Basic Notions Of Topology

Topological Spaces, Bases and Subbases, Induced Topologies

Let X be an arbitrary set. A system $\mathcal{O}$ of subsets of X is called a *topology* on X, if the following holds:

a) The union of every class of sets in $\mathcal{O}$ is a set in $\mathcal{O}$, i.e. for an arbitrary index set I we have:
$$O_i \in \mathcal{O} \text{ for all } i \in I \implies \bigcup_{i \in I} O_i \in \mathcal{O}.$$

b) The intersection of every finite class of sets in $\mathcal{O}$ is a set in $\mathcal{O}$, i.e. for all $n \in \mathbb{N}$ we have:
$$O_1, \ldots, O_n \in \mathcal{O} \implies \bigcap_{i=1}^{n} O_i \in \mathcal{O}.$$

c) $X, \emptyset \in \mathcal{O}$.

A *topological space* is a pair $(X, \mathcal{O})$ consisting of a set X and a topology $\mathcal{O}$ on X. The sets in $\mathcal{O}$ are called the *open sets*; the complement of an open subset of X is called *closed*.

A system $\mathcal{B}$ of open subsets of a topological space $(X, \mathcal{O})$ ist called an *open basis* of the topology, if every open set $O \in \mathcal{O}$ is the union of sets from $\mathcal{B}$, i.e. for all $O \in \mathcal{O}$ holds:

$$\forall\, x \in \mathcal{O}\, \exists\, B \in \mathcal{B} : B \subset O \text{ and } x \in B.$$

An *open subbasis* of $(X, \mathcal{O})$ is a system $\mathcal{S} \subset \mathcal{O}$ such that the set of all finite intersections of sets from $\mathcal{S}$ forms an open basis of $\mathcal{O}$. An open bases and respectively an open subbasis is called *countable*, if it only consists of countably many sets.

Let X be an arbitrary set and $\mathcal{S}$ any class of subsets of X. Then there exists only one topology $\mathcal{O}$ on X for which $\mathcal{S}$ forms a subbases. This topology simply consists of the unions of finite intersections of sets in $\mathcal{S}$.

Example: Let $\mathcal{F}$ be class of all closed subsets of $\mathbb{R}^n$. We equip $\mathcal{F}$ with the unique topology for which the system
$$\{\mathcal{F}^C : C \in \mathcal{C}\} \cup \{\mathcal{F}_G : G \in \mathcal{G}\}$$
forms an open subbasis. This topology is usually called the *topology of closed convergence*. The system
$$\tau := \{\mathcal{F}^C_{G_1,\ldots,G_k} : C \in \mathcal{C},\, G_1, \ldots, G_k \in \mathcal{G},\, k \in \mathbb{N}_0\},$$
where $\mathcal{F}^C_{G_1,\ldots,G_k} := \mathcal{F}^C \cap \mathcal{F}_{G_1} \cap \cdots \cap \mathcal{F}_{G_k}$, is an open basis of the topology of closed convergence. Furthermore there exists a subset $\tau' \subset \tau$ such that τ' is a countable open basis of the topology of closed convergence.

If $U \subset X$ is a subset of a topological space $(X, \mathcal{O})$, then the class $\mathcal{O}_U := \{O \cap U \mid O \in \mathcal{O}\}$ of subsets of U forms a topology on U, the so-called *induced topology*. $(U, \mathcal{O}_U)$ is usually referred to as a *subspace* of X.

Example: $\mathcal{C} \subset \mathcal{F}$ can be equipped with the topology induced by the topology of closed convergence.

If $\mathcal{O}$ and $\mathcal{O}'$ both are topologies on the same set X, we say that $\mathcal{O}$ is *finer* than $\mathcal{O}'$, if $\mathcal{O}' \subset \mathcal{O}$. $\mathcal{O}'$ on the other hand is then said to be *coarser* than $\mathcal{O}$.

Example: On $\mathcal{C}$ the topology generated by the Hausdorff metric is finer than the topology induced by the topology of closed convergence.

Neighbourhoods, Convergence, Haussdorff Spaces

Let $(X, \mathcal{O})$ be a topological space and $x \in X$. A subset $N \subset X$ is called a *neighbourhood of* x, if there exists an open set $O \in \mathcal{O}$ such that $O \subset N$ and $x \in O$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space $(X, \mathcal{O})$ is said to converge to $x \in X$, if for every neighbourhood U of x there exists a $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$.

