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Stochastic and Integral Geometry I (WS 06/07)

Exercise Sheet 3

Please, hand in your solutions at the end of the **lecture** on **Monday, Nov 20th**!

1. Let $k \in \mathbb{N}$, $\gamma_1, \dots, \gamma_k > 0$ be different positive real numbers and $p_1, \dots, p_k \in (0, 1]$ such that $\sum_{i=1}^k p_i = 1$. Define γ to be the discrete random variable taking values in $\{\gamma_1, \dots, \gamma_k\}$ given by
$$\mathbb{P}(\gamma = \gamma_i) = p_i \text{ for } i = 1, \dots, k.$$
Let X be a point process on $\mathbb{R}^d$ with conditional distribution
$$\mathbb{P}(X(A) = l \mid \gamma = \gamma_i) = e^{-\gamma_i \lambda_d(A)} \frac{(\gamma_i \lambda_d(A))^l}{l!}$$
for all $A \in \mathcal{B}$, $l \in \mathbb{N}_0$ and $i \in \{1, \dots, k\}$.
   (a) Show that X is stationary.
   (b) Show that X is *not* a Poisson process for $k \geq 2$.

2. Let X be a stationary Poisson process on $\mathbb{R}^2$ with intensity $\gamma > 0$. To each point $x \in X$ we attach a ball $B(x, 1)$ with centre x and radius 1. Let E denote the x_1-axis in $\mathbb{R}^2$ and $\pi_E : \mathbb{R}^2 \to E$ the orthogonal projection onto E.
   (a) X induces a point process X' on E given by
$$X' := \{\pi_E x : x \in X \text{ and } B(x, 1) \cap E \neq \varnothing\}.$$
Show that X' is a stationary Poisson process on E with intensity 2γ.
   (b) For $x \in \mathbb{R}^2$ denote by S_x the intersection of the ball $B(x, 1)$ with E. Obviously, if $B(x, 1) \cap E \neq \varnothing$, S_x is a line segment in E centered at $\pi_E x$ and its length $\lambda(S_x)$ depends on the distance of x to E. Let $B \subset \mathbb{R}$ be a Borel set with $\lambda(B) = 1$. Compute
$$\mathbb{E} \sum_{\substack{x \in X \\ \pi_E x \in B}} \lambda(S_x) \qquad \text{and} \qquad \mathbb{E} \sum_{x \in X} \lambda(S_x \cap B)$$
and compare both results.

3. Let X be a stationary Poisson process on $\mathbb{R}^2$ with intensity $\gamma > 0$. Let $B(r)$ be a 2-dimensional ball of radius r. Compute
$$\mathbb{E} \sum_{\substack{x, y \in X \cap B(r) \\ x \neq y}} \|x - y\| \qquad \text{and} \qquad \mathbb{E} \binom{|X \cap B(r)|}{2}^{-1} \sum_{\substack{x, y \in X \cap B(r) \\ x \neq y}} \|x - y\|$$
and compare. What is the meaning of these expectations?