{\mathcal{B}^{n-k}}) \right]
\]

"Points in \( S \)"

\[
= \mathbb{E} \left[ \Phi(\mathcal{F}_{\mathcal{B}^{n-k}}) \right] = \Theta (\mathcal{F}_{\mathcal{B}^{n-k}})
\]

"See pointed sets in \( S \)"

\[
= \gamma \int_{\mathbb{R}^n} \int_{\mathcal{L}^n} \mathbf{1}_{\left\{ L+X \in \mathcal{F}_{\mathcal{B}^{n-k}} \right\}} \lambda_L (dL) P_0 (dX)
\]

\[
= \gamma \int_{\mathbb{R}^n} \lambda_L \left( \mathcal{B}^{n-k} \setminus L^\perp \right) P_0 (dL)
\]

\[
= \gamma \int_{\mathbb{R}^n} \kappa_{n-k} \mathbf{1}_{\langle S, L^\perp \rangle > 1} P_0 (dL).
\]
A special case: \( k = 1 \)  

process of lines, \( \mu \in S^{n-1} \),

\[
S = \mu^\perp, \quad \phi \mu^\perp = \lambda \int_{S^{n-1}} |< \mu, L^\perp | \, P_0 \, (dL) \\
= \lambda \int_{S^{n-1}} |< \mu, L | \, P_0 \, (dL) \\
= \lambda \int_{S^{n-1}} |< \mu, \nu | \, \bar{P} \, (d\nu)
\]

where \( \bar{P} (A) = \frac{1}{2} \cdot P_0 (\{ R\mu : \mu \in A \}) \) for \( A \in \mathcal{B}(S) \)

such that \( A \cap (-A) = \emptyset \) and \( \bar{P} \) is an even probability measure on \( S^{n-1} \).

It is known that the even probability measure \( \bar{P} \) is uniquely determined by the collection \( \{ \phi \mu^\perp : \mu \in S^{n-1} \} \).

This follows from the uniqueness of the cosine transform of even measures.

Another special case: \( k = n-1 \)  

process of hyperplanes

Let again \( \mu \in S^{n-1} \) be fixed. Then

\[
\phi \mu^\perp = \lambda \int_{S^{n-1}} |< \mu, L^\perp | \, P_0 \, (dL) \\
= \lambda \int_{S^{n-1}} |< \mu, \nu | \, \bar{P} \, (d\nu)
\]
\[ \mathbf{P} = \mathbf{y} \cdot \int_{S^{n-1}} |\mathbf{u}, \mathbf{v}| \, \tilde{\mathbf{P}}(d\mathbf{v}), \]

where \( \tilde{\mathbf{P}}(A) = \frac{1}{2} \mathbb{P}_0 (\{ \mathbf{v} \in A \}) \), and \( A \) is as above and \( \tilde{\mathbf{P}} \) is an even probability measure on \( S^{n-1} \).

Again \( \mathbf{y}, \tilde{\mathbf{P}} \) and therefore \( \mathbf{P} \) and \( \tilde{\mathbf{P}} \) are uniquely determined by \( \mathbf{E} \phi \mathbf{f}_0 [\ln] \), \( \mathbf{u} \in S^{n-1} \).

This is a surprising feature of line processes and hyperplane processes!

Observe that for a Poisson process \( \mathbf{P} \) the characteristic functions \( \phi_{\mathbf{f}_0} \mathbf{f}_0 [\ln] , \mathbf{S} \in L_{\mathbf{e}^n} , \) resp. \( \mathbf{S} \in L_{\mathbf{e}_1} , \) already determine \( \mathbf{P} \) completely. This is no longer true for \( \mathbf{S} \in L_{\mathbf{e}_j} \).

The preceding analysis can be extended to cover the case \( j \neq 0 \) and to include the characteristic distribution of \( \phi_{\mathbf{f}_0} \mathbf{f}_0 [\ln] , \mathbf{S} \in L_{\mathbf{e}^n} . \)

**Definition:** For \( \mathbf{L} \in L_{\mathbf{e}_1} , \mathbf{S} \in L_{\mathbf{e}^n} , \) \( j \in \{1, \ldots, n-1\} \),

\[
[\mathbf{L}, \mathbf{S}] := \begin{cases} 
0 , & \text{if } \det (\mathbf{L} \mathbf{S}) > 0 , \\
\frac{1}{2} \det \left( \mathbf{u}_1 , \ldots , \mathbf{u}_j ; \mathbf{u}_{j+1} , \ldots , \mathbf{u}_{n-1} \right) , & \text{otherwise,} \\
\int_{S^{n-1}} |\mathbf{u}| \, \tilde{\mathbf{P}}(d\mathbf{u}) , & \text{if } \det (\mathbf{L} \mathbf{S}) < 0 .
\end{cases}
\]

The def. is independent of the special choice of \( \mathbf{O} \mathbf{W} \mathbf{B} . \)
Theorem 3.6. Let \( k \in \{2, \ldots, n-1\} \), let \( \Phi \) be a stationary process of \( k \)-flats in \( \mathbb{R}^n \) with intensity \( \gamma \) and direction distribution \( \mathcal{P}_0 \). Let \( \mathcal{F} \in \{1, \ldots, k-1\} \) and \( S \in \mathcal{C}_{k+n-k}^n \). Then, for \( A \in \mathcal{B}(L^z_j) \),

\[
\mathbb{P}_{\Phi, 0} \left[ \mathbb{P}_{\Phi} \left[ A \right] \right] = \gamma \int \left[ \left\{ L \in S \in A \right\} \right] L \mathcal{P}_0(dL)
\]

where \( \mathbb{P}_{\Phi, 0} := \mathbb{0} \) if \( \Phi \mathcal{P}_{\Phi} = \mathbb{0} \).

