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INSTITUT FÜR ALGEBRA UND GEOMETRIE

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Stochastic and Integral Geometry I (WS 06/07)

Exercise Sheet 1

Please, hand in your solutions at the end of the **lecture** on **Monday, Nov 6th**!

1. Show that, for any nonempty compact set $C \subset \mathbb{R}^d$, among the balls $B(x,r) := \{y \in \mathbb{R}^d : |x-y| \leq r\}$ with $x \in \mathbb{R}^d$ and $r \geq 0$ containing C there is a unique one of smallest radius (called the *circumball* of C).
   Hint: Existence *and* uniqueness have to be shown here!

2. Let $\xi_1, \dots, \xi_n$ be i.i.d. UR-points in the Borel set $K \subset \mathbb{R}^d$ and $X_n = \{\xi_1, \dots, \xi_n\}$. Show that it is justified to call X_n a point process, i.e. show that X_n is a measurable mapping of some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to $(N, \mathcal{N})$.

3. Let $\xi_1, \dots, \xi_n$ be i.i.d. UR-points in $K \subset \mathbb{R}^d$ and $X_n = \{\xi_1, \dots, \xi_n\}$. Show that for pairwise disjoint Borel sets $B_1, \dots, B_k \subset \mathbb{R}^d$ the random vector $(|X_n \cap B_1|, \dots, |X_n \cap B_k|)$ (taking values in $(\mathbb{N}_0)^k$) has a multinomial distribution. (compare Lemma 1.1.1(b) in your lecture notes)

4. Recall that random variables $X_1, \dots, X_k$ taking values in $\mathbb{N}_0$ are (stochastically) independent if and only if

$$\mathbb{P}(X_1 = i_1, \dots, X_k = i_k) = \prod_{j=1}^{k} \mathbb{P}(X_j = i_j) \text{ for all } i_1, \dots, i_k \in \mathbb{N}_0.$$

   For $j = 1, \dots, k$, assume that the random variable X_j is Poisson-distributed with parameter μ_j and that $X_1, \dots, X_k$ are independent. Show that the random variable $Y := X_1 + \ldots + X_k$ is Poisson-distributed with parameter $\mu_1 + \ldots + \mu_k$.

5*. (*will be discussed in the first problem class; not to be solved at home*)
Recall that for the space

$$N = \{\eta \subset \mathbb{R}^d : |\eta \cap C| < \infty \text{ for all compact sets } C \subset \mathbb{R}^d\}$$

of locally finite sets in $\mathbb{R}^d$, the σ-algebra $\mathcal{N}$ generated by the family

$$\mathcal{F} := \{N_{B,k} : B \text{ Borel subset of } \mathbb{R}^d, k \in \mathbb{N}_0\}$$

is considered, where $N_{B,k} = \{\eta \in N : |\eta \cap B| = k\}$. Show that already the subfamily

$$\mathcal{F}_0 := \{N_{B,0} : B \text{ Borel subset of } \mathbb{R}^d\}$$

generates $\mathcal{N}$ and, moreover, that $\mathcal{F}_0$ is intersection stable.