Example: Let $(F_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and $F \in \mathcal{F}$. Then the following statements a)and b) are equivalent:

a) $F_j \to F$ for $j \to \infty$ in the topology of closed convergence.

b) Both c1) and c2) hold:

  c1) For all $x \in F$ there exists for almost all $j \in \mathbb{N}$ an element $x_j \in F_j$ such that $x_j \to x$ for $j \to \infty$.

  c2) For every subsequence $(F_{j_k})_{k\in\mathbb{N}}$ and every convergent sequence $(x_{j_k})_{k\in\mathbb{N}}$ with $x_{j_k} \in F_{j_k}$ we have $\lim_{k\to\infty} x_{j_k} \in F$.

Let $(X, \mathcal{O})$ have a countable basis. Then the following are equivalent:

a) $A \subset X$ is closed.

b) For every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \to a$ we have $a \in A$.

A *Hausdorff space* is a topological space in which each pair of distinct points can be separated by disjoint neighbourhoods.

Example: $\mathcal{F}$ equipped with the topology of closed convergence is a Hausdorff space.

Continuous Mappings

A mapping $f : X \to Y$ between two topological spaces $(X, \mathcal{O}_1)$ and $(Y, \mathcal{O}_2)$ is called *continuous* if the preimage of every open set in Y is an open set in X, i.e.

$$f : X \to Y \text{ continuous} \quad :\Longleftrightarrow \quad \forall\, O \in \mathcal{O}_2 : f^{-1}(O) \in \mathcal{O}_1.$$

Let $\mathcal{S}_2$ be an open subbasis of $\mathcal{O}_2$. Then f is continuous if and only if

$$\forall S \in \mathcal{S}_2 : f^{-1}(S) \in \mathcal{O}_1.$$

Let $(X, \mathcal{O})$ have a countable basis. Then $f : X \to Y$ is continuous if and only if for all $x \in X$ holds:

$$\text{For all}\, (x_n)_{n\in\mathbb{N}} \text{ with}\, x_n \to x \text{ we have}\, f(x_n) \to f(x).$$

Compactness

A topological space $(X, \mathcal{O})$ is called *compact*, if every open cover of X has a finite subcover, i.e. for every index set I we have:

$$\bigcup_{i \in I} O_i = X \text{ and } O_i \in \mathcal{O} \text{ for all } i \in I \quad \Longrightarrow \quad \exists\, I' \subset I : | I' | < \infty \text{ and } \bigcup_{i \in I'} O_i = X.$$

We call a subset $A \subset X$ of X compact, if the subspace $(A, \mathcal{O}_A)$ is a compact.

$(X, \mathcal{O})$ is said to be *locally compact*, if it is a Hausdorff space and for each element $x \in X$ there is a compact neighbourhood of x.

Every compact Hausdorff space $(X, \mathcal{O})$ with a countable basis is *sequential compact*, i.e. every sequence $(x_n)_{n \in \mathbb{N}}$ in X has a convergent subsequence.

Example: $\mathcal{F}$ is compact, $\mathcal{F}' := \mathcal{F} \setminus \{\emptyset\}$ is locally compact but not compact.

Metric Spaces and Metrizability

A set X is called a *metric space* if there exists a mapping $d : X \times X \to [0, \infty)$ with the following properties:

a) $d(x, y) = 0 \quad \Longleftrightarrow \quad x = y$,

b) $d(x, y) = d(y, x)$ for all $x, y \in X$,

c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We say that a subset $O \subset X$ is open if for every $x \in X$ there is an $r > 0$ such that

$$B(x, r) := \{y \in X \mid d(x, y) < r\} \subset O.$$

The class $\mathcal{O} := \{O \subset X \mid O \text{ open}\}$ of all open subsets of X forms a topology on X, usually called the topology *generated by* d.

Example: $\mathcal{C}$ equipped with the Hausdorff metric d.

Let X be a set equipped with two metrics d and d'. Let $\mathcal{O}$ and $\mathcal{O}'$ denote the topologies generated by d and d' respectively. Then $\mathcal{O}$ is finer than $\mathcal{O}'$ if and only if every sequence $(x_n)_{n \in \mathbb{N}}$ in X that converges to $x \in X$ with respect to d converges to x with respect to d' as well.

A *metrizable* space is a topological space $(X, \mathcal{O})$ with the property that there exists a metric d on the set X whose generated topology is precisely $\mathcal{O}$.

Both compact Hausdorff spaces and locally compact spaces are metrizable if and only if they have a countable bases.

Example: $\mathcal{F}$ and $\mathcal{F}'$ are metrizable.