**Proof.** Define \( \mathcal{B}_S := \mathcal{B}^n \cap S \). Then

\[
\mathbb{P}_{\Phi, 0} \left[ \mathbb{P}_{\Phi} \left[ A \right] \right] = \frac{1}{k_{n-k}} \cdot \mathcal{E} \left[ \sum_{E \in \Phi} \frac{1}{k_{n-k}} \mathbb{P}_{\mathcal{B}_S} \left( \mathbb{P}_{\mathcal{B}} \left( (E \mathcal{S}) \right) \right) \right]
\]

where

\[
= \frac{1}{k_{n-k}} \cdot \mathcal{E} \left[ \sum_{E \in \Phi} \frac{1}{k_{n-k}} \mathbb{P}_{\mathcal{B}_S} \left( \mathbb{P}_{\mathcal{B}} \left( (E \mathcal{S}) \right) \right) \right]
\]

\[
= \frac{1}{k_{n-k}} \cdot \int_{E \in \Phi} \frac{1}{k_{n-k}} \mathbb{P}_{\mathcal{B}_S} \left( \mathbb{P}_{\mathcal{B}} \left( (E \mathcal{S}) \right) \right) \Theta(dE)
\]

\[
= \frac{1}{k_{n-k}} \cdot \int L \mathbb{P}_{\mathcal{B}_S} \left( \mathbb{P}_{\mathcal{B}} \left( (L + \mathcal{S}) \right) \right) \mathcal{P}_0(dL).
\]
The characteristic function is \( 1 \) if and only if \( \ln S \in A \) and \( x \in B_S \setminus L^1 \).

Hence

\[
\Phi_{\ln S} P_0, \Phi_{\ln S} (A) = \frac{1}{k_{n-k}} \int_{L^1} 1_{\{\ln S \in A\}} d_{n-k} (B_S \setminus L^1) P_0 (dL) 
\]

\[
= \frac{1}{k_{n-k}} \int_{L^1} 1_{\{\ln S \in A\}} [L, S] P_0 (dL).
\]

Observe that with \( T = (L \cap S)^\perp \cap S \), we have (if \( L \cap S \neq 0 \))

\[
B_S \setminus L^1 = (B_S \setminus T) \setminus L^1 = B_T \setminus L^1,
\]

and thus

\[
d_{n-k} (B_S \setminus L^1) = k_{n-k} |< T, L^1 >| = k_{n-k} [L, S].
\]

So far we considered a process of flats and its intersection with a fixed flat. Next we intersect any \( k \) hyperplanes of a hyperplane process with each other so as to obtain a process of \( n-k \) flats.
Description of hyperplanes

\( H(u, \tau) = \{ x \in \mathbb{R}^n : \langle x, u \rangle = \tau \} \), \( u \in S^{n-1}, \tau \in \mathbb{R} \)

\( H(u, 0) = u^\perp = H(u) \).

Let \( \overline{\Phi} \) be a hyperplane process with intensity \( \gamma \) and direction distribution \( \overline{P} \) resp. \( P \) for the intensity measure \( \Theta = \overline{E} \). We thus get:

\[
\int_{E_{n-1}} ^{n} \phi \ d\Theta = \int_{E_{n-1}} ^{n} \int_{L^+} ^{\infty} \phi(L+x) \lambda_n(dx) \overline{P}(dL)
\]

\[
= \frac{1}{\gamma} \int_{E_{n-1}} ^{n} \int_{\mathbb{R}} ^{\infty} \phi(H(u, \tau)) \lambda_n(dx) \overline{P}(d\tau).
\]

The section process of order \( \lambda \) of \( \overline{\Phi} \), denoted by \( \overline{\Phi}^\lambda \), is defined by:

\[
\overline{\Phi}^\lambda (\cdot) := \frac{1}{\lambda^n} \sum_{(H_n, H_k)} \mathbb{1}_{\{H_n \neq H_k \}} \mathbb{1}_{\{H_n \cap H_k \in E_{n-k}\}}
\]

It is clear that \( \overline{\Phi}^\lambda \) is measurable and locally finite, hence a point process. If \( \Phi \) is stationary, then \( \overline{\Phi}^\lambda \) is stationary. In general, \( \overline{\Phi}^\lambda \) need not be simple. If \( \Phi \) is Poisson, then \( \overline{\Phi}^\lambda \) is Poisson, but \( \overline{\Phi}^\lambda \) need not be Poisson.

For \( m \leq n \) and \( u_{m+1}, \ldots, u_m \in \mathbb{R}^n \), we define

\[
\Delta_m (u_{m+1}, \ldots, u_m) = |\text{det}(u_{m+1}, \ldots, u_m)|.
\]
Theorem 3.7: Let $F$ be a stationary Poisson hyperplane process in $\mathbb{R}^n$ with intensity $\gamma > 0$ and spherical direction distribution $\tilde{F}$. For $k \in \{2, \ldots, n\}$, let $F_k$ be the section process of order $k$ of $F$. Let $\Gamma$ be the intensity and $\tilde{R}_k$ the direction distribution of $F_k$. Then, for $A \in \mathcal{B}(\mathbb{R}^n)$, we get

$$\tilde{F}_k \tilde{P}_{F_k}(A) = \frac{1}{k!} \sum \int \int \int 1 \{m_1 \cap \cdots \cap m_k \in A\} \frac{1}{k!} \tilde{R}_k(m_1, \ldots, m_k) \tilde{P}(dm_1) \cdots \tilde{P}(dm_k)$$

Proof: For $E \in \mathcal{B}(L_{n-k}^n)$, we get

$$\Theta_k(E) = E \left[ \tilde{F}_k(E) \right] = \frac{1}{k!} \sum \int \int \int 1 \left\{ m_1 \cap \cdots \cap m_k \in E \right\} \tilde{R}_k(m_1, \ldots, m_k) \tilde{P}(dm_1) \cdots \tilde{P}(dm_k)$$

$$= \frac{1}{k!} \int \int \int 1 \left\{ m_1 \cap \cdots \cap m_k \in E \right\} \tilde{R}_k(d(m_1, \ldots, m_k))$$

Now we choose $A \in \mathcal{B}(L_{n-k}^n)$ and $E = \tilde{F}_k \cap \tilde{P}_{-1}(A)$. Then we get

$$\tilde{F}_k \tilde{P}_{F_k}(A) = \frac{1}{k!} \tilde{R}_k \left( \tilde{F}_k \cap \tilde{P}_{-1}(A) \right)$$

$$= \frac{1}{k!} \int \int \int 1 \left\{ m_1 \cap \cdots \cap m_k \in A \right\} \tilde{R}_k(d(m_1, \ldots, m_k))$$
Next we calculate the inner integral.

\[ I_k := \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbb{1} \left\{ \mathbb{1}(u_i, t_i) \cap \cdots \cap \mathbb{1}(u_k, t_k) \in \mathbb{R}^n \right\} dt_1 \cdots dt_k. \]

