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Stochastic and Integral Geometry I (WS 06/07)

Exercise Sheet 5

Please, hand in your solutions at the end of the **lecture** on **Monday, Dec 4th**!

1. Let Z be a stationary random closed set in $\mathbb{R}^d$ and $\overline{V}_d(Z) := \mathbb{P}(0 \in Z)$ the probability that Z contains the origin.

   (a) Show that, for all $B \in \mathcal{B}(\mathbb{R}^d)$,
   $$\mathbb{E}\lambda_d(Z \cap B) = \overline{V}_d(Z)\lambda_d(B).$$

   (b) Assume that $\overline{V}_d(Z) < 1$. Show that the contact distribution function H^B of Z with structuring element $B \in \mathcal{K}$ is then given by the formula
   $$H^B(r) = \frac{\overline{V}_d(Z + rB^*) - \overline{V}_d(Z)}{1 - \overline{V}_d(Z)} \quad \text{for } r > 0.$$

2. Let Z be the Boolean model in $\mathbb{R}^2$ with intensity $\gamma > 0$ and the trivial grain distribution $\mathbb{Q}$ that chooses the unit ball $B(1)$ with probability 1. Then the particles of Z are the balls $x + B(1)$ centered at the points $x \in X$ of the underlying Poisson process X.
   A point $z \in Z$ is called *visible* if $|\{x \in X : z \in x + B(1)\}| = 1$, i.e. if z is contained in exactly one particle of Z. Let $u = (0, -1)$ be the unit vector pointing in direction of the negative x_2-axis. Geometrically, u is the osculation point of the unit ball with a horizontal tangent line (touching the ball from below). Now let
   $$\tilde{X} := \{x + u : x \in X, x + u \text{ visible}\}$$
   be the point process of visible lower osculation points of Z with horizontal tangent lines. Show that, $\tilde{X}$ is a stationary point process in $\mathbb{R}^d$ with intensity $\tilde{\gamma} = \gamma e^{-\pi\gamma}$.

3. Recall that a functional $\varphi : \mathcal{R} \to \mathbb{R}$ is called *additive* if and only if $\varphi(\emptyset) = 0$ and, for all $K, M \in \mathcal{R}$,
   $$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M).$$
   Here $\mathcal{R}$ denotes the *convex ring*, i.e. the family of all finite unions of compact convex sets in $\mathbb{R}^d$. Prove the so called *inclusion-exclusion principle*, i.e. prove that for additive φ the formula
   $$\varphi\Big(\bigcup_{i=1}^m C_i\Big) = \sum_{k=1}^m (-1)^{k+1} \sum_{1 \le i_1 < \ldots < i_k \le m} \varphi(C_{i_1} \cap \ldots \cap C_{i_k})$$
   holds for each $m \in \mathbb{N}$ and $C_1, \ldots, C_m \in \mathcal{R}$.

4*. For $C \in \mathcal{C}$, let $A_C = \{D \in \mathcal{C} : D \cap C \neq \emptyset\}$. Show that the family $\{A_C : C \in \mathcal{C}\}$ generates the Borel σ-algebra $\mathcal{B}(\mathcal{C})$.