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# **Stochastic and Integral Geometry I** (WS 06/07)

Exercise Sheet 4

Please, hand in your solutions at the end of the **lecture** on **Monday, Nov 27th**!

1. Let $X$ be a stationary Poisson process on $\mathbb{R}^2$ with intensity $\gamma > 0$ and let $Y := X \cap I$ be the restriction of $X$ to the set $I = [0, \pi) \times \mathbb{R}$. For each $x = (\alpha, p) \in I$, denote by $g(\alpha, p)$ the line with angle $\alpha$ (to the $x_1$-axis) and (signed) distance $p$ to the origin (where $p$ is positive, if the orthogonal projection of the origin onto $g(\alpha, p)$ is either in the upper halfplane or on the positive $x_1$-axis, and negative otherwise.)

   (a) Show that, for $C \in \mathcal{C}$, the random variable
   $$Z_C = |\{(\alpha, p) \in Y : g(\alpha, p) \cap C \neq \varnothing\}|$$
   is almost surely finite und has a Poisson distribution.

   (b) Show that, for any compact, convex polygon $P \subset \mathbb{R}^2$, $\mathbb{E}Z_P = \gamma L(P)$, where $L(P)$ denotes the perimeter of $P$.

   (c) Show that, for any compact, convex set $K \subset \mathbb{R}^2$,
   $$\mathbb{E} \sum_{(\alpha,p)\in Y} \lambda_1(g(\alpha, p) \cap K) = \pi\gamma\lambda_2(K).$$

2. Show that the mapping $c : \mathcal{C} \to \mathbb{R}^d$ which assigns to each set $C \in \mathcal{C}$ its circumcenter $c(C)$ is continuous (with respect to the Hausdorff metric in $\mathcal{C}$).

3. Let $X$ be a stationary Poisson process on $\mathbb{R}^d$ with intensity $\gamma > 0$ and $\mathbb{Q}$ a probability measure on $\mathcal{C}_0$. As in the lecture in Section 1.3, we choose an enumeration $X = \{\xi_1, \xi_2, \ldots\}$ and a sequence $Z_1, Z_2, \ldots$ of i.i.d. random compact sets with distribution $\mathbb{Q}$ such that $X, Z_1, Z_2, \ldots$ are independent. Let $Y := \{\xi_1 + Z_1, \xi_2 + Z_2, \ldots\}$. In the lecture it has been shown that under the assumption
   $$\int_{\mathcal{C}_0} \lambda_d(C + D) d\mathbb{Q}(C) < \infty \quad \text{for } D \in \mathcal{C}$$
   the process $Y$ is locally finite in the sense that the mean number of sets in $Y$ intersecting a given set $D \in \mathcal{C}$ is finite:
   $$\mathbb{E} \sum_{i=1}^{\infty} \mathbf{1}_{\{(\xi_1+Z_i)\cap D\neq\varnothing\}} < \infty.$$
   Prove that $Y$ is a Poisson process on $\mathcal{C}$ with intensity measure $\Theta$ given by
   $$\Theta(A) = \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^d} \mathbf{1}_A(C + x) d\lambda_d(x) d\mathbb{Q}(C) \quad \text{for } A \in \mathcal{B}(\mathcal{C}),$$
   i.e. show that, for each set $A \in \mathcal{B}(\mathcal{C})$ and $k = 0, 1, 2, \ldots$,
   $$\mathbb{P}(|Y \cap A| = k) = e^{-\Theta(A)} \frac{\Theta(A)^k}{k!}.$$
   Hint: For $A \in \mathcal{B}(\mathcal{C})$ and $r > 0$, let $A_r := \{C \in A : c(C) \in B(r)\}$. Prove the assertion first for the sets $A_r$ and then let $r \to \infty$.