Let us first restrict to the case \( k = n \). If \( u_1, \ldots, u_n \) are linearly independent, then

\[ \left\{ T : \mathbb{R}^n \to \mathbb{R}^n, \ (t_1, \ldots, t_n) \mapsto x, \right\} \]

where \( \{x\} = \mathbb{1}(u_1, t_1) \cap \cdots \cap \mathbb{1}(u_n, t_n) \)

is bijective with inverse

\[ T^{-1}(x) = (\langle x, u_1 \rangle, \ldots, \langle x, u_n \rangle) \]

and Jacobian \( J_T^{-1}(x) = \det(u_1, \ldots, u_n) \). Hence

\[ I_n = \int_{\mathbb{R}^n} \mathbb{1} \left\{ T(t_1, \ldots, t_n) \in \mathbb{R}^n \right\} dt_1 \cdots dt_n \]

\[ = \int_{\mathbb{R}^n} \mathbb{1} \left\{ x \in \mathbb{R}^n \right\} J_T^{-1}(x) dx = \det(u_1, \ldots, u_n). \]

For \( k < n \) and \( u_1, \ldots, u_k \) an application of the preceding argument in \( \mathbb{R}^k \{u_1, \ldots, u_k\} \) shows that

\[ I_k = \det(u_1, \ldots, u_k). \]

If \( u_1, \ldots, u_k \) are linearly dependent, then \( \mathbb{1}(u_i, t_i) \cap \cdots \cap \mathbb{1}(u_k, t_k) \notin \mathbb{R}^n \).

and therefore again equality is obtained, since also \( \det(u_1, \ldots, u_k) = 0 \).

This proves the assertion. \( \square \)
3.2 Particle processes

A particle process \( \Phi \) in \( \mathbb{R}^n \) is a point process in \( \mathcal{F} = \mathcal{F}(\mathbb{R}^n) \) with \( \Theta \cap (\mathcal{F}' \setminus \mathcal{C}') = \emptyset \), where \( \Theta = \Xi \Phi \).

If \( \Theta (\mathcal{F}' \setminus \mathcal{K}) = \emptyset \), we call \( \Phi \) a particle process in \( \mathcal{K} \) or \( \Phi \cap (\mathcal{F}' \setminus \mathcal{R}) = \emptyset \), then \( \Phi \) is a particle process in \( \mathcal{R} \) (polyconvex nets).

As before, we assume that \( \Theta (\mathcal{F}') < \infty \) for \( \mathcal{C} \in \mathcal{C} \), i.e. \( \Theta \) is assumed to be locally finite.

**Process of facts:** decomposition of \( \Theta \) in translations and dilations

**Process of particles:** decomposition of \( \Theta \) in position and shape with respect to a centre function.

**Idea:**

\[ \begin{array}{c}
\text{K} \quad \rightarrow \\
\text{c(K)} \quad \rightarrow \\
\emptyset \quad \rightarrow \\
\text{k-c(K)}
\end{array} \]

By a map \( c : \mathcal{C} \rightarrow \mathbb{R}^n \) with \( c(K + x) = c(K) + x \) for \( K \in \mathcal{K} \) and \( x \in \mathbb{R}^n \), is called a centre function.

**Example:** For \( K \in \mathcal{K} \), let \( c(K) \) be the centre of the smallest ball which contains \( K \) (it is unique!)

In the following, we shall use this particular centre map.

**Lemma 3.8:** The map \( c \) is continuous.

From the example.
Set \[ C_0 = \{ C \in C : c(C) = 0 \}. \]

"Shape space".

Then \( C_0 \) is closed in \( C \), \( C_0 \subset \mathcal{D}(C) \approx B(C) \subset C R(C) \).

Similarly, we can define \( K', X', R, R_0 \).

**Lemma 3.9** The map

\[ \varphi : \mathbb{R}^n \times C_0 \to C', \quad (x, C) \mapsto x + C, \]

is a homeomorphism with inverse \( \varphi^{-1} : C' \to (C(C), C - c(C)) \).

**Theorem 3.10** Let \( \theta \) be an ordinary particle process in \( \mathbb{R}^n \) with \( \theta \neq 0 \). Then there is a \( \gamma \in (C, 0) \) and a probability measure \( P_0 \) on \( C_0 \) such that

\[ \theta = \gamma \star \varphi \left( \mathbb{R}^n \otimes P_0 \right), \]

where \( \gamma, P_0 \) are uniquely determined by this equation.

For \( \gamma : C' \to [0, \infty) \) measurable, we then obtain

\[ \int_B d\Theta = \gamma^{-1} \int_B (C(x, x) B_x d(x)) P_0 (dx) \]

**Proof.** We define \( \Theta = \varphi^{-1} (\theta) \) on \( \mathbb{R}^n \times C_0 \).

For \( B \in \mathcal{B}(\mathbb{R}^n) \) and \( A \in \mathcal{B}(C_0) \), we get \( (y \in \mathbb{R}^n) \)

\[ \Theta (B + y) \cdot A = \Theta (\varphi (B + y) \cdot A) = \Theta (\varphi (B \cdot A) + y) \]

\[ = \Theta (\varphi (B \cdot A) + y) = \Theta (B \cdot A), \]

since \( \varphi \) is bijective and therefore \( \Theta \) is translation invariant.
\[
C^n := \{0, 1\}^n, \quad \Theta^+ := \{x \in \mathbb{R}^n : \max x_i = 1\} \quad \text{and} \quad C_0 = C^n \backslash \Theta^+
\]

Let \( \mathbb{Z}^n := \{z_i \in \mathbb{Z}^n \} \). Then

\[
\Theta (C^n \times \mathbb{Z}^n) = \Theta (\{ \mathbf{c} \in \mathbb{R}^n : \text{c}(C) \in C_0^\circ \}) \\
\leq \sum_{\mathbf{c} \in \mathbb{R}^n} \Theta (\{ \mathbf{c} \in \mathbb{R}^n : \text{c}(C) \in C^n \times \{z_i\} \}) \\
= \sum_{z_i \in \mathbb{Z}^n} \Theta (\{ \mathbf{c} \in \mathbb{R}^n : \text{c}(C) \in C^n \times \{z_i\} \}) \\
= \Theta (\{ \mathbf{c} \in \mathbb{R}^n : \text{c}(C) \in C_0^{\circ} \}) \\
\leq \Theta (\mathbb{R}^n) < \infty
\]

We conclude that \( \Theta = \Theta^0 \circ (\mathbf{g}) = \rho : \lambda \mathbb{R} \rightarrow \mathbb{R} \) by a similar reasoning as before.

