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Stochastic and Integral Geometry I (WS 06/07)

Exercise Sheet 2

Please, hand in your solutions at the end of the **lecture** on **Monday, Nov 13th**!

1. Let
$$(\mathbb{R}^d)^m_{\neq} := \left\{(x_1, \dots, x_m) : x_i \in \mathbb{R}^d, x_i \neq x_j \text{ for all } i \neq j\right\}$$
and $\mathcal{B}_{\neq}$ be the Borel σ-algebra on $(\mathbb{R}^d)^m_{\neq}$. $\mathcal{B}_{\neq}$ is equivalent to the trace σ-algebra $\mathcal{B}((\mathbb{R}^d)^m) \cap (\mathbb{R}^d)^m_{\neq}$ of the Borel σ-algebra $\mathcal{B}((\mathbb{R}^d)^m)$ on the set $(\mathbb{R}^d)^m_{\neq}$. (Why?) Show that $\mathcal{B}_{\neq}$ is generated by the system
$$\mathcal{D} := \left\{A_1 \times \ldots \times A_m : A_i \in \mathcal{B}(\mathbb{R}^d), A_i \cap A_j = \emptyset \text{ for all } i \neq j\right\}.$$

2. Let $0 < \gamma < \infty$. Theorem 1.2.1 states the existence of a point process X on $\mathbb{R}^d$ (the Poisson process with intensity γ) characterized by the property that for each bounded Borel set $B \subset \mathbb{R}^d$
$$\mathbb{P}(|X \cap B| = k) = e^{-\gamma\lambda_d(B)} \frac{[\gamma\lambda_d(B)]^k}{k!} \quad \text{for } k = 0, 1, 2, \dots. \tag{1}$$
Show that X satisfies (1) for all Borel sets $B \subset \mathbb{R}^d$. If $\lambda_d(B)$ is infinite, (1) has to be interpreted as a *degenerated Poisson distribution* for which the random variable $|X \cap B|$ is allowed to assume values in $\mathbb{N}_0 \cup \{+\infty\}$ and which is given by $\mathbb{P}(|X \cap B| = k) = 0$ for $k \in \mathbb{N}_0$ and $\mathbb{P}(|X \cap B| = +\infty) = 1$.

3. For a stationary Poisson process X in $\mathbb{R}^d$ with intensity γ, let
$$d_X := \inf\{\|x\| : x \in X\}.$$
   (a) Show that d_X is a random variable and determine the distribution function H of d_X (given by $H(r) = \mathbb{P}(d_X \leq r)$).
   (b) Show that
$$H(r) = \lim_{\varepsilon \to 0} \mathbb{P}\left(|X \cap B(r)| \geq 2 \,\middle|\, |X \cap B(\varepsilon)| = 1\right),$$
   where $B(s)$ denotes the ball of radius s centered at the origin.

4. Let X be a stationary Poisson process on $\mathbb{R}^d$ with intensity γ and K, M compact subsets of $\mathbb{R}^d$.
   (a) Show that the mapping $x \mapsto \lambda_d(K \cap (M + x)), \mathbb{R}^d \to [0, +\infty)$ is measurable.
   (b) Prove that
$$\mathbb{E} \sum_{x \in X} \lambda_d(K \cap (M + x)) = \gamma\lambda_d(K)\lambda_d(M).$$