In the statement of Theorem 3.10, we call \( \mathbf{g} \) the intensity and \( P_0 \) the shape distribution of the stationary particle process \( \Phi \).

Remarks:
1) \( \Phi \) isomorphic \( \Rightarrow P_0 \) is rotation invariant.
2) Marked particle processes on \( \mathbb{R}^n \times C_0 \).
3) If \( \Phi \) is a stationary particle process with intensity \( \mathbf{g} \) and shape distribution \( P_0 \), then for \( K \in C_0 \), where
\[ \Theta (F_K) = \int \int \int_{F_K^n} \mathbb{1}_{\{C+x \in F_K\}} \mathbb{P}_o(\text{d}x) \mathbb{P}_o(\text{d}x) \]

\[ = \int \int_{F_K^n} V_n (K+C') \mathbb{P}_o(\text{d}C) , \]

where \( C' = \{-x, x \in C\} \). Note that

\[ \Theta (F_K) < \infty \forall K \in \mathbb{C} \quad \Rightarrow \quad \int_{F_K^n} V_n (C+rB^n) \mathbb{P}_o(\text{d}C) < \infty \]

for some \( r > 0 \).

4. If \( C \in \mathbb{C}^n \), then

\[ V_n (C+B^n) = \sum_{i=0}^{\infty} r^{-i} \lambda_{n-i} V_i (C) . \]

(Steele's formula)

**Corollary 3.11** Let \( \mathbb{C} (0, \infty) \) and \( P_o \) a probability measure on \( \mathbb{C} \), which satisfies (\( \ast \)). Then (up to equivalence) there is a unique stationary Poisson process \( \Phi \) in \( \mathbb{C} \) with intensity \( \lambda \) and shape distribution \( P_o \). Furthermore, \( \Phi \) is isotropic if and only if \( P_o \) is rotation invariant.

We use the following notation:

\[ E_o(A) = \left\{ C \in \mathbb{C} : c(C) \in A \right\} = \Phi (A \times E_o) , \quad A \in \mathbb{S}(\mathbb{R}) \]

\[ \pi_o : E^1 \rightarrow E_o , \quad C \mapsto C - c(C) . \]

The following result provides some intuitive description of intensity and distribution distribution of following particle process.
Intensity and shape distribution of a stationary particle process in $\mathbb{R}^n$ can be expressed via mean values:

**Theorem 3.12** Let $\xi$ be a stationary particle process in $\mathbb{R}^n$. Then, for $A \in \mathcal{B}(\mathbb{R}^n)$, we get

(a) $\gamma P_0(A) = \frac{1}{\lambda_n} E\left[ \Phi \left( \mathbb{E} \left( B^n \cap \pi^{-1} c(A) \right) \right) \right].$

(b) $\gamma P_0(A) = \lim_{n \to \infty} \frac{1}{\lambda_n(\pi K)} E\left[ \Phi \left( \mathbb{E}^{\pi K} \cap \pi^{-1} c(A) \right) \right].$

(c) $\gamma P_0(A) = \lim_{n \to \infty} \frac{1}{\lambda_n(\pi K)} E\left[ \Phi \left( \mathbb{E}^{\pi K} \cap \pi^{-1} c(A) \right) \right].$

**Proof** (a) Clearly,

$E\left[ \Phi \left( \mathbb{E} \left( B^n \cap \pi^{-1} c(A) \right) \right) \right] = \Theta \left( \mathbb{E} \left( B^n \cap \pi^{-1} c(A) \right) \right) = \Theta \left( \Phi \left( B^n \times A \right) \right) = \gamma \cdot \lambda_n(B^n) \cdot P_0(A) = \gamma \cdot \lambda_n \cdot P_0(A).$

(b), (c) This requires the estimate of boundary terms which can be shown to be negligible as $n \to \infty$. □

**Def.** Let $\xi$ be a stationary particle process with intensity $\gamma$ and shape distribution $P_0$. Let $\Phi: \mathcal{E} \to \mathcal{B}$ be translation invariant, measurable and nonnegative or $P_0$-integrable. Then we define the $\Phi$-density of $\xi$ by

$\Phi(\xi) := \gamma \cdot \int_{\mathcal{E}_0} \Phi dP_0.$
Alternative description of $\overline{\phi}$:

**Theorem 3.13** Let $\phi$ be a stationary particle process in $\mathbb{R}^n$ with shape distribution $\mathbb{P}_0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be as in the definition. Then:

a) $A \in \mathcal{B}(\mathbb{R}^n), \ 0 < \lambda_n(A) < \infty$:

$$\overline{\phi}(\overline{x}) = \frac{1}{\lambda_n(A)} \cdot \mathbb{E} \left[ \sum_{C \in \mathcal{F}, C \in A} \phi(C) \right].$$

b) $K \in \mathcal{K}^n, \ 0 < V_n(K) < \infty$:

$$\overline{\phi}(\overline{x}) = \lim_{n \to \infty} \frac{1}{V_n(\nu K)} \cdot \mathbb{E} \left[ \sum_{C \in \mathcal{F}, C \cap K \neq \emptyset} \phi(C) \right].$$

c) $K \in \mathcal{K}^n, \ 0 < V_n(K) < \infty$, $\int_{\mathbb{R}^n} |\phi(C)| V_n(C + \mathbb{R}^n) \mathbb{P}_0(dC) < \infty$:

$$\overline{\phi}(\overline{x}) = \lim_{n \to \infty} \frac{1}{V_n(\nu K)} \cdot \mathbb{E} \left[ \sum_{C \in \mathcal{F}, C \cap K \neq \emptyset} \phi(C) \right].$$

**Proof.** (a) Let $\nu$ be the intensity of $\phi$. By Campbell's theorem:

$$\mathbb{E} \left[ \sum_{C \in \mathcal{F}, C \in A} \phi(C) \right] = \mathbb{E} \sum_{C \in \mathcal{F}} \frac{1}{\lambda_n(A)} \cdot \nu \phi(C) = \mathbb{E} \sum_{C \in \mathcal{F}} \frac{1}{\lambda_n(A)} \cdot \nu \phi(C).$$

$$= \nu \cdot \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{\lambda_n(A)} \cdot \nu \phi(C) \lambda_n(d\omega) \mathbb{P}_0(dC).$$

$$= \nu \cdot \lambda_n(A) \int_{\mathbb{R}^n} \phi(C) \mathbb{P}_0(dC) = \lambda_n(A) \cdot \overline{\phi}(\overline{x}).$$
For a stationary particle process \( \Phi \), let

\[
\Phi^c := \sum_{C \in \Phi} \delta_C(C)
\]

which is a point process in \( \mathbb{R}^n \) with locally finite intensity measure. In particular, \( \Phi \) and \( \Phi^c \) have the same intensity. One could therefore try to obtain \( \Phi \) from \( \Phi^c \) by independent marking with random compact sets having distribution \( \Theta_0 \). For Poisson processes this can be done as follows.

**Theorem 3.14.** Let \( \Phi \) be a stationary Poisson particle process in \( \mathbb{R}^n \) with intensity \( \lambda \) and shape distribution \( \Theta_0 \). Then \( \Phi^c \) is a stationary Poisson process with intensity \( \lambda \) and the following holds: For each \( C \in \Phi \) with \( \lambda_\nu(C) > 0 \) and each \( k \in \mathbb{N} \), there are random points \( \xi_1, \ldots, \xi_k \in C \) with distribution \( (\lambda_\nu(C))/\lambda_\nu(C) \) and there are random closed sets \( Z_1, \ldots, Z_k \in \mathcal{C} \) with distribution \( \Theta_0 \) so that \( \xi_1, \ldots, \xi_k, Z_1, \ldots, Z_k \) are stochastically independent and

\[
\mathbb{P}( \Phi \subseteq C \cap \Theta_0(C) = k \mid \Phi \cap \Theta_0(C) = k ) = \mathbb{P}( \sum_{i=1}^k s_i C_i = \cdot )
\]

**Proof.** For \( A \in \Phi^c \), \( \mathbb{P}( \Phi^c(A) = k ) = \mathbb{P}( \Phi( \Theta_0(A)) = k ) \), hence \( \Phi^c \) is a Poisson process and stationary with intensity measure

\[
\Theta_0(A) = \Theta( \Theta_0(A)) = k \cdot \lambda_\nu(A).
\]

If \( C \in \mathcal{C} \) with \( \lambda_\nu(C) > 0 \), then \( 0 < \Theta( \Theta_0(A) ) < \infty \) and
There are stochastically independent random closed sets $\bar{z}_1, \ldots, \bar{z}_k$ with

$$\mathbb{P}(\Phi(L_\mathcal{C}(C)) \in \cdot \mid \Phi(L_\mathcal{C}(C)) = k) = \mathbb{P}(\sum_{i=1}^{k} \delta_{\bar{z}_i} \in \cdot)$$

where

$$\mathbb{P}(\bar{z}_i \in \cdot) = \frac{\Theta \left( L_\mathcal{C}(C) \right)}{\Theta \left( C_\mathcal{C}(C) \right)}.$$

Thus

$$\Theta \left( L_\mathcal{C}(C) \right) = \Phi \left( (\vartriangle L_\mathcal{C}) \otimes \gamma \cdot \mathcal{P}_0 \right)$$

and

$$\Theta \left( C_\mathcal{C}(C) \right) = \gamma \cdot \lambda_n (C)$$

it follows that

$$\mathbb{P}(\bar{z}_i \in \cdot) = \Phi \left( \frac{\vartriangle L_\mathcal{C}}{\lambda_n (C)} \otimes \mathcal{P}_0 \right).$$

We define $(\xi_i, \bar{z}_i) = \Phi^{-1} \circ \bar{z}_i$ and thus obtain a sequence of independent random elements having the required properties. \(\Box\)
The next theorem describes the connection between a stationary particle process, an associated marked point process and the corresponding unmarked point process.

**Theorem 3.15** Let $\Phi$ be a stationary particle process in $\mathbb{R}^n$ and let $c$ be a center function. Then

$$\Phi^c := \sum_{C \in \Phi} \delta_{c(C)}$$

is a stationary point process in $\mathbb{R}^n$, and

$$\Phi_c := \sum_{C \in \Phi} \delta_{c(C), C-c(C)}$$

is a stationary marked point process with mark space $\mathbb{R}$.

The intensities of all three processes are the same. The mark distribution of $X_c$ is the union of the shape distribution $\mathbb{P}_0$ of $\Phi$ under the map $C \mapsto C - c(C)$.

**Proof** Measurability of $\Phi_c$ is easy to see. Moreover,

$$\mathbb{E}\left[ \Phi_c \left( C_0 \times \mathbb{R} \right) \right] = \mathbb{E}\left[ \text{card} \left\{ C \in \Phi : c(C) \in C_0 \right\} \right]$$

$$= \mathbb{E}\left[ \text{card} \left\{ C \in \Phi : 32 : C \cap \left( C_0 + z_1 \right) \neq \emptyset , \ c(C) \in C_0 \right\} \right]$$

$$\leq \sum_{i \geq 1} \mathbb{E}\left[ \text{card} \left\{ C \in \Phi : C \cap \left( C_0 + z_i \right) \neq \emptyset , \ c(C) \in C_0 \right\} \right]$$

$$= \sum_{i \geq 1} \mathbb{E}\left[ \text{card} \left\{ C \in \Phi : C \cap C_0 \neq \emptyset , \ c(C) \in C_0 - z_i \right\} \right]$$
\[ = \mathbb{E} \text{card} \{ C \in \Phi : C \cap C_0^c \neq \emptyset \} < \infty, \]

hence \( \Phi_c \) satisfies the required finiteness assumption.

Stationarity: Let \( z \in \mathbb{R}^n \). Then

\[
\Phi_c + z = \sum_{C \in \Phi} \delta \left( (C + z) \cap C, C - c(C) \right).
\]

\[
= \sum_{C \in \Phi} \delta \left( (C + z) \cap C, C + z - c(C + z) \right).
\]

\[
= \sum_{C \in \Phi + z} \delta \left( (C) \cap C, C - c(C) \right) = (\Phi \cdot + z)_c.
\]

\[
\sim \Phi_c.
\]

Intuitively: Let \( c_0 \) denote the circumcentre - circ. Then

\[
g(\Phi_c) = g\left( \left( \Phi_c \right)^\circ \right) = g\left( \Phi_c^c \right) = \mathbb{E} \text{card} \{ C \in \Phi : c(C) \in C_0^c \}
\]

\[
= \sum_{i \geq 1} \mathbb{E} \text{card} \{ C \in \Phi : c(C) \in C_0^c, \ c_0(C) \in C_0^c + 2i \}
\]

\[
= \sum_{i \geq 1} \mathbb{E} \text{card} \{ C \in \Phi : c(C) \in C_0^c - 2i, \ c_0(C) \in C_0^c \}
\]

\[
= \mathbb{E} \text{card} \{ C \in \Phi : c_0(C) \in C_0 \}
\]

\[
= g(\Phi).
\]

The remaining assertion is left as an exercise. \( \square \)
Theorem 3.16 Let $\Phi$ be a stationary Poisson particle process in $\mathbb{R}^n$. Then $\Phi_c$ is an independently marked stationary Poisson process.

Proof Let $\varphi : \mathbb{C} \to \mathbb{R}^n \times \mathbb{C}^o$, $c \mapsto (c(c), c - c(c))$. Then

$$\{ \Phi_c (G) = k \} = \{ \varphi^{-1} (G) \} = k$$

for $G \in \mathcal{F}(\mathbb{R}^n \times \mathbb{C}^o)$, $k \in \mathbb{N}_0$. Hence, by Theorem 3.15 the process $\Phi_c$ is a stationary marked Poisson process by $\oplus$. From Theorems 2.23 and 2.26 we get that $\Phi_c$ is independently marked. $\square$

Let $\tilde{\Phi}$ be a marked point process with mark space $\mathbb{C}$. Let $\tilde{\varphi} : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(x, c) = c + x$. Then

$$\tilde{\Phi} = \tilde{\varphi} (\tilde{\Phi}) = \sum_{(x, c) \in \tilde{\Phi}} \delta_{c + x}$$

is a particle process provided it is locally finite with locally finite intensity measure. In this case, $\tilde{\Phi}$ is called a gran-grain process.

If $\tilde{\Phi}$ is such a gran-grain-process and stationary with mark distribution $\mathcal{Q}$, then, for $K \in \mathcal{C}$, we have

$$\mathbb{E} [\Phi (F_K)] = \mathbb{E} [\tilde{\varphi} (\tilde{\Phi})] = \int_{c} \int_{\mathbb{R}^n} \mathbb{1}_{c} \omega (c, c - c(c)) d \mathcal{Q} (c) d \omega (c)$$

$$= \int_{c} \mathbb{E} (c + K) \omega (c)$$

Note, $E \tilde{\Phi}$ is locally finite if $\int_{c} \mathbb{E} (c + K) \omega (c) < \infty$ for some $\omega > 0$. 
Subsequently, we investigate RACS which arise from particle processes via taking unions. If $\Phi$ is a stationary particle process, then

$$Z_\Phi = \bigcup_{C \in \Phi} C$$

is a random closed set and stationary. There is also a converse statement. For $K \in \mathbb{R} \setminus \{\emptyset\}$, let

$$N(K) = \min \left\{ m \in \mathbb{N} : K = \bigcup_{i=1}^m K_i, K_i \in X \right\}.$$ 

Note that $N : \mathbb{R} \rightarrow \mathbb{N}$ is measurable.

**Theorem 3.12.** For each RACS $Z$ in $\mathbb{R}^n$ there is a unique particle process $\Phi$ with $Z = Z_\Phi$, for which $\Phi \sim g(\Phi)$ whenever $Z \sim g(Z)$, for some $g \in \text{Bin}$.

If $Z$ is a random $\Phi$-set with $\mathbb{E} [N(Z \setminus C)] < \infty$ for all $C \in \Phi$, then $\Phi$ can be chosen as a particle process with convex particles.

In particular, a stationary RACS $Z$ can be generated starting from a stationary germ-grain process $\Phi$ via

$$Z_\Phi = \bigcup_{(x, C) \in \Phi} (C + x).$$
Definition. A RACS \( Z \) is called a germ-grain-model if there is an independently marked germ-grain-process \( \bar{\Phi} \) with \( Z \sim Z\bar{\Phi} \). If \( \bar{\Phi} \) is a Poisson process, then \( Z \) is called a Boolean model. In this situation, we write \( Z = Z(\bar{\Phi}, \mathcal{Q}) \) if \( \bar{\Phi} = \bar{\Phi} \otimes \mathcal{Q} \).

Theorem 3.13. A RACS \( Z \) is a stationary Boolean model if \( Z \) is the union set of a stationary Poisson particle process.

Proof. Let \( Z \) be a stationary Boolean model. There is an independently marked germ-grain-process \( \bar{\Phi} \) with \( Z \sim Z\bar{\Phi} \) and \( \bar{\Phi} \) is a Poisson process. By Theorem 1.25, \( \bar{\Phi} \) is a Poisson process as well, and hence also

\[
\bar{\Phi} := \sum_{(X,C) \in \bar{\Phi}} \delta_{X+C}
\]

is a Poisson process. Thus \( Z = Z\bar{\Phi} \). By Theorem 2.30, \( \bar{\Phi} \) is also stationary.

If \( Z = Z\bar{\Phi} \) and \( \bar{\Phi} \) is a stationary Poisson particle process, then \( \bar{\Phi} = \Phi \) is an independently marked, stationary Poisson process (cf. Theorem 3.16), and \( Z \) is stationary, since \( Z\bar{\Phi} = Z\bar{\Phi} \), it follows that \( Z \) is a stationary Boolean model. \( \square \)
Examples. Let \( Z = Z(y, Q) \) be a stationary Boolean model, let \( E \subset X \) with \( 0 \in M \). Let \( \Phi \) be a stationary particle process with intensity \( \gamma \) and shape distribution \( Q \) such that \( Z \sim Z_{\Phi} \).

1)\[ P (0 \in Z) = P (\Phi (C_{103}) = 0) = \exp \left( -\gamma \cdot \int_{E_0} \int_{C_{103}} 1 \{ C + z \in C_{103} \} \, d\sigma (dC) \right) = \exp \left( -\gamma \cdot \int_{E_0} V_n (C) \, d\sigma (dC) \right) = \exp \left( -V_n (\Phi) \right).\]

2) Contact distribution function:

\[ H (r) = P (d_n (0, z) \leq r \mid 0 \in Z) \]

By definition, \( d_n (0, z) \leq r \Leftrightarrow (0 + r M) \cap z \neq \emptyset \Leftrightarrow 0 \in Z + r M^+ \).

We obtain

\[ H (r) = \frac{P (0 \in Z, 0 \in Z + r M^+)}{P (0 \in Z)} = \frac{P (0 \in Z) - P (0 \in Z + r M^+)}{P (0 \in Z)} \]

\[ = 1 - \frac{P (0 \in Z + r M^+)}{P (0 \in Z)} \]

\[ > 1 - e^{-V_n (\Phi)} \]
Further, we define

\[ H^i (r) = 1 - \exp \left( \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right) \]

\[ = 1 - \frac{\exp \left( -\gamma \int_{\phi_\Phi} \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right)}{\exp \left( -\gamma \int_{\phi_\Phi} \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right)} \]

\[ = 1 - \frac{\exp \left( -\gamma \int_{\phi_\Phi} \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right)}{\exp \left( -\gamma \int_{\phi_\Phi} \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right)} \]

hence

\[ H^i (r) = 1 - \exp \left( -\gamma \int_{\phi_\Phi} \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right) \]

If \( M = B^n \) and \( \Phi \) is concentrated on \( X_0 \), then

\[ H^i (r) = 1 - \exp \left( -\gamma \sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right) \]

\[ = 1 - \exp \left( -\sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right) \]

\[ = 1 - \exp \left( -\sum_{\Phi (F+M) = 0} \phi_\Phi (\Phi) \right) \]

i.e.

\[ \lim_{r \to \infty} (1 - H^i (r)) = \sum_{j=1}^{n} k_j \alpha^j \bar{v}_{n-j} (\Phi) \]

Interpretation!
3) Special formulas for additive functionals of Boolean models

Let $Z$ be a stationary Boolean model with convex grains. Let $\gamma$ be the intensity and $\xi$ the direction distribution of $\xi$ with $Z = Z_{\xi}$. If $\gamma : \mathbb{R} \rightarrow \mathbb{R}_+$ is a geometric functional, we want to calculate

$$E \left[ \gamma (Z \cap K) \right]$$

for a fixed $K \subseteq \mathbb{R}^d$.

In this case, calculating this mean value means to express it in terms of the underlying data of the given particle process, i.e., $\gamma$ and $\xi$. For this, we need an additional property of $\gamma$ which is called additivity:

**Def.** A functional $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ with $\gamma (\emptyset) = 0$ is called additive if

$$\gamma (K \cup L) + \gamma (K \cap L) = \gamma (K) + \gamma (L) \quad \text{for all } K, L \subseteq \mathbb{R}^d.$$

Another property, which will be needed is conditionally boundedness. We call $\gamma$ conditionally bounded, if

$$\sup \left\{ |\gamma (K)| : K \in \mathcal{K}, \; K \subseteq L \right\} < \infty \quad \text{for all } L \subseteq \mathbb{R}^d.$$

Clearly, if $\gamma$ is monotone on $\mathcal{K}$, then it is also conditionally bounded.
For each realization of $\Phi$, there are $K_i$, $i \geq 1$, such that

$$\Phi = \sum_{i \geq 1} S_{K_i}, \quad \Xi = \bigcup_i K_i.$$  

If $K \in \Xi$ is fixed, then only finitely many of the sets $K_i$, $i \geq 1$, hit $K$, say $\Phi(F_k) = m$ and $K_i$, $K_j$ have nonempty intersection with $K$. Then

$$\varphi(\Xi \cap K) = \varphi\left(\bigcup_{i \geq 1} K_i \cap K\right) = \varphi\left(\bigcup_{i \in I} (K_i \cap K)\right)$$

$$= \sum_{k=1}^m \frac{m!}{k!} \sum_{|I|=k} \varphi\left(K \cap \bigcap_{i \in I} K_i\right)$$

$$= \sum_{k=1}^m \frac{m!}{k!} \sum_{(K_1, \ldots, K_k) \in \Phi} \varphi\left(K \cap \bigcap_{i=1}^k K_i \cap K_1 \cap \cdots \cap K_k\right).$$

Here clearly $m$ and $K_1, \ldots, K_m \in \Phi$ depend on the particular realization. Next we use the expansion obtained so far to derive an integrability condition that will be needed:

$$|\varphi(\Xi \cap K)| \leq \sum_{k=1}^m \frac{1}{k!} \sum_{|I|=k} \frac{1}{\prod_{i \in I} (m_i)} \varphi(K \cap \bigcap_{i \in I} K_i) \leq c(K) \sum_{k=1}^m \frac{1}{k!} \left(\begin{array}{c} m \\ k \end{array}\right) \leq c(K) 2^m = c(K) 2^\varphi(\Xi) \Phi(F_k).$$
But then
\[ \mathbb{E}[\varphi(2^n K)] \leq c(k) \mathbb{E}\left[ z^{\Phi(\bar{T}_k)} \right] \]
\[ = c(k) \cdot \sum_{k=0}^{\infty} 2^k \mathbb{P}(\Phi(\bar{T}_k) = k) \]
\[ = c(k) \sum_{k=0}^{\infty} 2^k \frac{\Theta(\bar{T}_k)^k}{k!} e^{-\Theta(\bar{T}_k)} \]
\[ = c(k) e \]
\[ \mathbb{E}[\varphi(2^n K)] < \infty. \]

This integrability condition will allow us to interchange summation and integration. Hence,
\[ \mathbb{E}[\varphi(2^n K)] = \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} \sum_{(K_{1},...,K_{k})} \varphi(K_{1},...,K_{k}) \]
\[ = \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} \int_{X} \cdots \int_{X} \varphi(K_{1},...,K_{k}) \Theta(dk_{1}) \cdots \Theta(dk_{k}) \]

From the special form of the intensity measure \( \Theta \) of \( \bar{T} \), we deduce
\[ \mathbb{E}[\varphi(2^n K)] = \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} x^{k} \int_{X_{0}} \cdots \int_{X_{0}} \varphi(K_{1},...,K_{k},(K_{k+1},...,K_{2^n})) \]
\[ = \mathbb{P}_{0}(dk_{1}) \cdots \mathbb{P}_{0}(dk_{k}). \]
The evaluation of these integrals requires further techniques from integral geometry. Here we only consider some special cases which can be treated more directly.

**Special cases:**

1) \( \Phi = X, K = \{0\} \)

\[ E \left( \Phi(2 \cap 101) \right) = \mathbb{P}(0 \in \mathcal{W}) \quad \text{and} \]

\[ \int_{X_0} \int_{X_0} \chi \left( \{0\} \cap (K_i + x_1) \cap \left( K_i + x_1 \right) \right) \, dx_1 \, dx_2 \]

\[ = \sum_{i=1}^{n} \, \frac{k}{n} \, V_n(K_i) . \]

The right-hand side thus yields

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \left( \int_{X_0} \, V_n(C) \, P_0(\mathcal{W}) \right)^k \]

\[ \overline{V_n(\Phi)} \]

\[ = - \sum_{k=1}^{\infty} \frac{(-\overline{V_n(\Phi)})^k}{k!} = 1 - e^{-\overline{V_n(\Phi)}} . \]

Next, we consider the case \( \Phi = V_{n-1} (= \frac{1}{2} \text{surface area}) \). We start with the following observation:
\[ V_{n-1} (K \cap (K_1 + x_1)) \]
\[ = \frac{1}{2} \mathcal{H}^{n-1} (\partial K \cap (K_1 + x_1)) + \frac{1}{2} \mathcal{H}^{n-1} (\partial (K_1 + x_1) \cap K), \]

Hence

\[ \int_{\mathbb{R}^n} V_{n-1} (K \cap (K_1 + x_1)) \, dx_1 \]
\[ = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\partial K} \frac{1}{1 + \mathcal{H}^{n-1} (z)} \mathcal{H}^{n-1} \left( (z-x_1) \right) \, dx_1 \, dz \]
\[ + \frac{1}{2} \int_{\mathbb{R}^n} \int_{\partial (K_1 + x_1)} \frac{1}{1 + \mathcal{H}^{n-1} (z)} \, dx_1 \, dz \]
\[ = \frac{1}{2} \int_{\partial K} \int_{\mathbb{R}^n} \frac{1}{1 + \mathcal{H}^{n-1} (z)} \mathcal{H}^{n-1} (x_1) \, dx_1 \, dz \]
\[ + \frac{1}{2} \int_{\partial K_1} \int_{\mathbb{R}^n} \frac{1}{1 + \mathcal{H}^{n-1} (z+x_1)} \mathcal{H}^{n-1} (dz) \, dx_1 \]
\[ = \frac{1}{2} \int_{\partial K_1} V_n (K_1) \mathcal{H}^{n-1} (dz) + \frac{1}{2} \int_{\partial K_1} V_n (K) \mathcal{H}^{n-1} (dz) \]
\[ = V_n (K_1) V_{n-1} (K) + V_n (K) V_{n-1} (K_1). \]

This relation can now be iterated and finally leads to
\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{i=0}^{k} \int_{\chi_0}^{\infty} \sum_{i=0}^{\infty} \int_{\chi_0}^{\infty} \int_{\chi_0}^{\infty} \cdots \int_{\chi_0}^{\infty} V_{n-1}(K \cap (K_{i+k_0}) \cap \cdots \cap (K_{k+1})) \, dx_1 \cdots dx_k
\]

where \( K_{0_i} = K \). Hence we obtain

\[
\mathbb{E} \left[ V_{n-1}(Z \cap K) \right] = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_{\chi_0}^{\infty} \left( \sum_{i=0}^{k-1} \mathbb{P}(dK_i) \mathbb{P}(dK) \right)
\]

\[
= \sum_{k=n}^{\infty} \frac{(-1)^{k-1}}{k!} V_{n-1}(K) \left( \frac{1}{k} \int_{\chi_0}^{\infty} \mathbb{P}(dK) \right)
\]

\[
+ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} V_{n}(K) \left( k \cdot \int_{\chi_0}^{\infty} \mathbb{P}(dK) \right) \int_{\chi_0}^{\infty} V_{n-1}(K) \mathbb{P}(dK)
\]

\[
= -V_{n-1}(K) \sum_{k=1}^{\infty} \frac{(-\mathbb{V}(\Phi))^k}{k!} + V_{n}(K) \mathbb{V}_{n-1}(\Phi) \sum_{k=1}^{\infty} \frac{(-\mathbb{V}(\Phi))^k}{(k-1)!}
\]

\[
= V_{n-1}(K) \left( 1 - e^{-\mathbb{V}(\Phi)} \right) + V_{n}(K) \mathbb{V}_{n-1}(\Phi) e^{-\mathbb{V}(\Phi)}
\]
Special cases:

\[ m = 2 : \]

\[
E \left[ A(z \Phi) \right] = A(K) \left( 1 - e^{-\frac{A(\Phi)}{K}} \right)
\]

\[
E \left[ u(z \Phi) \right] = A(K) \overline{u}(\Phi) e^{-\frac{A(\Phi)}{K}} + u(K) \left( 1 - e^{-\frac{A(\Phi)}{K}} \right)
\]

If \( z \) and \( \Phi \) are also isotropic, i.e. \( Q \) is not taken into account, then we also have

\[
E \left[ X(z \Phi) \right] = A(K) e^{-\frac{A(\Phi)}{K}} \left( \frac{1}{2\pi} \overline{u}(\Phi) e^{-\frac{A(\Phi)}{K}} \right)^2 + \frac{1}{2\pi} u(K) \overline{u}(\Phi) e^{-\frac{A(\Phi)}{K}} + 1 - e^{-\frac{A(\Phi)}{K}}
\]

If we define densities also for RACS via a special limit:

\[
\overline{u}(z) = \lim_{v \to \infty} \frac{E \left[ u(z \Phi) \right]}{A(vK)}
\]

which can be shown to exist in the present situation, then

\[
\overline{A}(z) = 1 - e^{-\frac{A(\Phi)}{K}}
\]

\[
\overline{u}(z) = \overline{u}(\Phi). e^{-\frac{A(\Phi)}{K}}
\]

\( \overline{u}(z) \) is also isotropic

\[
\overline{x}(z) = e^{-\frac{A(\Phi)}{K}} \left( 1 - \frac{1}{4\pi} \overline{u}(\Phi)^2 \right)
\]
If \( Z \) is a stationary and isotropic Boolean model and \( n = 2 \) and if \( \Lambda(\varepsilon), \varpi(\varepsilon), \Xi(\varepsilon) \) are considered to be observable, then \( \Lambda(\varepsilon), \varpi(\varepsilon) \) and \( \varpi \) can be determined from these equations. In particular, an estimator \( \hat{\varpi} \) for \( \varpi \) is given by

\[
\hat{\varpi} = \frac{1}{1 - \hat{\rho}} \left( \hat{\Lambda}(\varepsilon) + \frac{1}{4\pi (1 - \hat{\rho})^2} \hat{\Xi}(\varepsilon)^2 \right).
\]

Similar formulas exist for \( n = 3 \) and a stationary and isotropic Boolean model \( Z \).

More general integral geometric formulas are required in order to treat stationary Boolean models which are not necessarily isotropic. Here one enters early new territory at the frontier of research.